HW 5, due Monday June 7

Chapter 26 Problems 30, 32, 33 and

(4) Let X_n be a branching process with expected number of offsprings of each member of population > 1, so that $c := P^1(\tau_0 < \infty) < 1$. Prove that $\lim_{n\to\infty} c^{X_n}$ exists with probability 1 and conclude that on the set $\{\tau = \infty\}$ we have $\lim_{n\to\infty} X_n = \infty$ (a.s.). (Hint: use that c^x is harmonic and use Theorem 24-19)

(5) Let $Z_n = (X_n, Y_n)$ be defined by $X_0 = Y_0 = 0$, $X_n = \xi_1 + ... + \xi_n$ and $Y_n = \eta_1 + ... + \eta_n$ for $n \geq 1$, where ξ_n, η_n are iid with $P\{\xi_n = \pm 1\} = P\{\eta_n = \pm 1\} = 1/2$. Prove that Z_n is a Markov sequence on the state space $\mathbb{Z}^2$ (two dimensional lattice) with transition probabilities given by

$$P^{(x,y)}(x+1, y+1) = P^{(x,y)}(x-1, y-1) = \frac{1}{4},$$

and

$$P^{(x,y)}(x+1, y-1) = P^{(x,y)}(x-1, y+1) = \frac{1}{4}.$$

(6) (Recurrence of two-dimensional random walk) For the process from (5), by remembering Bernoulli, prove that

$$u_{2n} := P^{(0,0)}\{Z_{2n} = 0\} = 4^{-2n}\binom{2n}{n}^2.$$

By using Stirling's formula show that $\sum_n u_{2n} = \infty$. On the basis of renewal theory derive from here that the number of returns of Z_n to the origin is infinite with probability one.

(7) Let w_t be a Wiener process. Prove that $\int_0^1 w_t\, dt$ is a Gaussian random variable and find its parameters. (*Hint*: You may need to use $(\int_0^1 w_t\, dt)^2 = \int_0^1 \int_0^1 w_t w_s\, ds dt$. It is also a good idea to represent $\int_0^1 w_t\, dt$ as a limit of partial sums.)

(8) (Self-similarity of the Wiener process) Let w_t be a Wiener process and $c \neq 0$ be a constant. Prove that $c w_{t/c^2}$ is again a Wiener process.

(9) Let w_t be a Wiener process, $\tau_a = \inf\{t \geq 0 : w_t = a\}$. We know that $P\{\tau_a < \infty\} = 1$. In particular, $P\{\tau(a,b) < \infty\} = 1$ where $a < 0 < b$ and $\tau(a,b)$ is the first time w_t hits either a or b. By using that w_t is a martingale, prove that

$$P\{w_{\tau(a,b)} = b\} = \frac{|a|}{|a| + b}.$$