# A REPRESENTATION OF NONNEGATIVE SUBMARTINGALES AND ITS APPLICATIONS

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ABSTRACT. We suggest a representation of nonnegative submartingales by means of increasing processes and give a simpler proof of the Doob-Meyer theorem.

**1**. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $\{\mathcal{F}(t), t \in [0, \infty]\}$ an increasing filtration of $\mathcal{F}$-complete $\sigma$-fields $\mathcal{F}(t) \subset \mathcal{F}$, $\xi(t)$ a positive (that is nonnegative) submartingale with respect to $\{\mathcal{F}(t)\}$ defined for $t \in [0, \infty]$. Under certain rather general hypotheses Smirnov [3] proved the existence of a nonnegative increasing process $\eta(t)$ such that

$$\xi(t) = E\{\eta(t)|\mathcal{F}(t)\} \quad \text{(a.s.)} \quad \forall t \in [0, \infty]. \tag{1}$$

If $\xi(t)$ is a martingale, obviously one can take $\eta(t) = \xi(\infty)$. On the other hand, if $\eta(t)$ is any increasing process such that $E|\eta(t)| < \infty$, $\forall t$, then the right-hand side of (1) is a submartingale. Therefore representation (1) is a quite natural generalization of the usual representation of martingales.

In this note we first prove that representation (1) is valid if we impose only the hypotheses stated above in the first phrase. Then we use the representation to prove the Doob-Meyer theorem. In our opinion this proof is simpler and shorter than the known ones being only based on the most elementary facts of the theory of martingales. However, we admit that we only prove the Doob-Meyer theorem in the form given by Meyer [2], that is without Doléans's assertion that natural increasing processes are predictable (see [1]).

Observe that the existence of representation (1) can be easily explained by using the multiplicative representation: $\xi(t) = A(t)M(t)$, where $A(t)$ is $\mathcal{F}_t$-adapted and increasing and $M(t) = E\{M(\infty)|\mathcal{F}_t\}$. In that situation one can take $\eta(t) = A(t)M(\infty)$.

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Translation of "Une représentation des sousmartingales positives et ses applications", Lecture Notes in Math., Vol. 1426, 1990, 473-476. In the published paper some text from the submitted manuscript was deleted by the editors. It is included back in this translation.

We also point out that these notes appeared as a result of several conversations with S.N. Smirnov to whom the author is sincerely grateful.

**2**. Let $Q=\{q_1,q_2,...\}$ be a dense subset of $[0,\infty]$ containing all discontinuity points of the increasing function $E\xi(t)$. We let $q_1=0,q_2=\infty$. For each integer $n\geq 2$ we denote by $Q_n$ the subset of $Q$ consisting of its $n$ first elements, which we arrange in the increasing order: $q_n(i),i=1,...,n$. For $i\leq n-1$ choose functions $f_n(i)$ such that they are $\mathcal{F}(q_n(i))$-measurable and (a.s.)

$$\xi(q_n(i))=f_n(i)E\{\xi(q_n(i+1))|\mathcal{F}(q_n(i))\}. \tag{2}$$

Since by definition for $s\leq t$ (a.s.)

$$\xi(s)\leq E\{\xi(t)|\mathcal{F}(s)\}, \tag{3}$$

we may assume that $0\leq f_n(i)\leq 1$. Finally, for $i=1,...,n-1$ let

$$\zeta_n(t)=f_n(i)\cdot...\cdot f_n(n-1)\quad\text{for}\quad t\in[q_n(i),q_n(i+1)).$$

For $i=n$ we have $q_n(i)=\infty$ and we let $\zeta_n(\infty):=1$. Iterating (2) yields

$$\xi(t)=E\{\xi(\infty)\zeta_n(t)|\mathcal{F}(t)\}\quad\text{(a.s.)}\quad\forall t\in Q_n. \tag{4}$$

Furthermore, obviously $\zeta_n(t)$ is an increasing function of $t$ and $0\leq\zeta_n(t)\leq 1$. It follows that there exists a subsequence $\{n'\}\subset\{n\}$ such that $n'\to\infty$ and for each $q\in Q$ the sequence $\zeta_{n'}(q)$ converges weakly in $L_2(\Omega,\nu)$, where $\nu(d\omega)=\xi(\omega,\infty)\,P(d\omega)$, to a limit denoted by $\zeta(q)$. For each $t\in[0,\infty]$ introduce

$$\alpha(t)=\inf_{q\geq t}\zeta(q), \tag{5}$$

where and everywhere below by $q$ we denote a generic element of $Q$.

Since $\zeta_n(t)$ increases in $t$, on $Q$ we have $\alpha(q)=\zeta(q)$ (a.s.). For $t\notin Q$ we have

$$\alpha(t)=\lim_{q\downarrow t}\zeta(q)=\lim_{q\downarrow t}\alpha(q), \tag{6}$$

where the former limit exists almost surely and the latter one for any $\omega\in\Omega$. We multiply through (4) by $I_A$, where $A\in\mathcal{F}(t)$, take expectations and after replacing $n$ with $n'$ let $n'\to\infty$. Then for $q\in Q$ we find

$$\xi(q)=E\{\xi(\infty)\zeta(q)|\mathcal{F}(q)\}=E\{\xi(\infty)\alpha(q)|\mathcal{F}(q)\}\quad\text{(a.s.)}. \tag{7}$$

In addition, due to (3) and (7) for $t<q$ (a.s.)

$$\xi(t)\leq E\{\xi(q)|\mathcal{F}(t)\}=E\{\xi(\infty)\alpha(q)|\mathcal{F}(t)\},$$

which along with (6) yield (a.s.)

$$\xi(t)\leq E\{\xi(\infty)\alpha(t)|\mathcal{F}(t)\}. \tag{8}$$

To prove that actually here we have an equality instead of the inequality it suffices to prove that the expectations of both sides coincide. They coincide indeed for $t \in Q$ (see (7)) and for $t \notin Q$ the choice of $Q$ and (7) and (6) imply

$$E\xi(t) = \lim_{q \downarrow t} E\xi(q) = \lim_{q \downarrow t} E\xi(\infty)\alpha(q) = E\xi(\infty)\alpha(t).$$

Thus, (1) holds with $\eta(t) = \xi(\infty)\alpha(t)$.

*Remark* 1 (cf. [3]). [1] If $E\xi(t)$ is right (left) continuous, then $\eta(t)$ can be taken to be right (respectively, left) continuous.

Indeed, in the first case introduce $\beta(t)$ according to (5) with $q \geq t$ replaced with $q > t$ if $t < \infty$ and set $\beta(\infty) = 1$. Then $\beta(t)$ is right continuous and $\alpha(t) \leq \beta(t) \leq 1$. Furthermore, for $t < \infty$ owing to (7) and (1) we have

$$E\{\xi(\infty)\beta(t)\} = \lim_{q \downarrow t} E\{\xi(\infty)\zeta(q)\} = \lim_{q \downarrow t} E\xi(q) = E\xi(t) = E\{\xi(\infty)\alpha(t)\}.$$

Hence $\xi(\infty)\beta(t) = \xi(\infty)\alpha(t)$ (a.s.) for $t < \infty$. This is also true for $t = \infty$ since $\beta(\infty) = \alpha(\infty) = 1$.

If $E\xi(t)$ is left continuous, introduce

$$\beta(t) = \sup_{q<t} \zeta(q), \quad t > 0, \quad \beta(0) = \alpha(0).$$

Now $\beta(t)$ is left continuous, $\beta(t) \leq \alpha(t)$ for $t \in [0, \infty]$, and again $\xi(\infty)\beta(t) = \xi(\infty)\alpha(t)$ (a.s.) since for $t = 0$ this is obvious and for $t > 0$ due to (7) and (1) we have

$$E\{\xi(\infty)\beta(t)\} = \lim_{q \uparrow t} E\{\xi(\infty)\zeta(q)\} = E\xi(t) = E\{\xi(\infty)\alpha(t)\}.$$

*Remark* 2. [2] In contrast with Remark 1 the continuity of $E\xi(t)$ does not imply that $\eta(t)$ can be chosen continuous. Indeed, if $\mathcal{F}(t) \equiv \mathcal{F}$, then $\eta(t) = \xi(t)$ (a.s.). Moreover, in this situation any bounded increasing process $\xi(t)$ is a submartingale and it can have discontinuous trajectories even if $E\xi(t)$ is continuous.

**3**. We pass to the Doob-Meyer theorem. Additionally to the hypotheses of §1 assume that $E\xi(t)$ *is right continuous* and take $\eta(t)$ in (1) to be right continuous. For any bounded random variable $\lambda$ let $m_t(\lambda) = E\{\lambda | \mathcal{F}(t)\}$. From Doob's upcrossing theorem we know that, with probability one, for all $t \in (0, \infty]$ at once there exists

$$m_{t-}(\lambda) = \lim_{q \uparrow t} m_q(\lambda)$$

[1] Strangely enough the original text of the remark was somewhat changed in print although the essence remained the same. We prefer to translate the original text.

[2] This remark was deleted by the editors.

and the process $m_{t-}(\lambda)$ is left continuous. For simplicity of notation, if $\lambda = I_B$ with $B \in \mathcal{F}$, we write $m_t(B)$ and $m_{t-}(B)$ instead of $m_t(\lambda)$ and $m_{t-}(\lambda)$, respectively. Observe that for any sequence of events $B_n$ such that $B_n \downarrow \emptyset$ we have (a.s.)

$$\sup_{t>0} m_{t-}(B_n) \leq \sup_{q \in Q} m_q(B_n) \downarrow 0.$$ [3]

Upon adding that $m_{r-}(B) \leq 1$ and $\eta(\infty) = \xi(\infty)$ is integrable, we see that the formula

$$\mu_t(B) = E\int_0^t m_{r-}(B)\, d\eta(r) \tag{9}$$

introduces a finite measure on $\mathcal{F}$, which is absolutely continuous with respect to $P$. #[4] Therefore, the following objects are well defined:

$$\gamma(t) = \frac{\mu_t(d\omega)}{P(d\omega)}, \quad A(t) = \inf_{q>t} \gamma(q), \quad t < \infty, \quad A(\infty) = \gamma(\infty).$$

Clearly $A(t)$ is increasing and right continuous. Moreover, $\mu_t \leq \mu_s$ when $t \leq s$ and $\gamma(t) \leq \gamma(s)$ (a.s.) if $t \leq s$. Hence, for $t < \infty$

$$A(t) = \lim_{q \downarrow t} \gamma(q) \quad \text{(a.s.)}.$$

Also observe that $\gamma(t)$ is majorized by an integrable variable $\gamma(\infty)$ and by the dominated convergence theorem

$$E\{I_B A(t)\} = \lim_{q\downarrow t} E\{I_B \gamma(q)\} = \lim_{q \downarrow t} E\{\int_0^q m_{r-}(B)\, d\eta(r)\}$$
$$= E\{\int_0^t m_{r-}(B)\, d\eta(r)\} = E\{I_B\gamma(t)\}.$$

It follows that $A(t) = \gamma(t)$ (a.s.). In particular, $A(0) = 0$ (a.s.).# A classical approximation argument shows that, for any bounded random variable $\lambda$ and $0 \leq s \leq t \leq \infty$, we have

$$E\{\lambda A(t)\} = E\{\int_0^t m_{r-}(\lambda)\, d\eta(r)\},$$
$$E\{\lambda(A(t) - A(s))\} = E\{\int_s^t m_{r-}(\lambda)\, d\eta(r)\}. \tag{10}$$

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[3]Added in translation: Observe that by the Kolmogorov-Doob maximal inequality this holds in probability. After that use that the sequence is decreasing.

[4]The text between two signs # was replaced by the editors with "Soit $A(t)$ la densité de $\mu_t$ par rapport à $P$. En procédant comme au §2, nous pouvons supposer que $A(t)$ est, pour tout $\omega \in \Omega$ une fonction croissante et continue à droite de $t$ et que $A(\infty) = \lim_{t\uparrow\infty} A(t)$. (The latter equality is generally wrong and is not used in the paper.)

If $\lambda$ is $\mathcal{F}(s)$-measurable, then $m_{r-}(\lambda) = \lambda$ for $r > s$ and it follows from (10) and (1) that

$$E\{\lambda(A(t) - A(s))\} = E\{\lambda(\eta(t) - \eta(s))\} = E\{\lambda(\xi(t) - \xi(s))\}. \quad (11)$$

Furthermore obviously, $m_{r-}(m_t(\lambda)) = m_{r-}(\lambda)$ (a.s.) for any $r \le t$ which implies that

$$E\{\lambda A(t)\} = E\{\int_0^t m_{r-}(m_t(\lambda))\, d\eta(r)\} = E\{m_t(\lambda)A(t)\} = E\{\lambda m_t(A(t))\}.$$

This means that $A(t)$ is $\mathcal{F}(t)$-measurable. Now equation (11) implies that $\xi(t) - A(t)$ is a martingale, say $M(t)$. Thus we get the Doob-Meyer decomposition: $\xi(t) = A(t) + M(t)$, where $M(t)$ is a martingale and $A(t)$ is an adapted increasing process.

Finally, starting from (10) and approximating the integral of a bounded left-continuous function by Riemannian sums we obtain that[5]

$$E\{\int_0^\infty m_{r-}(\lambda)\, dA(r)\} = E\{\int_0^\infty m_{r-}(\lambda)\, d\eta(r)\} = E\{\lambda A(\infty)\}.$$

This means that the increasing process $A(t)$ is natural in the sense of Meyer [2].

## References


[1] C. Dellacherie, Capacités et Processus Stochastiques, Springer,1972.

[2] P.-A. Meyer, Probability and Potentials, Blaisdell, 1966.

[3] M.Yu. Svertchkov and S.N. Smirnov, *A representation for supermartingales*, Vestnik Moskov. Univ. Ser. XV Vychisl. Mat. Kibernet., 1989, no. 3, 46-50, 74 in Rusian; English translation: Moscow Univ. Comput. Math. Cybernet. 1989, no. 3, 52-56.



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[5]The following explanation was deleted by the editors from the original manuscript: Introduce $\kappa_n(t) = q_n(i)$ for $t \in (q_n(i), q_n(i+1)]$, $i = 1, ..., n-1$. Clearly, $\kappa_n(t) < t$, $\kappa_n(t) \uparrow t$ and for bounded $\lambda$

$$\sum_{i=1}^{n-1} m_{q_n(i)}(\lambda)(A(q_n(i+1)) - A(q_n(i))) = \int_0^\infty m_{\kappa_n(t)}(\lambda)\, dA(t) \to \int_0^\infty m_{t-}(\lambda)\, dA(t).$$

Then by (11) and the dominated convergence theorem we get