Uniform Provability in Classical Logic[†]

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Abstract

Uniform proofs are sequent calculus proofs with the following characteristic: the last step in the derivation of a complex formula at any stage in the proof is always the introduction of the top-level logical symbol of that formula. We investigate the relevance of this uniform proof notion to structuring proof search in classical logic. A logical language in whose context provability is equivalent to uniform provability admits of a goal-directed proof procedure that interprets logical symbols as search directives whose meanings are given by the corresponding inference rules. While this uniform provability property does not hold directly of classical logic, we show that it holds of a fragment of it that only excludes essentially positive occurrences of universal quantifiers under a modest, sound, modification to the set of assumptions: the addition to them of the negation of the formula being proved. We further note that all uses of the added formula can be factored into certain derived rules. The resulting proof system and the uniform provability property that holds of it are used to outline a proof procedure for classical logic. An interesting aspect of this proof procedure is that it incorporates within it previously proposed mechanisms for dealing with disjunctive information in assumptions and for handling hypotheticals. Our analysis sheds light on the relationship between these mechanisms and the notion of uniform proofs.

Key Words: classical logic, proof theory, proof search, uniform provability, logic programming.

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1 Introduction

Uniform proofs as identified in [MNPS91] capture a goal-directedness in proof search. In essence, a uniform proof is a sequent calculus proof that is found by constructing, at each stage, a proof for a *single* "goal" formula from a collection of assumptions. Further, if the goal formula is non-atomic, then the search for a uniform proof for it may proceed by first simplifying the formula in accordance with the inference rule pertaining to its top-level logical symbol. One reason for interest in this category of proofs is that it provides a framework for interpreting the logical symbols in the formulas being proved as primitives for directing search and the inference rules pertaining to these symbols as specifications of their search semantics. This viewpoint is exploited in [MNPS91] in describing a proof-theoretic foundation for logic programming. In particular, classes of formulas and proof relations are thought to constitute a satisfactory basis for logic programming just in case provability in their context is equivalent to the existence of a uniform proof. The virtue of this "uniform provability property" is that it permits a duality between a declarative and a search-related reading for logical symbols that appears to be central to a programming use of logic. This criterion for logic programming has turned out to be of actual practical interest: it is satisfied by the logic of Horn clauses that underlies Prolog and has also been instrumental in the discovery of rich and useful but yet logically principled extensions to this language [HM94, Mil94, MNPS91, NM94, Pfe89].

Our interest in this paper is in a different, but related, utility for uniform proofs, namely, as a device for structuring the search for proofs of formulas. A fair degree of determinism can be imparted to such a search in situations where the uniform provability property holds of a logical language, and this fact has been utilized in the past in describing efficient proof procedures for suggested extensions to logic programming; see, for instance, [Mil91, Nad93]. However, there are logics of which the uniform provability property does not hold directly. For example, suppose that our assumption set contains the formula $p(a) \vee p(b)$ and that our desire is to prove $\exists x\, p(x)$; our assumption set contains disjunctive information in this case, typifying the situation in disjunctive logic programming. We observe first that $\exists x\, p(x)$ is provable from $p(a) \vee p(b)$ in classical, intuitionistic and minimal logics. However, there is no uniform proof in any of these systems for the given formula from the relevant assumption set; for such a proof to exist, it is necessary that $p(t)$ be provable from the same assumption set for some specific term t, and this requirement clearly does not hold. As another example, consider the formula $(p \supset q) \vee p$ in classical logic. While this formula has a proof, it does not have a uniform one; the latter kind of proof would exist only if either $(p \supset q)$ or p is provable and, once again, clearly, neither is. The broad question motivating the discussions in this paper is whether some benefit may be derived from the uniform proof notion in structuring proof search even in situations such as these where the uniform provability property does not hold of the underlying logic.

We answer this question below relative to classical logic. The main observations we make are the following. Suppose that we wish to show that a formula G follows from a set of assumptions Γ in classical logic. We may not be able to do this immediately by looking for a uniform proof. However, under a modest restriction in the syntax of Γ and G, there is a simple augmentation of Γ that makes the search for uniform proofs a complete strategy. In particular, we show that if universal quantifiers do not occur positively in G or negatively in Γ, then there is a proof for G from Γ in classical logic if and only if there is a uniform proof for G from $\Gamma, (G \supset \bot)$. This result is actually a strengthening of the one in [NL95] in that this "modified" uniform provability property is shown to hold for an extension of disjunctive logic programming that permits hypothetical goals. We further note that all uses of the added formula can be factored into certain derived rules. These observations are then used to describe a simplified proof system for classical logic.[1] The resulting proof system provides the basis for a proof procedure that generalizes the one usually employed in logic programming towards dealing with all of classical logic. An interesting aspect of this proof procedure is that its rule for "backchaining" incorporates within it the restart mechanism of nH-Prolog [Lov87, Lov91] for dealing with disjunctive information in assumption sets and the mechanism with the same name of QNR-Prolog [Gab85] for handling hypotheticals in goals.

2 Logical preliminaries

We will work within the framework of a first-order logic in this paper. The logical symbols that we assume as primitive are $\top$, $\bot$, $\wedge$, $\vee$, $\supset$, $\exists$, and $\forall$. The first two symbols in this collection denote the tautologous and the contradictory propositions, respectively. The symbol $\neg$ is not primitive to our language, but it can be easily defined using other symbols that are primitive: $\neg A$ can be thought of as an abbreviation for $(A \supset \bot)$.

Notions of derivation that are of interest to us are formalized by sequent calculi. A sequent in our context is a pair of multisets of formulas. Assuming that Γ and Δ are its elements, the pair is written as $\Gamma \longrightarrow \Delta$ and Γ and Δ are referred to as its antecedent

[1] The presentation of this proof system assumes a syntactic transformation of formulas. As we note later, that the only indispensable aspect of this transformation is the elimination of essentially positive occurrences of universal quantifiers.

$$\frac{B, B, \Gamma \longrightarrow \Delta}{B, \Gamma \longrightarrow \Delta}\ \text{contr-L} \qquad \frac{\Gamma \longrightarrow \Delta, B, B}{\Gamma \longrightarrow \Delta, B}\ \text{contr-R}$$

$$\frac{\Gamma \longrightarrow \Delta, \bot}{\Gamma \longrightarrow \Delta, D}\ \bot\text{-R}$$

$$\frac{B, D, B \wedge D, \Gamma \longrightarrow \Delta}{B \wedge D, \Gamma \longrightarrow \Delta}\ \wedge\text{-L} \qquad \frac{\Gamma \longrightarrow \Delta, B \qquad \Gamma \longrightarrow \Delta, D}{\Gamma \longrightarrow \Delta, B \wedge D}\ \wedge\text{-R}$$

$$\frac{B, \Gamma \longrightarrow \Delta \qquad D, \Gamma \longrightarrow \Delta}{B \vee D, \Gamma \longrightarrow \Delta}\ \vee\text{-L}$$

$$\frac{\Gamma \longrightarrow \Delta, B}{\Gamma \longrightarrow \Delta, B \vee D}\ \vee\text{-R} \qquad \frac{\Gamma \longrightarrow \Delta, D}{\Gamma \longrightarrow \Delta, B \vee D}\ \vee\text{-R}$$

$$\frac{B \supset D, \Gamma \longrightarrow B, \Delta \qquad D, \Gamma \longrightarrow \Theta}{B \supset D, \Gamma \longrightarrow \Delta, \Theta}\ \supset\text{-L} \qquad \frac{B, \Gamma \longrightarrow \Delta, D}{\Gamma \longrightarrow \Delta, B \supset D}\ \supset\text{-R}$$

$$\frac{[t/x]B, \forall x\, B, \Gamma \longrightarrow \Delta}{\forall x\, B, \Gamma \longrightarrow \Delta}\ \forall\text{-L} \qquad \frac{\Gamma \longrightarrow \Delta, [t/x]B}{\Gamma \longrightarrow \Delta, \exists x\, B}\ \exists\text{-R}$$

$$\frac{[c/x]B, \Gamma \longrightarrow \Delta}{\exists x\, B, \Gamma \longrightarrow \Delta}\ \exists\text{-L} \qquad \frac{\Gamma \longrightarrow \Delta, [c/x]B}{\Gamma \longrightarrow \Delta, \forall x\, B}\ \forall\text{-R}$$

Figure 1: Rules for Deriving Sequents

and succedent, respectively. Such a sequent is an axiom if either $\top \in \Delta$ or for some A that is either $\bot$ or an atomic formula,[2] it is the case that $A \in \Gamma$ and $A \in \Delta$. The rules that may be used in constructing sequent proofs are those that can be obtained from the schemata shown in Figure 1. In these schemata, Γ, Δ and Θ stand for multisets of formulas, B and D stand for formulas, c stands for a constant, x stands for a variable and t stands for a term. The notation B, Γ (Δ, B) is used here for a multiset containing the formula B whose remaining elements form the multiset Γ (respectively, Δ). Further, expressions of the form $[t/x]B$ are used to denote the result of replacing all free occurrences of x in B by t, with bound variables being renamed as needed to ensure the logical correctness of these replacements. There is the usual proviso with respect to the rules produced from the schemata $\exists$-L and $\forall$-R: the constant that replaces c should not appear in the formulas that form the lower sequent. The purpose of the schemata contr-L and contr-R is to blur the distinction between sets and multisets, and so we will be ambivalent about this difference at times.

We are interested in three notions of derivability for sequents of the form $\Gamma \longrightarrow B$. A **C**-proof for such a sequent is a derivation obtained by making arbitrary uses of the inference rules. We denote the existence of such a proof, which is a classical proof, for the sequent by writing $\Gamma \vdash_C B$. **I**-proofs, that formalize the notion of intuitionistic derivability, are **C**-proofs in which every sequent has exactly one formula in its succedent. We write $\Gamma \vdash_I B$ to indicate the existence of an **I**-proof for $\Gamma \longrightarrow B$. Finally, a *uniform proof* is an **I**-proof in which any sequent whose succedent contains a non-atomic formula occurs only as the lower sequent of an inference rule that introduces the top-level logical symbol of that formula. Notice that if $\Gamma \longrightarrow B$ has a uniform proof, then the following must be true with respect to this proof:

(1) If B is $C \wedge D$, then the sequent must be inferred by $\wedge$-R from $\Gamma \longrightarrow C$ and $\Gamma \longrightarrow D$.

(2) If B is $C \vee D$ then the sequent must be inferred by $\vee$-R from either $\Gamma \longrightarrow C$ or $\Gamma \longrightarrow D$.

(3) If B is $\exists x\, P$ then the sequent must be inferred by $\exists$-R from $\Gamma \longrightarrow [t/x]P$ for some term t.

(4) If B is $C \supset D$ then the sequent must be inferred by $\supset$-R from $C, \Gamma \longrightarrow D$.

(5) If G is $\forall x\, P$ then, for some constant c that does not occur in the given sequent, it must be the case that the sequent is inferred by $\forall$-R from $\Gamma \longrightarrow [c/x]P$.

[2]The logical constants $\top$ and $\bot$ are not considered atomic formulas under our definition.

These properties permit the search for a uniform proof to proceed in a goal-directed fashion with the top-level structure of the goal, *i.e.*, the formula being proved, controlling the next step in the search at each stage.

We shall write $\Gamma \vdash_O B$ to denote the existence of a uniform proof for $\Gamma \longrightarrow B$. Letting $\mathcal{D}$ and $\mathcal{G}$ denote collections of formulas and $\vdash$ denote a chosen proof relation, an abstract logic programming language is defined in [MNPS91] as a triple $\langle \mathcal{D}, \mathcal{G}, \vdash \rangle$ such that, for all finite subsets $\mathcal{P}$ of $\mathcal{D}$ and all $G \in \mathcal{G}$, $\mathcal{P} \vdash G$ if and only if $\mathcal{P} \vdash_O G$. In the programming interpretation of such a triple, elements of $\mathcal{D}$ function as program clauses and elements of $\mathcal{G}$ serve as queries or goals and we therefore refer to each of these as such.

The $\forall$-L rule usually included in sequent calculi has the form

$$\frac{[t/x]B, \Gamma \longrightarrow \Delta}{\forall x\, B, \Gamma \longrightarrow \Delta}$$

In our presentation, we have combined this version of the rule with the application of a contr-L rule. It is easily seen that the various provability relations of interest are the same under either version of the $\forall$-L rule. An analogous remark applies to the $\wedge$-L rule. A comment of some interest is that our presentation of the $\forall$-L rule actually renders the contr-L rule redundant. However, we do not use this fact in this paper.

Our final observation concerns the so-called *Cut* rule that has the following form:

$$\frac{\Gamma_1 \longrightarrow B, \Delta_1 \qquad\qquad B, \Gamma_2 \longrightarrow \Delta_2}{\Gamma_1, \Gamma_2 \longrightarrow \Delta_1, \Delta_2}$$

It is well-known that this rule is admissible with respect to classical and intuitionistic provability, *i.e.*, the same set of sequents have derivations with and without this rule. We use this fact in the next section.

3 Relating classical provability to intuitionistic provability

In considering the issue of uniform provability, it is usually necessary to distinguish between the sets of logical symbols that are permitted to appear positively and negatively in formulas. This distinction is, in fact, at the heart of the difference between the goals and program clauses in an abstract logic programming language. Our interest in this paper is in collections of formulas in classical logic that turn out not to define an abstract logic programming language. However, it is still useful to present the language that is of interest to us using the vocabulary of goals and program clauses. This language is, in fact, the one in which these respective classes of formulas are given by the syntax rules

$$\begin{array}{rcl} G & ::= & \top \mid \bot \mid A \mid G \wedge G \mid G \vee G \mid D \supset G \mid \exists x\, G \\ D & ::= & \top \mid \bot \mid A \mid G \supset D \mid D \wedge D \mid D \vee D \mid \exists x\, D \mid \forall x\, D \end{array}$$

in which A represents an atomic formula. The collections described by these rules deviate from the set of all formulas in that universal quantifiers are not permitted to appear positively in G-formulas and negatively in D-formulas. However, there is a simple syntactic transformation that can be applied to any given sequent to produce a new sequent whose antecedent contains only D-formulas and whose succedent contains only G-formulas and that is equivalent to the original sequent from the perspective of classical provability; this transformation is the dual of (static) Skolemization and is referred to as Herbrandization in [Sha92]. The language presented above is also related at a syntactic level to others that have been proposed previously. The logic of Horn clauses is obtained from it by not permitting (a) implications to appear as top-level symbols in G-formulas and (b) $\bot$, $\vee$ and $\exists$ to appear as top-level symbols in D-formulas. The language of hereditary Harrop formulas [MNPS91] retains the second restriction but removes the first and, in addition, permits universal quantifiers to appear as the top-level symbol in G-formulas. (The declarative content of the resulting collections of formulas is, in addition, clarified by intuitionistic provability.) Finally, the logic underlying disjunctive logic programming [LMR92, NL95] retains the G-formulas of Horn clause logic but permits $\vee$ and $\exists$ to appear at the top-level in D-formulas.

We are ultimately interested in a uniform provability property for the language described above. As a first step in this direction, we consider the relationship between classical and intuitionistic provability for sequents of the form $\Gamma \longrightarrow G$ where Γ is a collection of D-formulas and G is a G-formula. The category of G-formulas includes a large subset of the formulas in first-order logic, and so it is to be expected that these notions of provability do not coincide for the sequents that are of interest. That this is in fact the case is seen by considering the sequent $\longrightarrow ((p \supset q) \supset p) \supset p$; we assume here that p and q are propositional symbols. As witnessed by the following derivation, this sequent has a **C**-proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{p, (p \supset q) \supset p \longrightarrow q, p}{(p \supset q) \supset p \longrightarrow (p \supset q), p}\supset\text{-R} \qquad p \longrightarrow p}{(p \supset q) \supset p \longrightarrow p, p}\supset\text{-L}}{(p \supset q) \supset p \longrightarrow p}\text{contr-R}}{\longrightarrow ((p \supset q) \supset p) \supset p}\supset\text{-R}$$

However, it is well-known that the sequent in question does not have an **I**-proof. This situation is in contrast to the one that holds in the context of the other mentioned languages whose interpretation is based on classical logic: classical and intuitionistic provability are indistinguishable relative to the Horn clause language [MNPS91] and the language underlying disjunctive logic programming [NL95].

The distinction between the two notions of provability notwithstanding, there is a correspondence between the classical provability of a sequent of the kind being considered and the intuitionistic provability of a closely related sequent. In particular, a sequent of the form $\Gamma \longrightarrow G$ has a **C**-proof if and only if the sequent $G \supset \bot, \Gamma \longrightarrow G$ has an **I**-proof. We establish this fact in this section and use it later to extract a uniform provability property for our language.

We observe first that the mentioned augmentation of the set of assumptions is one that is sound with respect to classical logic and, in fact, without restrictions on the syntax of formulas.

Lemma 1 *Let Γ be a multiset of formulas and let F be a formula. Then $F \supset \bot, \Gamma \vdash_C F$ if and only if $\Gamma \vdash_C F$.*

Proof. The if direction is obvious. For the only if direction, we note that $F, \Gamma \vdash_C F$ and so, if $F \supset \bot, \Gamma \vdash_C F$, then $F \vee (F \supset \bot), \Gamma \vdash_C F$. Noting that $\vdash_C (F \supset \bot) \vee F$ and using the *Cut* rule, we see that $\Gamma \vdash_C F$.

□

Let Γ represent a collection of D-formulas as defined above and, similarly, let G be a G-formula. In light of Lemma 1, the first step in the suggested reduction of classical provability to uniform provability may be justified by showing that a sequent of the form $G \supset \bot, \Gamma \longrightarrow G$ has a **C**-proof if and only if it has an **I**-proof. It is this course that we follow below. Anticipating this conclusion, we observe that Lemma 1 cannot be true if the relation $\vdash_C$ is replaced in it by $\vdash_I$ even in our restricted context for otherwise the sequent $\longrightarrow ((p \supset q) \supset p) \supset p$ would be intuitionistically provable.

Definition 1. Let Ξ be a **C**-proof.

(i) An inference rule of the form

$$\frac{B, \Gamma \longrightarrow \Delta \qquad\qquad D, \Gamma \longrightarrow \Delta}{B \vee D, \Gamma \longrightarrow \Delta}$$

that appears in Ξ is said to be a nonconstructive occurrence of an $\vee$-L rule just in case there is no F in Δ such that $B, \Gamma \longrightarrow F$ and $D, \Gamma \longrightarrow F$ have **I**-proofs.

(ii) An inference rule of the form

$$\frac{B, \Gamma \longrightarrow \Delta, D}{\Gamma \longrightarrow \Delta, B \supset D}$$

that appears in Ξ is said to be a nonconstructive occurrence of an $\supset$-R rule just in case $B, \Gamma \longrightarrow D$ does not have an **I**-proof.

The nonconstructiveness measure of Ξ, denoted by $\mu(\Xi)$, is the number of nonconstructive occurrences of $\vee$-L and $\supset$-R rules in Ξ.

The following lemma explains the reason for singling out the $\vee$-L and $\supset$-R rules and also casts light on the terminology of Definition 1.

Lemma 2 *If the sequent $\Gamma \longrightarrow \Delta$ has a **C**-proof with nonconstructiveness measure* 0*, then there is some formula $F \in \Delta$ such that $\Gamma \longrightarrow F$ has an **I**-proof.*

Proof. By an induction on the height of **C**-proofs.

□

A converse to Lemma 2 also holds. We state this below in a more general form that is useful in subsequent discussions. Note that a **I**-proof is a **C**-proof whose nonconstructiveness measure is 0.

Lemma 3 *Let Γ and Δ be multisets of formulas that are sub(multi)sets of Γ' and Δ' respectively. If $\Gamma \longrightarrow \Delta$ has a **C**-proof of nonconstructiveness measure n, then $\Gamma' \longrightarrow \Delta'$ has a **C**-proof of nonconstructiveness measure n or less.*

Proof. By an induction on the height of the **C**-proof of $\Gamma \longrightarrow \Delta$. The essential idea is to show that the sequents in the **C**-proof of $\Gamma \longrightarrow \Delta$ can be "padded" with new formulas while preserving the applicability of the inference rules. The constants used in some of the $\exists$-L and $\forall$-R rules may have to be "renamed" to facilitate this, but it is easily seen that this can be done without altering the height or the nonconstructiveness measure of the derivation.

□

We show the main result of this section by arguing that there can be no really nonconstructive occurrence of the rules $\vee$-L and $\supset$-R in a proof of a sequent of the form $G \supset \bot, \Gamma \longrightarrow G$, where G is a G-formula and Γ is a multiset of D-formulas. Towards this end, we develop machinery for transforming apparently nonconstructive occurrences of the mentioned rules into transparently "constructive" ones.

Definition 2. We define an ordering on formulas that is intended to measure their strength as assumptions: $F_1 \succeq F_2$ just in case $F_1 = F_2$ or

(i) F_2 is $A \supset B$ and $F_1 \succeq B$,

(ii) F_2 is $A \vee B$ and $F_1 \succeq A$ or $F_1 \succeq B$, or

(iii) F_2 is $\exists x\, P$ and, for some constant c, $F_1 \succeq [c/x]P$.

This ordering is extended to multisets of formulas: $\Gamma_1 \succeq \Gamma_2$ just in case there is a 1-1 mapping $\kappa : \Gamma_2 \mapsto \Gamma_1$ such that $\kappa(F) \succeq F$.

Lemma 4 *If $\Gamma_1 \longrightarrow \Delta_1$ and $\Gamma_2 \longrightarrow \Delta_2$ are two sequents appearing along a common path in a **C**-proof (**I**-proof) with the first appearing before the second, then $\Gamma_1 \succeq \Gamma_2$.*

Proof. By induction on the distance between the two sequents and an examination of the inference rules.

□

Lemma 5 *Let Γ and Γ' be two multisets of formulas such that $\Gamma' \succeq \Gamma$. For any formula F, $\Gamma \longrightarrow F$ has an **I**-proof only if $\Gamma' \longrightarrow F$ has one. For any multiset Δ of formulas, if $\Gamma \longrightarrow \Delta$ has a **C**-proof Ξ, then $\Gamma' \longrightarrow \Delta$ has a **C**-proof whose nonconstructiveness measure is at most that of Ξ.*

Proof. The essential idea is to construct a proof of $\Gamma' \longrightarrow F$ ($\Gamma' \longrightarrow \Delta$) by mimicking the given proof of $\Gamma' \longrightarrow F$ ($\Gamma' \longrightarrow \Delta$), possibly dropping some $\supset$-L, $\vee$-L and $\exists$-L rules and thereby also pruning some branches. At a level of detail, we use an induction on the height of the given proof, showing the claim about **I**-proofs first and then using this relative to "constructive" uses of $\vee$-L and $\supset$-L rules in proving the claim about **C**-proofs.

□

Lemma 6 *Let Γ and Δ be multisets of D- and G-formulas respectively. Further, let $\Gamma \longrightarrow \Delta$ have a **C**-proof Ξ in which an $\supset$-R rule of the form*

$$\frac{B, \Sigma \longrightarrow \Pi, D}{\Sigma \longrightarrow \Pi, B \supset D}$$

occurs with the following characteristic: $B, \Sigma \longrightarrow D$ *does not have an* ***I****-proof but for some* $F \in \Pi$*,* $B, \Sigma \longrightarrow F$ *has an* ***I****-proof. Then* $B, \Sigma \longrightarrow \Delta$ *has a* ***C****-proof whose nonconstructiveness measure is smaller than that of* Ξ*.*

Proof. By induction on the height of Ξ. At least one inference rule must have been used in Ξ. We consider first the possibility that the last such rule pertains to a formula in the antecedent and then that it pertains to a formula in the succedent.

The argument in the case of antecedent rules that have only one upper sequent — *i.e.*, in the case of the rules contr-L, $\wedge$-L, $\exists$-L and $\forall$-L— takes a common form. In all these cases, the proof at the end has the structure

$$\frac{\Gamma' \longrightarrow \Delta}{\Gamma \longrightarrow \Delta}$$

It is easily seen that all the formulas in Γ' must be D-formulas if those in Γ are. Further, the $\supset$-R rule mentioned in the lemma appears in the proof of $\Gamma' \longrightarrow \Delta$. Thus, the induction hypothesis can be used to conclude that $B, \Sigma \longrightarrow \Delta$ has a **C**-proof of smaller nonconstructiveness measure than that of the proof of $\Gamma' \longrightarrow \Delta$. But the latter is actually identical to $\mu(\Xi)$.

Suppose that the last rule is an $\vee$-L, *i.e.*, one of the form

$$\frac{E, \Gamma' \longrightarrow \Delta \qquad F, \Gamma' \longrightarrow \Delta}{E \vee F, \Gamma' \longrightarrow \Delta}$$

Now, the $\supset$-R rule mentioned in the lemma appears in the proof of either $E, \Gamma' \longrightarrow \Delta$ or $F, \Gamma' \longrightarrow \Delta$. Without loss of generality, suppose the former. The induction hypothesis is again seen to be applicable relative to the proof of $E, \Gamma' \longrightarrow \Delta$. Using it and noting that the nonconstructiveness measure of this proof is at most $\mu(\Xi)$ yields the desired conclusion.

The only remaining possibility for an antecedent rule is $\supset$-L. In this case, Ξ has the form

$$\frac{E \supset F, \Gamma' \longrightarrow \Delta_1, E \qquad F, \Gamma' \longrightarrow \Delta_2}{E \supset F, \Gamma' \longrightarrow \Delta_1, \Delta_2}$$

at the end. The $\supset$-R rule mentioned in the lemma could appear either above the left upper sequent or the right upper sequent of the rule displayed. Suppose it is the latter. Noting that F must be a D-formula if $E \supset F$ is one and using the induction hypothesis, we see that $B, \Sigma \longrightarrow \Delta_2$ has a **C**-proof of smaller nonconstructiveness measure than the one for $F, \Gamma' \longrightarrow \Delta_2$. The desired conclusion is now reached by observing that the latter proof is a part of Ξ and by employing Lemma 3.

To complete the consideration of the case when an $\supset$-L is the last rule used, suppose that the $\supset$-R rule mentioned in the lemma appears above $E \supset F, \Gamma' \longrightarrow \Delta_1, E$. Using the induction hypothesis that is easily seen to be applicable, it follows that $B, \Sigma \longrightarrow \Delta_1, E$ has a **C**-proof whose nonconstructiveness measure is smaller than that of the given **C**-proof of $E \supset F, \Gamma' \longrightarrow \Delta_1, E$. By Lemma 4, $B, \Sigma \succeq (E \supset F), \Gamma'$ and so B, Σ can be written in the form F', Σ' where $\Sigma' \succeq \Gamma'$ and either F' is identical to $E \supset F$ or $F' \succeq F$. In the former case, $F, \Sigma' \succeq F, \Gamma'$ and so, by Lemma 5, $F, \Sigma' \longrightarrow \Delta_2$ has a **C**-proof of nonconstructiveness measure at most that of the **C**-proof of $F, \Gamma' \longrightarrow \Delta_2$. Combining this with the **C**-proof for $B, \Sigma \longrightarrow \Delta_1, E$ yields one for $B, \Sigma \longrightarrow \Delta_1, \Delta_2$ with lower nonconstructiveness measure than $\mu(\Xi)$. In the other case, *i.e.*, when $F' \succeq F$, it follows that $B, \Sigma \succeq F, \Gamma'$. Hence, by Lemmas 5 and 3, $F, B, \Sigma \longrightarrow \Delta_2$ has a **C**-proof with nonconstructiveness measure at most that of $F, \Gamma' \longrightarrow \Delta_2$. By combining this **C**-proof with that of $B, \Sigma \longrightarrow \Delta_1, E$ we get one for $E \supset F, B, \Sigma \longrightarrow \Delta_1, \Delta_2$ that has a nonconstructiveness measure less than $\mu(\Xi)$. By Lemma 5, there is a **C**-proof with the same characteristic for $F', B, \Sigma \longrightarrow \Delta_1, \Delta_2$ and, hence, using contr-L, one for $B, \Sigma \longrightarrow \Delta_1, \Delta_2$ as required.

We now consider the possibilities for a succedent rule being the last one in Ξ. The restriction in the syntax of the formulas in Δ ensures that this rule cannot be an $\forall$-R. If the last rule is one of $\bot$-R, $\vee$-R, $\exists$-R and contr-R, the same rule could be the last one in a purported **C**-proof of $B, \Sigma \longrightarrow \Delta$ as well. Further the upper sequent of such a rule application bears a relationship to the upper sequent of the corresponding rule application in the **C**-proof of $\Gamma \longrightarrow \Delta$ that permits the induction hypothesis to be used. The desired conclusion follows easily from these observations in these cases.

An argument similar to the one for the succedent rules considered above can also be provided in the case that the last rule in Ξ is an $\wedge$-R. The only possibility that remains to be considered, then, is that when an $\supset$-R rule is the last one. Here there are two subcases to contend with: this rule may or may not be be the one mentioned in the lemma. In the first situation, by Lemma 3, $B, \Sigma \longrightarrow \Delta$ has a **C**-proof whose nonconstructiveness measure is 0 and hence certainly less than $\mu(\Xi)$. In the other situation, an argument similar to that for the other succedent rules with a single upper sequent can be provided to show that $B, \Sigma \longrightarrow \Delta$ has a **C**-proof of nonconstructiveness measure less than $\mu(\Xi)$.

All the relevant cases having been considered, it follows that the lemma must be true.

□

Lemma 7 *Let Γ and Δ be multisets of D- and G-formulas respectively. Further, let $\Gamma \longrightarrow \Delta$ have a **C**-proof Ξ in which an $\vee$-L rule of the form*

$$\frac{B, \Sigma \longrightarrow \Pi \qquad D, \Sigma \longrightarrow \Pi}{B \vee D, \Sigma \longrightarrow \Pi}$$

*occurs with the following characteristic: for no $F \in \Pi$ is it the case that $B \vee D, \Sigma \longrightarrow F$ has an **I**-proof but for some $F \in \Pi$ it is the case that $D, \Sigma \longrightarrow F$ has an **I**-proof. Then $D, \Sigma \longrightarrow \Delta$ has a **C**-proof whose nonconstructiveness measure is smaller than that of Ξ.*

Proof. By an argument similar to that for Lemma 6.

□

The restriction in the syntax of D- and G-formulas is essential to the truth of Lemmas 6 and 7. For instance, consider the following **C**-proof of $\longrightarrow \forall x\, q(x) \vee \exists x\, (q(x) \supset \bot)$, assuming that q represents a unary predicate symbol in this sequent:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{q(c) \longrightarrow q(c), \bot}{\longrightarrow q(c), q(c) \supset \bot}\ \bot\text{-R}}{\longrightarrow q(c), \exists x\, (q(x) \supset \bot)}\ \exists\text{-R}}{\longrightarrow \forall x\, q(x), \exists x\, (q(x) \supset \bot)}\ \forall\text{-R}}{\longrightarrow \forall x\, q(x), \forall x\, q(x) \vee \exists x\, (q(x) \supset \bot)}\ \vee\text{-R}}{\longrightarrow \forall x\, q(x) \vee \exists x\, (q(x) \supset \bot), \forall x\, q(x) \vee \exists x\, (q(x) \supset \bot)}\ \vee\text{-R}}{\longrightarrow \forall x\, q(x) \vee \exists x\, (q(x) \supset \bot)}\ \text{contr-R}$$

It is easily seen that $q(c) \longrightarrow \forall x\, q(x) \vee \exists x\, (q(x) \supset \bot)$ does not have an **I**-proof as would be needed if Lemma 6 were to hold without restrictions. A similar observation can be made relative to Lemma 7 using the sequent $\forall x\, (p \vee q(x)) \longrightarrow (p \vee \forall x\, q(x))$ in which p is assumed to be a proposition symbol and q a unary predicate symbol.

Lemma 8 *Let Γ be a multiset of D-formulas and let G be a G-formula such that*

$$G \supset \bot, \Gamma \longrightarrow G$$

*has a **C**-proof. If $\Sigma \longrightarrow \Pi$ is a sequent that appears in this proof, then there is some $F \in \Pi$ such that $\Sigma \longrightarrow F$ has an **I**-proof.*

Proof. Suppose that the lemma is not true. Let Ξ be a **C**-proof for a sequent of the form $G \supset \bot, \Gamma \longrightarrow G$ that falsifies the lemma and, further, let Ξ have the smallest nonconstructiveness measure amongst **C**-proofs with this characteristic. By Lemma 2, $\mu(\Xi)$ cannot be 0. If $\mu(\Xi)$ is nonzero, there must be an $\vee$-L or an $\supset$-R rule in Ξ that is the first nonconstructive occurrence of a rule of either kind along a branch. We consider each possibility below.

Suppose that the rule in question is an $\vee$-L rule of the form

$$\frac{B, \Sigma \longrightarrow \Pi \qquad\qquad D, \Sigma \longrightarrow \Pi}{B \vee D, \Sigma \longrightarrow \Pi}$$

By assumption, for no $F \in \Pi$ is it the case that an **I**-proof exists for both $B, \Sigma \longrightarrow F$ and $D, \Sigma \longrightarrow F$. However, by Lemma 2 and our assumption concerning the structure of Ξ prior to this rule, there must be some $F, F' \in \Pi$ such that $B, \Sigma \longrightarrow F$ and $D, \Sigma \longrightarrow F'$ have **I**-proofs. From the latter, using Lemma 7, it follows that $D, \Sigma \longrightarrow G$ has a **C**-proof with smaller nonconstructiveness measure than $\mu(\Xi)$. Noting that the antecedent of every sequent in a **C**-proof of $G \supset \bot, \Gamma \longrightarrow G$ must be a multiset of D-formulas and then using the leastness assumption pertaining to $\mu(\Xi)$, we conclude that $D, \Sigma \longrightarrow G$ has an **I**-proof. From Lemma 4 and the fact that $B \vee D, \Sigma \longrightarrow \Pi$ appears in a **C**-proof of $G \supset \bot, \Gamma \longrightarrow G$, it follows that Σ is either of the form $G \supset \bot, \Sigma'$ or of the form $\bot, \Sigma'$. We assume the former, noting that the argument is simpler if the latter is true. Now, we can construct the following subderivation:

$$\dfrac{D, \Sigma \longrightarrow G \qquad \dfrac{\bot, \Sigma' \longrightarrow \bot}{\bot, \Sigma' \longrightarrow F}\ \bot\text{-R}}{D, \Sigma \longrightarrow F}\ \supset\text{-L}$$

Using the **I**-proof that exists for $D, \Sigma \longrightarrow G$ together with this, we can obtain an **I**-proof for the sequent $D, \Sigma \longrightarrow F$. But this is obviously a contradiction.

Suppose instead that the rule of interest was a $\supset$-R rule of the form

$$\frac{B, \Sigma \longrightarrow \Pi, D}{\Sigma \longrightarrow \Pi, B \supset D}$$

By our assumptions and Lemma 2, we have the following: $B, \Sigma \longrightarrow D$ does not have an **I**-proof, but for some $F \in \Delta$ it is the case that $B, \Sigma \longrightarrow F$ has an **I**-proof. From the latter and Lemma 6 it follows that $B, \Sigma \longrightarrow G$ has a **C**-proof whose nonconstructiveness measure is less than $\mu(\Xi)$. We can, once again, conclude from this that $B, \Sigma \longrightarrow G$ has an **I**-proof. From Lemma 4 it follows that Σ can be written as either $G \supset \bot, \Sigma'$ or $\bot, \Sigma'$. We assume the former, noting as before that the argument becomes simpler if the latter is true. Now, the following subderivation can be constructed:

$$\dfrac{B, \Sigma \longrightarrow G \qquad \dfrac{\bot, B, \Sigma' \longrightarrow \bot}{\bot, B, \Sigma' \longrightarrow D}\ \bot\text{-R}}{B, \Sigma \longrightarrow D}\ \supset\text{-L}$$

Using the **I**-proof of $B, \Sigma \longrightarrow G$ together with this, we obtain an **I**-proof for $B, \Sigma \longrightarrow D$, yielding, once again, a contradiction.

It is thus untenable that the lemma is false and so it must, in fact, be true.

□

The main conclusion that we desire in this section is an easy corollary of Lemmas 1 and 8.

Theorem 9 *Let Γ be any collection of D-formulas and let G be a G-formula. Then $\Gamma \vdash_C G$ if and only if $G \supset \bot, \Gamma \vdash_I G$.*

The restriction in the syntax of D- and G-formulas is important to the truth of Theorem 9, a fact that we became aware of through the comments of Robert Stärk. To see that this is the case, consider the sequent $\forall x\,((p(x) \supset \bot) \supset \bot) \longrightarrow \forall x\, p(x)$ in which p is a unary predicate symbol. This sequent has a **C**-proof but the sequent

$$(\forall x\, p(x) \supset \bot), \forall x\,((p(x) \supset \bot) \supset \bot) \longrightarrow \forall x\, p(x)$$

does *not* have an **I**-proof. A question of interest is whether the theorem can be strengthened in any way. In particular, are there alternative restrictions that can be placed on the syntax of G and the formulas in Γ that do not presuppose specific knowledge of these formulas but still ensure that $(G \supset \bot), \Gamma \longrightarrow G$ has an **I**-proof whenever $\Gamma \longrightarrow G$ has a **C**-proof? In response to this question, we note that this assurance can be given under only two circumstances: when the syntactic restrictions guarantee that $\Gamma \longrightarrow G$ itself has an **I**-proof (and these restrictions do not always preclude the use of the $\forall$-R rule) and when they ensure that the $\forall$-R rule will not be utilized. Thus, the restrictions assumed in Theorem 9 reflect the most liberal ones that allow classical provability to be reduced to intuitionistic provability through the indicated augmentation to the assumption set and where this reduction is a non-trivial one. A detailed discussion of these and other matters is planned for a sequel to this paper.

The proofs of the various lemmas in this section, culminating in that of Lemma 8, contain more information than is utilized in proving Theorem 9. One particular aspect that we note here is their constructive content: under a suitable interpretation, they provide the basis for a procedure that takes a **C**-proof for a sequent of the form $\Gamma \longrightarrow G$ and, by working downward from the leaves in this proof, that transforms this into an **I**-proof for $G \supset \bot, \Gamma \longrightarrow G$. For example, consider the **C**-proof for $\longrightarrow ((p \supset q) \supset p) \supset p$ displayed at the beginning of this section. Assuming that $G \supset \bot$ denotes the formula

$$(((p \supset q) \supset p) \supset p) \supset \bot,$$

the mentioned procedure would transform this **C**-proof into the following **I**-proof for a suitably augmented sequent:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(p \supset q) \supset p, p, (p \supset q) \supset p, G \supset \bot \;\longrightarrow\; p}{p, (p \supset q) \supset p, G \supset \bot \;\longrightarrow\; ((p \supset q) \supset p) \supset p}\,\supset\text{-R} \qquad \dfrac{\bot, p, (p \supset q) \supset p \;\longrightarrow\; \bot}{\bot, p, (p \supset q) \supset p \;\longrightarrow\; q}\,\bot\text{-R}}{p, (p \supset q) \supset p, G \supset \bot \;\longrightarrow\; q}\,\supset\text{-L}}{(p \supset q) \supset p, G \supset \bot \;\longrightarrow\; p \supset q}\,\supset\text{-R} \qquad p, G \supset \bot \;\longrightarrow\; p}{(p \supset q) \supset p, G \supset \bot \;\longrightarrow\; p}\,\supset\text{-L}}{G \supset \bot \;\longrightarrow\; ((p \supset q) \supset p) \supset p}\,\supset\text{-R}$$

This observation can be further sharpened by noting that only very restricted uses are made of the added formula in the transformation process. We utilize this fact in the next section in describing a modified deductive calculus for our language in which the augmentation of sequents is made implicit.

4 A uniform provability property

The uniform provability property fails to hold in an immediate sense for our fragment of classical logic. This is not a surprising fact, given that intuitionistic provability is already a more restrictive relation than classical provability relative to our language. Furthermore, intuitionistic provability is itself distinct in this context from uniform provability. This latter difference arises from the possibility for disjunctive and existential information to be present in assumptions. Thus, consider the sequent $p(a) \vee p(b) \;\longrightarrow\; \exists x\, p(x)$. This sequent has the following **I**-proof:

$$\dfrac{\dfrac{p(a) \;\longrightarrow\; p(a)}{p(a) \;\longrightarrow\; \exists x\, p(x)}\,\exists\text{-R} \qquad \dfrac{p(b) \;\longrightarrow\; p(b)}{p(b) \;\longrightarrow\; \exists x\, p(x)}\,\exists\text{-R}}{p(a) \vee p(b) \;\longrightarrow\; \exists x\, p(x)}\,\vee\text{-L}$$

However, as already noted, there can be no uniform proof for this sequent.

While the uniform provability property does not hold in a strict sense for our fragment of classical logic, it does hold of it in a derivative sense: assuming that Γ is a set of D-formulas and G is a G-formula, a **C**-proof exists for $\Gamma \;\longrightarrow\; G$ if and only if a uniform proof exists for $G \supset \bot, \Gamma \;\longrightarrow\; G$. In the previous section, we have already observed that the indicated augmentation of the assumption set yields a correspondence between classical and intuitionistic provability. Thus, one way to establish the above uniform provability property is to show that the same augmentation also leads to a coincidence between intuitionistic and uniform provability. A proof of this fact relative to the logic underlying disjunctive logic programming is provided in [NL95] and it turns out that this argument can be extended to the present context as well. We do this below, taking care to cast our discussions in a form that supports a subsequent extraction from them of a proof procedure for classical logic.

Our first step in the indicated direction is to refine the deductive calculus to be used for constructing derivations for the kinds of sequents of interest to us. In particular, consider the following inference rules that are parameterized by a specific formula G:

$$\frac{B, \Delta \longrightarrow F \qquad D, \Delta \longrightarrow G}{B \vee D, \Delta \longrightarrow F} \vee\text{-L}_G$$

$$\frac{\Delta \longrightarrow G}{\Delta \longrightarrow F} \text{res}_G$$

We assume that B, D and F are schema variables for formulas in these rules and that Δ denotes a multiset of formulas. It is easily seen that these rules are derived ones relative to the sequent calculus for intuitionistic logic in the case that Δ contains the formula $G \supset \bot$. Now, as noted at the end of the last section, the transformation procedure implicit in the proof of Lemma 8 yields an **I**-proof for $G \supset \bot, \Gamma \longrightarrow G$ in which *every* use that is made of the formula $G \supset \bot$ that is added to the assumptions can be seen to be embedded within one of these derived rules. Thus, by using these rules and by strengthening the proviso on the $\exists$-L and $\forall$-R rules to disallow the use of constants appearing in G, the augmentation of sequents can be made implicit.

Let us tentatively refer to derivations constructed in a sequent calculus obtained from that for intuitionistic logic through the above modifications as $\mathbf{I}_G$-proofs. We now make the following further observation: the $\vee$-L rule is redundant from the perspective of constructing $\mathbf{I}_G$-proofs for the kinds of sequents of interest to us. This observation is a consequence of the lemma below whose proof, when viewed constructively, provides the basis for transforming $\vee$-L rule occurrences into occurrences of the $\vee$-L_G rule.

Lemma 10 *Let Γ be a multiset of D-formulas and let G be a G-formula. Further, let $\Gamma \longrightarrow G$ have an $\mathbf{I}_G$-proof Ξ in which an $\vee$-L rule of the form*

$$\frac{B, \Sigma \longrightarrow F \qquad D, \Sigma \longrightarrow F}{B \vee D, \Sigma \longrightarrow F}$$

appears. Then $D, \Sigma \longrightarrow G$ has an $\mathbf{I}_G$-proof in which there are fewer occurrences of the $\vee$-L rule than in Ξ.

Proof. By induction on the height of the given $\mathbf{I}_G$-proof, using the analogue of Lemma 4 for $\mathbf{I}_G$-proofs and the easily established fact that if $\Delta' \succeq \Delta$ and $\Delta \longrightarrow F$ has an $\mathbf{I}_G$-proof with n occurrences of the $\vee$-L rule, then $\Delta' \longrightarrow F$ has an $\mathbf{I}_G$-proof with n or fewer occurrences of the $\vee$-L rule. We omit the details of the argument, noting that they are similar to those in the proofs of Lemmas 6 and 7.

$\square$

The restriction in the syntax of D- and G-formulas is essential to the truth of the above lemma. Thus, consider the following $\mathbf{I}_G$-proof for $\longrightarrow\ \forall x\,((p(x,a) \vee p(x,b)) \supset \exists y\, p(x,y))$, assuming that p is a binary predicate symbol and a, b and c are constant symbols:

$$\dfrac{\dfrac{\dfrac{\dfrac{p(c,a) \longrightarrow p(c,a)}{p(c,a) \longrightarrow \exists y\, p(c,y)}\ \exists\text{-R} \qquad \dfrac{p(c,b) \longrightarrow p(c,b)}{p(c,b) \longrightarrow \exists y\, p(c,y)}\ \exists\text{-R}}{(p(c,a) \vee p(c,b)) \longrightarrow \exists y\, p(c,y)}\ \vee\text{-L}}{\longrightarrow\ (p(c,a) \vee p(c,b)) \supset \exists y\, p(c,y)}\ \supset\text{-R}}{\longrightarrow\ \forall x\,((p(x,a) \vee p(x,b)) \supset \exists y\, p(x,y))}\ \forall\text{-R}$$

This derivation has one occurrence of an $\vee$-L rule and it is easily seen that there is no $\mathbf{I}_G$-proof for the sequent $p(c,b) \longrightarrow \forall x\,((p(x,a) \vee p(x,b)) \supset \exists y\, p(x,y))$ in which there are no occurrences of the $\vee$-L rule.

On the strength of Lemma 10 and the comments preceding it, we assume henceforth that the $\mathbf{I}_G$-proofs that we consider *do not* contain occurrences of the $\vee$-L rule. The results of the previous section can now be summarized in the context of our present discussion as follows:

Theorem 11 *Let Γ be a multiset of D-formulas and let G be a G-formula. Then the sequent $\Gamma \longrightarrow G$ has a $\boldsymbol{C}$-proof if and only if it has a $\boldsymbol{I}_G$-proof.*

We now relativize the notion of a uniform proof to our modified calculus. In particular, let an $\mathbf{O}_G$-proof be an $\mathbf{I}_G$-proof with the following characteristic: if there is a sequent in this proof whose succedent contains a non-atomic formula, then that sequent occurs as the lower sequent of an inference rule that introduces the top-level logical symbol of that formula. The following may then be observed:

Lemma 12 *Let G be a G-formula and let Γ be a multiset of D-formulas. Then $\Gamma \longrightarrow G$ has a $\boldsymbol{I}_G$-proof only if it has an $\boldsymbol{O}_G$-proof.*

Proof. Suppose that $\Gamma \longrightarrow G$ has an $\mathbf{I}_G$-proof. It must then have an $\mathbf{I}_G$-proof in which there is a rule introducing the top-level logical symbol of every non-atomic formula appearing in the succedent of a sequent; to ensure that this is the case, we only need to introduce some inference steps right after $\perp$-R and res_G rules in a manner whose details are entirely transparent. Further, an $\mathbf{I}_G$-proof of this kind exists that also satisfies the following additional condition: no antecedent rule immediately succeeds a succedent rule pertaining to a top-level logical symbol of a formula in a common sequent except in the case that the antecedent rule is $\exists$-L and the succedent rule is $\exists$-R. To see that this is so, we first observe, by an easy induction on the heights of $\mathbf{I}_G$-proofs, that (a) if a sequent of the form $\Sigma \longrightarrow A \wedge B$ has an $\mathbf{I}_G$-proof of height h, then both $\Sigma \longrightarrow A$ and $\Sigma \longrightarrow B$ have $\mathbf{I}_G$-proofs of height h or less, (b) if a sequent of the form $\Sigma \longrightarrow A \supset B$ has an $\mathbf{I}_G$-proof of height h, then $A, \Sigma \longrightarrow B$ has an $\mathbf{I}_G$-proof of height h or less, and (c) if a sequent of the form $\Sigma \longrightarrow \forall x\, B$ has an $\mathbf{I}_G$-proof of height h, then, for any constant c, $\Sigma \longrightarrow [c/x]B$ has an $\mathbf{I}_G$-proof of height h or less. Now, given an $\mathbf{I}_G$-proof for a sequent of the form $\Sigma \longrightarrow F$, let us associate with this proof the pair of natural numbers $\langle n_1, n_2\rangle$ in which n_1 is the height of the given $\mathbf{I}_G$-proof and n_2 is the count of the number of logical symbols in F and let us consider an ordering on $\mathbf{I}_G$-proofs that is based on the the extension of the usual ordering on the natural numbers to a lexicographic ordering on the pairs of numbers corresponding to the proofs. The existence of an $\mathbf{I}_G$-proof of the required form is established by an induction on the mentioned ordering.

Let us call an $\mathbf{I}_G$-proof satisfying the requirements mentioned above an $\mathbf{I}'_G$-proof. We then define the *nonuniformity measure* of a $\exists$-L rule as the count of the number of connectives and quantifiers that appear in the succedent of the lower sequent of the rule, and the nonuniformity measure of an $\mathbf{I}'_G$-proof as the sum of the (nonuniformity) measures of the $\exists$-L rules that appear in it. We claim that any $\mathbf{I}'_G$-proof of $\Gamma \longrightarrow G$ that has a nonzero nonuniformity measure can be transformed into an $\mathbf{I}'_G$-proof of smaller measure. It follows from this that $\Gamma \longrightarrow G$ has an $\mathbf{O}_G$-proof.

To show the claim, suppose that the $\mathbf{I}'_G$-proof of $\Gamma \longrightarrow G$ in fact has a nonzero nonuniformity measure. It must then be the case that somewhere in the derivation an $\exists$-L rule appears right after an $\exists$-R rule. In other words, there is a subderivation of the form

$$\cfrac{\cfrac{[c/x]B, \Gamma' \longrightarrow [t/y]D}{[c/x]B, \Gamma' \longrightarrow \exists y\, D}\ \exists\text{-R}}{\exists x\, B, \Gamma' \longrightarrow \exists y\, D}\ \exists\text{-L}$$

at some point in the given $\mathbf{I}'_G$-proof. Let us assume that D is atomic — this assumption is not really essential and can be dispensed with in a more detailed argument. Using the fact that what is displayed above is a subpart of an $\mathbf{I}'_G$-proof of $\Gamma \longrightarrow G$, it can be shown that $[c/x]B, \Gamma' \longrightarrow G$ has an $\mathbf{I}'_G$-proof of smaller nonuniformity measure than that of the one for $\Gamma \longrightarrow G$; as in the case of Lemma 6, the essential idea is to mimic the structure of the given proof of $\Gamma \longrightarrow G$ while noting that at least one occurrence of an $\exists$-L rule — the one shown above — that makes a nonzero contribution to the nonuniformity measure can be eliminated. By induction it follows then that $[c/x]B, \Gamma' \longrightarrow G$ has an $\mathbf{O}_G$-proof. We further observe that the proviso on a $\exists$-L rule ensures that c does not occur in B, Γ' or G. From this it is easily seen, for any constant c', $[c'/x]B, \Gamma' \longrightarrow G$ has an $\mathbf{O}_G$-proof. Let c' be a constant that does not occur in t in addition to not appearing in B, Γ' and G. Then we can construct the following subderivation:

$$\dfrac{\dfrac{\dfrac{[c'/x]B, \Gamma' \longrightarrow G}{[c'/x]B, \Gamma' \longrightarrow [t/y]D}\ \mathrm{res}_G}{\exists x\, B, \Gamma' \longrightarrow [t/y]D}\ \exists\text{-L}}{\exists x\, B, \Gamma' \longrightarrow \exists y\, D}\ \exists\text{-R}$$

Using the known $\mathbf{O}_G$-proof for $[c'/x]B, \Gamma' \longrightarrow G$ together with this to replace the earlier subderivation, we obtain the desired $\mathbf{I}'_G$-proof of reduced measure.

□

The syntactic restrictions on D- and G-formulas are, once again, necessary for the truth of Lemma 12: assuming that p and q are binary predicate symbols, it can be seen, for instance, that the sequent

$$\forall x\, \forall y\, p(x,y) \longrightarrow \forall x\, ((\exists y\, (p(x,y) \supset q(x,y))) \supset \exists y\, q(x,y))$$

has an $\mathbf{I}_G$-proof but *does not* have an $\mathbf{O}_G$-proof.

The uniform provability property is an easy consequence of Lemma 12.

Theorem 13 *Let G be a G-formula and let Γ be a multiset of D-formulas. Then $\Gamma \vdash_C G$ if and only if $G \supset \bot, \Gamma \vdash_O G$.*

Proof. Given Lemma 1, the if direction is obvious. For the only if direction, we use Theorem 11, Lemma 12 and the fact that an $\mathbf{O}_G$-proof for $\Gamma \longrightarrow G$ can be translated into a uniform proof for $G \supset \bot, \Gamma \longrightarrow G$.

□

There is a constructive content to the proofs of Lemmas 10 and 12 and it is useful to understand this. For this purpose, consider the proof for the sequent

$$p(a) \vee p(b) \longrightarrow \exists x\, p(x)$$

that is shown at the beginning of this section. Construing the inference rule labelled as $\vee$-L as an $\vee$-L_G rule instead, this proof is seen also to be an $\mathbf{I}_G$-proof. Based on the argument provided for Lemma 12, this proof can be transformed into the $\mathbf{O}_G$-proof

$$
\dfrac{p(a) \longrightarrow p(a) \qquad \dfrac{p(b) \longrightarrow p(b)}{p(b) \longrightarrow \exists x\, p(x)}\ \exists\text{-R}}{\dfrac{p(a) \vee p(b) \longrightarrow p(a)}{p(a) \vee p(b) \longrightarrow \exists x\, p(x)}\ \exists\text{-R}}\ \vee\text{-L}_G
$$

5 A reduced proof system for classical logic

The uniform provability property that was established in the previous section is useful in describing a proof procedure for classical logic. The starting point for such a procedure is a formula from which essentially positive occurrences of universal quantifiers have been eliminated by the process of Herbrandization. Now, whenever the procedure is required to find a proof for a non-atomic formula, it uses the top-level logical symbol in this formula to determine the next step in proof search. However, the way to proceed is not quite so clear when an atomic formula has been produced through this process: the $\perp$-R rule and a variety of antecedent rules may be applicable at this point and there is at present no mechanism for picking between these. We outline an approach to dealing with this situation in this section. This approach is based on combining the antecedent rules and the $\perp$-R rule into a generalization of the backchaining rule that is known from Horn clause logic and that, in a sense, is controlled by the atomic formula for which a proof is sought. In our context there will be three different manifestations of this rule, and, as is typically the case, more than one instance of these forms of the rule might be applicable at a relevant stage in the proof search process. The manner in which a choice is made between these different possibilities could have a substantial impact on the behavior of an actual proof procedure. However, we stop short of considering the pragmatically important question of how this choice is to be made, presenting only the basic structure of the proof procedure through a reduced proof system.

Our main objective, then, is the enunciation of a suitable backchaining rule. In stating this rule and in manifesting its intuitive content, it is preferable to use a simplified syntax for D- and G-formulas. In particular, we assume from now on that our goals and program clauses are given by the rules

$$
\begin{array}{rcl}
G & ::= & A \mid G \wedge G \mid G \vee G \mid D \supset G \mid \exists x\, G \\
D & ::= & (A \vee \ldots \vee A) \mid G \supset (A \vee \ldots \vee A) \mid \forall x\, D
\end{array}
$$

in which the symbol A is assumed to represent the category of atomic formulas augmented by the logical constants $\top$ and $\perp$. Using known logical equivalences and the notion of (static) Herbrandization, the question of classical provability of a sequent of the form $\Gamma \longrightarrow F$ in which the formulas are permitted to have an arbitrary syntax can be transformed into an identical question for a similar sequent in which the assumption and goal formulas adhere to the respective simplified syntax. The use of this "reduced" syntax therefore does not constitute a loss of generality in our discussions.[3]

Our backchaining rule will be based, as usual, on the notion of an instance of a program clause. In the present setting this notion is explicated as follows:

Definition 3. Let D be a program clause. Then $[D]$ denotes a collection of pairs of sets of formulas given as follows:

(1) If D is $A_1 \vee \ldots \vee A_n$, then $[D] = \{\langle \emptyset, \{A_1, \ldots A_n\}\rangle\}$.

(2) If D is $G \supset (A_1 \vee \ldots \vee A_n)$, then $[D] = \{\langle \{G\}, \{A_1, \ldots, A_n\}\rangle\}$.

(3) If D is $\forall x\, D_1$, then $[D] = \bigcup\{[[t/x]D_1] \mid t \text{ is a term}\}$.

This notation is extended to a (multi)set Γ of program clauses as follows:

$$[\Gamma] = \bigcup\{[D] \mid D \in \Gamma\}.$$

The starting point for our proof search is represented by a sequent of the form $\Gamma \longrightarrow G$ in which Γ is a multiset of D-formulas and G is a G-formula. In the discussions that follow, we assume a calculus for $\mathbf{O}_G$-proofs that is relativized to this starting sequent; in particular, the G-formula in the $\vee$-L_G and res_G rules is chosen to coincide with its succedent formula.

The following lemma underlies our generalization of the backchaining rule. We adopt a harmless abuse of notation in the statement of this lemma and in the subsequent discussions in that we permit n to be 0 in a listing $A_1, \ldots, A_n$ of formulas, assuming, in this case, that the "listing" denotes an empty sequence.

Lemma 14 *Let Γ be a multiset of program clauses and let C be an atomic formula or $\bot$. Then $\Gamma \longrightarrow C$ has an $\boldsymbol{O}_G$-proof with l sequents appearing in it just in case one of the following holds:*

[3]With the exception of the elimination of certain occurrences of universal quantifiers, the simplification in the syntax of formulas is also not essential and is chosen mainly for reasons of perspicuity. An alternative approach would be to incorporate the mentioned syntactic transformation of formulas implicitly into the definition of the instances of a program clause that follows. Notice, however, that the proper treatment of existential quantification in program clauses under this approach would require the relativization of the definition of clause instances to a given signature.

(1) $\Gamma \longrightarrow G$ *has an* $\mathbf{O}_G$*-proof with fewer than* l *sequents in it.*

(2) *For some* $A_1, \ldots, A_n$, *either* $\langle \emptyset, \{C, A_1, \ldots, A_n\}\rangle \in [\Gamma]$ *or* $\langle \emptyset, \{\bot, A_1, \ldots, A_n\}\rangle \in [\Gamma]$ *and, if* $n > 0$ *then, for* $1 \leq i \leq n$, $A_i, \Gamma \longrightarrow G$ *has an* $\mathbf{O}_G$*-proof with fewer than* l *sequents.*

(3) *For some* G' *and* $A_1, \ldots, A_n$ *either* $\langle \{G'\}, \{C, A_1, \ldots, A_n\}\rangle$ *or* $\langle \{G'\}, \{\bot, A_1, \ldots, A_n\}\rangle$ *is a member of* $[\Gamma]$ *and* $\Gamma \longrightarrow G'$ *and, if* $n > 0$ *then, for* $1 \leq i \leq n$, $A_i, \Gamma \longrightarrow G$ *have* $\mathbf{O}_G$*-proofs with fewer than* l *sequents.*

Proof. An easy induction on the size of an $\mathbf{O}_G$-proof for a sequent of the kind that is of interest.

□

The content of the above lemma from the perspective of proof search is abstracted into the following definition.

Definition 4. Let Γ represent a multiset of program clauses and let C represent an atomic formula or $\bot$. We describe three rules below that are relativized to a particular choice of goal G.

(1) The RESTART rule is the following:

$$\frac{\Gamma \longrightarrow G}{\Gamma \longrightarrow C}$$

(2) The ATOMIC rule is the following

$$\frac{A_1, \Gamma \longrightarrow G \qquad \ldots \qquad A_n, \Gamma \longrightarrow G}{\Gamma \longrightarrow C}$$

provided that $\langle \emptyset, \{C, A_1, \ldots, A_n\}\rangle \in [\Gamma]$ or $\langle \emptyset, \{\bot, A_1, \ldots, A_n\}\rangle \in [\Gamma]$. In the degenerate case, *i.e.*, when the second component of the pair shown is simply $\{C\}$ or $\{\bot\}$, this rule has no upper sequents and, in this case, constitutes an axiom.

(3) The BACKCHAIN rule is the following

$$\frac{\Gamma \longrightarrow G' \quad A_1, \Gamma \longrightarrow G \quad \ldots \quad A_n, \Gamma \longrightarrow G}{\Gamma \longrightarrow C}$$

provided that $\langle\{G'\},\{C,A_1,\ldots,A_n\}\rangle\in[\Gamma]$ or $\langle\{G'\},\{\bot,A_1,\ldots,A_n\}\rangle\in[\Gamma]$. In the degenerate case, *i.e.*, when the second component of the pair shown is simply $\{C\}$ or $\{\bot\}$, this rule has $\Gamma\longrightarrow G'$ as its only upper sequent.

By a "reduced proof system" relative to a goal G let us mean a calculus whose axioms are of the form $\Delta\longrightarrow\top$ and whose rules are the RESTART, ATOMIC and BACKCHAIN rules relativized to G, $\vee$-R, $\wedge$-R, $\supset$-R and $\exists$-R. The main result of this section is then the following:

Theorem 15 *Let Γ be a (multi)set of program clauses and let G be a goal under the syntax described for such formulas in this section. Then $\Gamma\longrightarrow G$ has a* **C***-proof if and only if it has a proof in the reduced proof system relative to G.*

Proof. An immediate consequence of Theorem 11 and Lemmas 12 and 14.

□

The reduced proof system provides the basic structure of the promised procedure for constructing proofs for formulas in classical logic. This procedure would simplify complex goals based on the rules $\vee$-R, $\wedge$-R, $\supset$-R and $\exists$-R and would use an instance of the RESTART, ATOMIC or BACKCHAIN rule on reaching an atomic formula. In a practical rendition of this procedure, it will be necessary to delay the choice of term to be used relative to the $\exists$-R rule. A suitable delaying ability can, as usual, be obtained by using a variable that can be later instantiated in conjunction with this rule and by carrying out the instantiation by using unification in the implementation of the ATOMIC and BACKCHAIN rules.

The procedure described above can, of course, also be used to find proofs for sequents of the form $\Gamma\longrightarrow G$. An interesting aspect of this procedure is that it reduces to others described in the literature when (further) restrictions are placed on the syntax of G and the formulas in Γ. For example, suppose that disjunction and the symbol $\bot$ are disallowed in the heads of program clauses and implication is disallowed at the top-level in goals. The logic being considered reduces in this case to that of Horn clauses. From Lemma 2 and an examination of the proof of Lemma 12, it is easily seen that the RESTART rule is redundant in this context. Further, only the degenerate forms of the ATOMIC and BACKCHAIN rules are relevant in this situation and that too in a form where the possibility of $\bot$ being the head of a clause instance need not be considered. Our procedure is equivalent under these observations to the usual one employed for Horn clause logic. Along a different direction, suppose the syntax of program clauses in the Horn clause setting is enriched by permitting disjunctions in their heads, thereby producing the logic underlying disjunctive logic programming [NL95]. From an examination of the proofs of Lemmas 8 and 12, it becomes apparent that the RESTART rule is redundant in this situation as well. The exclusion of this rule from our proof procedure yields one that has the essential structure of the Inheritance Near-Horn Prolog procedure (InH-Prolog) [LR91, RLS91]. Finally suppose that disjunction and $\bot$ is disallowed in the heads of program clauses but that the syntax for these formulas and goals is otherwise unaltered from the one presented at the beginning of this section. The resulting goals subsume (conjunctions of) the N-clauses of [GR84]. In this context, the RESTART rule and only (restricted versions of) the degenerate forms of the ATOMIC and BACKCHAIN rules are relevant and our proof procedure reduces to (a simple generalization of) the QNR-Prolog procedure described in [Gab85].

An important aspect of the proof procedure we have outlined above is the directionality present in the backchaining rules used in it. We note that the ability to impart this directionality to these rules is also dependent of the augmentation of the assumption set with the negation of the original goal formula. To see this, suppose that the formulas in the antecedents of the sequents whose proofs we seek are either disjunctions of atoms or of the form

$$(B_1 \wedge \ldots \wedge B_n) \supset (A_1 \vee \ldots \vee A_m)$$

where the B_is and A_js are atomic and the succedents of these sequents are conjunctions of atoms; despite the apparently severe syntactic restrictions on the formulas, this context is of interest because it corresponds to propositional classical logic under a translation to clausal form. Now, it is easily seen that classical, intuitionistic and uniform provability coincide for sequents of the described kind. However, when a proof is sought for an atomic formula in this context, this formula does not always help in determining what should be used from the assumption set. For example, consider the sequent

$$p \vee q, p \supset r, q \supset r \longrightarrow r.$$

In constructing a **I**-proof for this sequent, the last rule that must be used is an $\vee$-L that introduces the top-level connective in the assumption formula $p \vee q$. At a deeper level, the inability to use the (atomic) succedent formula to drive the proof search in this situation arises from the fact that $\vee$-L rules may sometimes have to appear after $\supset$-L rules in **I**-proofs. The augmentation of the assumption set with the negation of the goal formula permits the $\vee$-L rule to be replaced by the $\vee$-L$_G$ rule, leading eventually to an elimination of the mentioned dependency. An alternative approach, which works within the original proof system, is to proceed as if $\vee$-L rules are not required in the proof being constructed and, when they are determined to be actually necessary, to attempt to insert them at an appropriate point in the proof. The modified problem reduction format of Plaisted [Pla88] is based on this approach and on noting that the use of assumption formulas of the form $(B_1 \wedge \ldots \wedge B_n) \supset A$ where A is atomic can be driven, even in this context, by the atomic formula for which proof is sought.

6 Conclusion

We have examined the applicability of the notion of uniform provability to classical logic in this paper. It is easily observed that this form of derivation diverges from classical provability in the general case. However, we have shown that if there are no essentially positive occurrences of universal quantifiers in our formulas, then a modest, sound, modification to the set of assumptions — in particular, the addition to them of the negation of the formula to be proved — results in a coincidence between uniform and classical provability. We have exploited this fact in outlining a proof procedure for classical logic. The procedure that we have described subsumes several previously proposed ones for different subsets of classical logic. The uniform proof notion appears, in this sense, to be a unifying principle behind proof search in this logical setting.

The discussions in this paper suggest other directions for further investigation. At a pragmatic level, it is of interest to develop, and to experiment with, an actual proof procedure based on the ideas presented here. Another matter worthy of consideration is the usefulness of the uniform proof notion and the general approach described here in structuring proof search in intuitionistic logic.

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