Take home final due Thursday June 10

1. Can there be two characteristic functions which coincide only for $|u| \leq 1$? Give an example of a sequences of chracteristic functions which converges to a given characteristic function only for $|u| \leq 1$. (Hint: Use Polya.)

2. Explain why the function defined by $(1 - v^2)^{1/2}$ for $|v| \leq 1$ and 0 for $|v| \geq 1$ is not a characteristic function.

3. Give an example of iid $X_1, X_2, ...$ such that $(X_1 + ... + X_n)/n$ converges in distribution but not in probability. (Hint: upon assuming that $(X_1 + ... + X_n)/n$ tends in probability to a random variable ξ, prove that ξ is constant (a.s.) due to Kolmogorov's 0-1 law.)

4. Give an example of a sequence of moment generating functions $\varphi_n(u)$ such that $\lim_{n\to\infty} \varphi_n(u)$ exists for any $u \geq 0$, but this limit is not a continuous function of $u \in [0, \infty)$.

5. Let (X, Y, Z) be a 3 dimensional normal variable. Prove that X, Y, Z are independent iff they are pairwise uncorrelated.

6. Let w_t be a Wiener process, $\tau_a = \inf\{t \geq 0 : w_t = a\}$. We know that $P\{\tau_a < \infty\} = 1$. In particular, $P\{\tau(a, b) < \infty\} = 1$ where $a < 0 < b$ and $\tau(a, b)$ is the first time w_t hits either a or b. By using that $w_t^2 - t$ is a martingale, prove that

$$E\tau(a, b) = |a|b.$$

7. For any constant b and $a, T > 0$ define

$$P_T(a, b) = P\{\max_{s\leq T}(w_s + bs) < a\}.$$

We know that $P_T(a, b) =$

$$\frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{-bT} e^{-\frac{1}{2T}(x+a)^2} \, dx - e^{2ba} \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{-bT} e^{-\frac{1}{2T}(x-a)^2} \, dx.$$

By using change of variables show that $P_T(a, b) =$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a/\sqrt{T} - b\sqrt{T}} e^{-\frac{1}{2}x^2} \, dx - e^{2ba} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-a/\sqrt{T} - b\sqrt{T}} e^{-\frac{1}{2}x^2} \, dx.$$

By letting $T \to \infty$ conclude that $P\{\max_{s<\infty}(w_s + bs) < a\}$ equals 0 if $b \geq 0$ and $1 - e^{2ba}$ if $b < 0$. (Warning: the case $b = 0$ requires a little bit extra attention.)