RIESZ-MARKOV-KAKUTANI THEOREM

Theorem 0.1. *Let X be a compact Polish space and C be the set of real-valued continuous functions on X provided with uniform norm. Let L be a linear continuous functional on C such that $Lf \geq 0$ if $f \geq 0$. Then there exists a nonnegative finite measure μ on Borel subsets of X such that*

$$Lf = \int_X f(x)\,\mu(dx). \tag{0.1}$$

We need a few auxiliary results for the proof. For $F \subset X$ define

$$\nu(F) = \inf\{Lf : f \in C, f \geq I_F\}.$$

Obviously, for any $A, B \subset X$,

$$\nu(A \cup B) \leq \nu(A) + \nu(B),$$

and $\nu(A) \leq \nu(B)$ if $A \subset B$. We write $F \in \mathfrak{B}_0$ if $\nu(\partial F) = 0$.

Lemma 0.2. *The collection $\mathfrak{B}_0$ is an algebra and the set-function ν is an additive function of $\mathfrak{B}_0$.*

Proof. Observe that $\nu(\partial F) = \nu(\partial(F^c))$ and

$$\nu(\partial(A \cup B)) \leq \nu((\partial A) \cup \partial B) \leq \nu(\partial A) + \nu(\partial B)$$

implying that $\mathfrak{B}_0$ is an algebra.

To prove the additivity of ν on $\mathfrak{B}_0$ take $A, B \in \mathfrak{B}_0$ such that $AB = \emptyset$. Then

$$\bar{A}\bar{B} = (\partial A \cup A)(\partial B \cup B) = [(\partial A)\partial B] \cup [(\partial A)B] \cup [A\partial B] \subset (\partial A) \cup \partial B,$$
$$\nu(\bar{A}\bar{B}) = 0.$$

Hence, for any $\varepsilon > 0$ one can find $f, \phi \in C$ such that

$$f \geq I_{A\cup B}, \quad 1 \geq \phi \geq I_{\bar{A}\bar{B}}, \quad Lf \leq \nu(A \cup B) + \varepsilon, \quad L\phi \leq \varepsilon.$$

Define

$$f_1(x) = 1 \quad \text{for} \quad x \in \bar{A}, \quad f_1(x) = \phi(x) \quad \text{for} \quad x \in \bar{B},$$

and extend f_1 as a continuous function on X such that

$$0 \leq f_1(x) \leq f(x) + \phi(x)$$

(take any continuous extension $g \geq 0$ and to satisfy the above inequality consider $\min(g, f + \phi)$). Also set $f_2 = f + \phi - f_1$. Then $f_2 \in C$, $f_2 \geq 0$, and $f_2 = f \geq 1$ on B. Hence,

$$\nu(A) + \nu(B) \leq Lf_1 + Lf_2 = Lf + L\phi \leq \nu(A \cup B) + 2\varepsilon.$$

This finishes the proof of the lemma.

Remark 0.1. If $\bar{A}\bar{B} = \emptyset$, one can take $\phi \equiv 0$ and in that case even for $A, B \notin \mathfrak{B}_0$ we have

$$\nu(A \cup B) = \nu(A) + \nu(B).$$

Lemma 0.3. *The set-function ν is regular on $\mathfrak{B}_0$, that is, for any $A \in \mathfrak{B}_0$ and $\varepsilon > 0$ there exists a closed set $\Gamma \subset A$ and an open set $G \supset A$ such that $\Gamma, G \in \mathfrak{B}_0$ and $\nu(G) - \varepsilon \le \nu(A) \le \nu(\Gamma) + \varepsilon$.*

Proof. Since $\mathfrak{B}_0$ is an algebra, it suffices to concentrate on open $G \supset A$. Observe that for any $f \in C$ we have

$$\nu(\{x : f(x) = \lambda\}) = 0$$

for all λ apart perhaps from countably many of them. This follows from Remark 0.1 since the sets $\{x : f(x) = \lambda\}$ are closed and disjoint for different λ so that for different λ_i

$$\sum_i \nu(\{x : f(x) = \lambda_i\}) = \nu\big(\cup_i \{x : f(x) = \lambda_i\}\big) \le \nu(X) < \infty.$$

It follows that open sets $\{x : f(x) > \lambda\}$ belong to $\mathfrak{B}_0$ for all λ apart perhaps from countably many of them.

Now take a nonnegative $f \in C$ such that $f \ge I_A$ and

$$Lf \le \nu(A) + \varepsilon/2.$$

Take a $\lambda \in (0, 1)$ such that the open set $G := \{x : f(x) > \lambda\} \in \mathfrak{B}_0$. Observe that $G \supset A$. Also

$$\nu(G) \le L(f/\lambda) \le \nu(A)/\lambda + \varepsilon/(2\lambda) = \nu(A) + \frac{1-\lambda}{\lambda}\nu(A) + \varepsilon/(2\lambda),$$

and it only remains to take λ so close to 1 that what we add to $\nu(A)$ above is less than ε. The lemma is proved.

Lemma 0.4. *The additive set-function ν on $\mathfrak{B}_0$ has a σ-additive extension as a measure on $\sigma(\mathfrak{B}_0)$.*

Proof. It suffices to show that for any decreasing sequence of sets $A_n \in \mathfrak{B}_0$ such that $\cap_n A_n = \emptyset$, we have $\nu(A_n) \to 0$ as $n \to \infty$.

We argue by contradiction and assume that there exists an $\varepsilon > 0$ and a decreasing sequence of sets $A_n \in \mathfrak{B}_0$ such that $\cap_n A_n = \emptyset$ but $\nu(A_n) \ge \varepsilon$. Then we take closed $\Gamma_n \in \mathfrak{B}_0$ such that $\Gamma_n \subset A_n$ and $\nu(\Gamma_n) \ge \nu(A_n) - \varepsilon/2^{n+1}$.

The closed sets $B_n = \cap_{k \le n}\Gamma_k$ are nested and their intersection is empty. Then B_n is empty for an n (X is a compact set). However,

$$A_n \setminus B_n = \Big(\bigcap_{k \le n} A_k\Big) \setminus \bigcap_{k \le n} \Gamma_k \subset \bigcup_{k \le n} (A_k \setminus \Gamma_k),$$

$$\nu(A_n \setminus B_n) \le \varepsilon \sum_k 2^{-k-1} = \varepsilon/2, \quad \mu(B_n) \ge \nu(A_n) - \varepsilon/2 \ge \varepsilon/2.$$

and this is the desired contradiction.

We call μ the extended ν.

The end of proof of the theorem. The only things which remain to be proved are that $\sigma(\mathfrak{B}_0)$ contains the Borel σ-field and that (0.1) holds. Observe that $\rho(x, x_0)$ is a continuous function of x for any x_0. By what was said above, $B_r(x_0) \in \mathfrak{B}_0 \subset \sigma(\mathfrak{B}_0)$ for almost all $r > 0$. Since $\sigma(\mathfrak{B}_0)$ is a σ-field we conclude that $B_r(x_0) \in \sigma(\mathfrak{B}_0)$ for all $r > 0$ and all Borel sets are also in $\sigma(\mathfrak{B}_0)$.

To prove (0.1) take $f \in C$ such that $0 \leq f \leq 1$. Let $c_0 < 0 < c_1 < ... < c_{n-1} < 1 < c_n$ be such that $\{x : f(x) < c_i\} \in \mathfrak{B}_0$. Then for any $\varepsilon > 0$ one can find $\phi_i \in C$ such that

$$\phi_i \geq I_{E_i}, \quad L\phi_i \leq \nu(E_i) + \varepsilon/n = \mu(E_i) + \varepsilon/n,$$

where $E_i = \{x : c_i \leq f(x) < c_{i+1}\}$. Then

$$f \leq \sum_{i=0}^{n-1} c_{k+1}\phi_k, \quad Lf \leq \sum_{i=0}^{n-1} c_{k+1}L\phi_k \leq \sum_{i=0}^{n-1} c_{k+1}\mu(E_i) + \varepsilon$$

$$\leq \int_X f(x)\,\mu(dx) + \varepsilon + \max(c_{k+1} - c_k).$$

Since what we add can be made arbitrarily small

$$Lf \leq \int_X f(x)\,\mu(dx), \quad L(1-f) \leq \int_X (1 - f(x))\,\mu(dx)$$

and the theorem is proved because $L1 = \nu(X) = \mu(X)$.

Something like Section 19.3

Theorem 0.5 (Bachelier)**.** *For every* $t \in (0, 1]$ *we have* $\max_{s \leq t} w_s \sim |w_t|$, *which is to say that for every* $x \geq 0$

$$P\{\max_{s\leq t} w_s \leq x\} = \frac{2}{\sqrt{2\pi t}} \int_0^x e^{-y^2/(2t)}\,dy.$$

Proof. Take independent identically distributed random variables η_k so that $P(\eta_k = 1) = P(\eta_k = -1) = 1/2$, and define ξ_t^n by

$$\xi_t^n := S_{[nt]}/\sqrt{n} + (nt - [nt])\eta_{[nt]+1}/\sqrt{n},$$

where $S_k := \eta_1 + ... + \eta_k$. First we want to find the distribution of

$$\zeta^n = \max_{[0,1]} \xi_t^n = n^{-1/2} \max_{k\leq n} S_k.$$

Observe that, for each n, the sequence $(S_1, ..., S_n)$ takes its every particular value with the same probability 2^{-n}. In addition, for each integer $i > 0$, the number of sequences favorable for the events

$$\{\max_{k\leq n} S_k \geq i, S_n < i\} \quad \text{and} \quad \{\max_{k\leq n} S_k \geq i, S_n > i\} \tag{0.2}$$

is the same. One proves this by using the reflection principle; that is, one takes each sequence favorable for the first event, keeps it until the moment when it reaches the level i for the last time before n and then *reflects* its remaining part about this level. This implies equality of the probabilities of the events in (0.2). Furthermore, due to the fact that i is an integer, we have

$$\{\zeta^n \geq in^{-1/2},\ \xi_1^n < in^{-1/2}\} = \{\max_{k\leq n} S_k \geq i, S_n < i\}$$

and

$$\{\zeta^n \geq in^{-1/2},\ \xi_1^n > in^{-1/2}\} = \{\max_{k\leq n} S_k \geq i, S_n > i\}.$$

Hence,

$$P\{\zeta^n \geq in^{-1/2},\ \xi_1^n < in^{-1/2}\} = P\{\zeta^n \geq in^{-1/2},\ \xi_1^n > in^{-1/2}\}.$$

Moreover, obviously,

$$P\{\zeta^n \geq in^{-1/2},\ \xi_1^n > in^{-1/2}\} = P\{\xi_1^n > in^{-1/2}\},$$

$$P\{\zeta^n \geq in^{-1/2}\} = P\{\zeta^n \geq in^{-1/2},\ \xi_1^n > in^{-1/2}\} + P\{\zeta^n \geq in^{-1/2},\ \xi_1^n < in^{-1/2}\} + P\{\xi_1^n = in^{-1/2}\}.$$

It follows that

$$P\{\zeta^n \geq in^{-1/2}\} = 2P\{\xi_1^n > in^{-1/2}\} + P\{\xi_1^n = in^{-1/2}\} \tag{0.3}$$

for every integer $i > 0$. The last equality also obviously holds for $i = 0$. We see that for numbers a of type $in^{-1/2}$, where i is a nonnegative integer, we have

$$P\{\zeta^n \geq a\} = 2P\{\xi_1^n > a\} + P\{\xi_1^n = a\}. \tag{0.4}$$

Certainly, the last probability goes to zero as $n \to \infty$ since ξ_1^n is asymptotically normal with parameters $(0, 1)$. Also, keeping in mind Donsker's theorem, it is natural to think that

$$P\{\max_{s\leq 1} \xi_s^n \geq a\} \to P\{\max_{s\leq 1} w_s \geq a\}, \quad 2P\{\xi_1^n > a\} \to 2P\{w_1 > a\}.$$

Therefore, (0.4) naturally leads to the conclusion that

$$P\{\max_{s\leq 1} w_s \geq a\} = 2P\{w_1 > a\} = P\{|w_1| > a\} \quad \forall a \geq 0,$$

and this is our statement for $t = 1$.

To justify the above argument, notice that (0.3) implies that

$$P\{\zeta^n = in^{-1/2}\} = P\{\zeta^n \geq in^{-1/2}\} - P\{\zeta^n \geq (i+1)n^{-1/2}\}$$

$$= 2P\{\xi_1^n = (i+1)n^{-1/2}\} + P\{\xi_1^n = in^{-1/2}\} - P\{\xi_1^n = (i+1)n^{-1/2}\}$$

$$= P\{\xi_1^n = (i+1)n^{-1/2}\} + P\{\xi_1^n = in^{-1/2}\}, \quad i \geq 0.$$

Now for every bounded continuous function $f(x)$ which vanishes for $x < 0$ we get

$$Ef(\zeta^n) = \sum_{i=0}^{\infty} f(in^{-1/2})P\{\zeta^n = in^{-1/2}\} = Ef(\xi_1^n - n^{-1/2}) + Ef(\xi_1^n).$$

By Donsker's theorem and by the continuity of the function $x_\cdot \to \max_{[0,1]} x_t$ we have

$$Ef(\max_{[0,1]} w_t) = 2Ef(w_1) = Ef(|w_1|).$$

We have proved our statement for $t = 1$. For other values of t one uses that cw_{s/c^2} is a Wiener process if $c > 0$. The theorem is proved.

Theorem 0.6. *Let u be a bounded continuous and continuously differentiable function on $\mathbb{R}$ such that u' is piece-wise differentiable and its derivative is bounded. Let c be a bounded Borel function on $\mathbb{R}$ such that $c > \delta$, where $\delta > 0$ is a constant. Denote*

$$f = cu - (1/2)u''.$$

Then

$$u(0) = E\int_0^\infty e^{-\phi_t} f(w_t)\, dt.$$

where

$$\phi_t = \int_0^t c(w_s)\, ds.$$

Comment on $u(x)$ for $x \neq 0$.

Example 0.1. Denote

$$m_t = \max_{s \leq t} w_s.$$

Then for $\lambda, \mu > 0$

$$E\int_0^\infty e^{-\mu m_t - \lambda t}\, dt = E\int_0^\infty e^{-\mu|w_t| - \lambda t}\, dt$$

is the value at 0 of the solution of the following equation

$$(1/2)u'' - \lambda u = -e^{-\mu|x|}.$$

Assuming that $\mu^2 \neq 2\lambda$ on finds that

$$u(x) = \frac{2\mu}{\sqrt{2\lambda}(\mu^2 - 2\lambda)} e^{-|x|\sqrt{2\lambda}} - \frac{2}{\mu^2 - 2\lambda} e^{-\mu|x|},$$

$$u(0)=\frac{2}{\mu\sqrt{2\lambda}+2\lambda}.$$

Example 0.2. Let $a>0$. Introduce $T_a(s)$ as the time spent by w_t inside $(-a,a)$ during the time period $[0,s]$:

$$T_a(s)=\int_0^s I_{(-a,a)}(w_s)\,ds.$$

Then for $\lambda,\mu>0$

$$I_a(\mu,\lambda):=E\int_0^\infty e^{-\mu T_a(t)-\lambda t}\,dt$$

is the value at 0 of the solution of

$$(1/2)u''-(\lambda+\mu I_{(-a,a)})u=-1.$$

We find that u is an even function and

$$u(x)=\frac{1}{\lambda}+c_1e^{-|x|\sqrt{2\lambda}}\quad\text{for}\quad|x|\geq a,$$

$$u(x)=\frac{1}{\lambda+\mu}+c_2\cosh(x\sqrt{2(\lambda+\mu)})\quad\text{for}\quad|x|\leq a,$$

where c_1 and c_2 are found from the conditions that $u(a+)=u(a-)$ and $u'(a+)=u'(a-)$:

$$c_1=-c_2e^{a\sqrt{2\lambda}}\sqrt{(\lambda+\mu)\lambda^{-1}}\sinh(a\sqrt{2(\lambda+\mu)}),$$

$$c_2=\frac{\mu}{\lambda(\lambda+\mu)}\big[\cosh(a\sqrt{2(\lambda+\mu)})+\sqrt{(\lambda+\mu)\lambda^{-1}}\sinh(a\sqrt{2(\lambda+\mu)})\big]^{-1},$$

so that

$$I_a(\mu,\lambda)=\frac{1}{\lambda+\mu}+c_2.$$

Take $\mu=(2a)^{-1}\nu$ and let $a\downarrow 0$. We will find the distribution of the so-called local time of w_t at the origine.

One finds that

$$\frac{1}{\lambda+\mu}+c_2\to\frac{2}{\nu\sqrt{2\lambda}+2\lambda},$$

which as we know is the Laplace transform of

$$Ee^{-\nu m_t}.$$

Hence, the Laplace transforms of

$$Ee^{-\nu(2a)^{-1}T_a(t)}$$

converge and by the general theory for $t>\varepsilon>0$

$$\overline{\lim_{a\downarrow 0}}\,Ee^{-\nu(2a)^{-1}T_a(t)}\leq\varepsilon^{-1}\overline{\lim_{a\downarrow 0}}\int_{t-\varepsilon}^t Ee^{-\nu(2a)^{-1}T_a(s)}\,ds=\varepsilon^{-1}\int_{t-\varepsilon}^t Ee^{-\nu m_s}\,ds,$$

which for $\varepsilon\downarrow 0$ along with a similar estimate from below yields that

$$\lim_{a\downarrow 0}Ee^{-\nu(2a)^{-1}T_a(t)}=Ee^{-\nu m_t}.$$

This by the general theory implies that the distributions of $(2a)^{-1}T_a(t)$ converge weakly to that of m_t or $|w_t|$. It turns out that the distribution of the limit of $(2a)^{-1}T_a(t)$ in C is the same as that of m_t!

Example 0.3. We are going to find the distribution of the time spent by w_t on the positive half-line on the time interval $[0,s]$:

$$T_+(s)=\int_0^s I_{(0,\infty)}(w_t)\,dt.$$

We know that for $\lambda,\mu>0$

$$E\int_0^\infty e^{-\mu T_+(t)-\lambda t}\,dt$$

is the value at 0 of the solution of

$$(1/2)u''-(\lambda+\mu I_{(0,\infty)})u=-1.$$

We have

$$u(x)=\frac{1}{\lambda+\mu}+c_1e^{-x\sqrt{2(\lambda+\mu)}}\quad\text{for}\quad x\geq 0,$$
$$u(x)=\frac{1}{\lambda}+c_2e^{x\sqrt{2\lambda}}\quad\text{for}\quad x\leq 0,$$

where c_1 and c_2 are found from the conditions that $u(0+)=u(0-)$ and $u'(0+)=u'(0-)$:

$$c_1=c_2+\frac{1}{\lambda}-\frac{1}{\lambda+\mu}=c_2+\frac{\mu}{\lambda(\lambda+\mu)},$$
$$c_2\sqrt{2(\lambda+\mu)}+\frac{\mu\sqrt{2(\lambda+\mu)}}{\lambda(\lambda+\mu)}=-c_2\sqrt{2\lambda},$$
$$c_2\sqrt{\lambda+\mu}+\frac{\mu}{\lambda\sqrt{\lambda+\mu}}=-c_2\sqrt{\lambda},$$
$$c_2=-\frac{\mu}{\lambda\sqrt{\lambda+\mu}(\sqrt{\lambda+\mu}+\sqrt{\lambda})}=-\frac{\sqrt{\lambda+\mu}-\sqrt{\lambda}}{\lambda\sqrt{\lambda+\mu}},$$
$$u(0)=\frac{1}{\lambda}+c_2=\frac{1}{\sqrt{\lambda}\sqrt{\lambda+\mu}}.$$

Observe that $1/\sqrt{\lambda}$ is the Laplace transform of a constant times $t^{-1/2}$ and $1/\sqrt{\lambda+\mu}$, as a function of λ, is the Laplace transform of a constant (independent of μ) times $t^{-1/2}e^{-\mu t}$. Hence, $u(0)$, as a function of λ is the La[lace trnsform of the convolutions of the above two functions, that is equal to a constant (independent of μ) times

$$\int_0^t\frac{1}{\sqrt{t-s}}\frac{1}{\sqrt{s}}e^{-\mu s}\,ds.$$

It follows that

$$Ee^{-\mu T_+(t)}=\alpha\int_0^t\frac{1}{\sqrt{t-s}}\frac{1}{\sqrt{s}}e^{-\mu s}\,ds,$$

where α is a constant, the distribution of $T_+(t)$ has a density

$$\alpha \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s}}$$

and for $r \in (0,t)$

$$P(T_+(t) \leq r) = \alpha \int_0^r \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s}}\, ds = 2\alpha \arcsin \sqrt{r/t},$$

implying that $2\alpha = 2/\pi$.