A REMARK ON A PAPER OF F. CHIARENZA AND M. FRASCA

N.V. KRYLOV

Abstract. In 1990 F. Chiarenza and M. Frasca published a paper in which they generalized a result of C. Fefferman on estimates of the integral of $|bu|^p$ through the integral of $|Du|^p$ for $p>1$. Formally their proof is valid only for $d\geq 3$. We present here further generalization with a different proof in which D is replaced with the fractional power of the Laplacian for any dimension $d\geq 1$.

Let an integer $d\geq 1$ and let $\mathbb{R}^d$ be a Euclidean space of points $x=(x^1,...,x^d)$. Fix $\alpha\in(0,d)$ and consider the Riesz potential

$$R_\alpha f(x)=\int_{\mathbb{R}^d}\frac{f(x+y)}{|y|^{d-\alpha}}\,dy.$$

We denote by $B_r(x)$ the open ball of radius r centered at x, $B_r=B_r(0)$, $\mathbb{B}_r$ the collection of $B_r(x)$, $S_1=\{|x|=1\}$. Our main result is the following, in which r,p,A are some numbers and $b=b(x)$ is a measurable function.

Theorem 1. *Assume $\alpha\leq r$, $1<r<p\leq d$, $b\geq 0$, $f\in L_r$, and for any $\rho>0$ and $B\in\mathbb{B}_\rho$.*

$$\Big(\fint_B b^p\,dx\Big)^{1/p}\leq A\rho^{-\alpha}.$$

Then

$$I:=\int_{\mathbb{R}^d}b^r|R_\alpha f|^r\,dx\leq N(\alpha,d,r,p)A^r\int_{\mathbb{R}^d}|f|^r\,dx. \tag{0.1}$$

Below by N we denote generic constants depending only on α,d,r,p,q.

Corollary 2. *If $u\in C_0^\infty(\mathbb{R}^d)$, then*

$$\int_{\mathbb{R}^d}b^r|u|^r\,dx\leq NA^r\int_{\mathbb{R}^d}\big|(-\Delta)^{\alpha/2}u|^r\,dx$$

and, if $\alpha=1$ and hence $d\geq 2$, (the Chiarenza-Frasca result)

$$\int_{\mathbb{R}^d}b^r|u|^r\,dx\leq NA^r\int_{\mathbb{R}^d}|Du|^r\,dx.$$

Indeed, $f:=(-\Delta)^{\alpha/2}u$ satisfies $f\in L_r$ and $R_\alpha f=u$ and the L_r-norms of Du and $(-\Delta)^{1/2}u$ are equivalent.

2010 *Mathematics Subject Classification.* 46E35, 42B25.

Key words and phrases. Fractional Laplacian, Chiarenza-Frasca theorem, Fefferman's theorem.

Remark 3. The author have used the Chiarenza-Frasca theorem in a few papers leading to [5] about strong solutions of Itô's equations. Theorem 1 paves the way to treat equations driven by Lévy rather than Wiener processes.

We prove Theorem 1 adapting to the "elliptic" setting the proof of Theorem 4.1 of [4]. We need two auxiliary and certainly well-known results in which $\mathbb{M}$ is the Hardy-Littlewood maximal operator.

Lemma 4. *If $1 < q \leq p$, it holds that*

$$R_\alpha(b^q) \leq NA(\mathbb{M}(b^q))^{1-1/q}.$$

Proof. We have

$$\begin{aligned} R_\alpha b^q(0) = N\int_0^\infty r^{\alpha-1}\int_{S_1} b^q(r\theta)\,\sigma(d\theta)\,dr \\ = N\int_0^\infty r^{\alpha-d}\frac{d}{dr}\int_{B_r} b^q\,dx\,dr \\ \leq N\int_0^\infty r^{\alpha-d-1}\int_{B_r} b^q\,dx\,dr = N\int_0^\rho + N\int_\rho^\infty \\ \leq N\rho^\alpha\mathbb{M}(b^q) + N\rho^{\alpha-q\alpha}A^q, \end{aligned}$$

where we used that

$$\fint_{B_r} b^q\,dx \leq \Big(\fint_{B_r} b^p\,dx\Big)^{q/p} \leq A^q r^{-q\alpha}.$$

For

$$\rho^{-q\alpha} = \mathbb{M}(b^q)/A^{-q}, \quad \rho^\alpha = A[\mathbb{M}(b^q)]^{-1/q}$$

we get the result. □

Lemma 5. *For any $\rho > 0$*

$$I := \int_{\mathbb{R}^d} b^p\mathbb{M}I_{B_\rho}\,dx \leq NA^p\rho^{d-p\alpha}. \tag{0.2}$$

Proof. We have

$$\mathbb{M}I_{B_\rho} \leq N(I_{B_\rho} + I_{|x|>\rho}\frac{\rho^d}{|x|^d})$$

and

$$\int_{\mathbb{R}^d} b^p I_{B_\rho} \leq NA^p\rho^{d-p\alpha},$$

$$\begin{aligned} \rho^d\int_\rho^\infty r^{-d}\frac{d}{dr}\int_{B_r} b^p\,dx\,dr \leq d\rho^d\int_\rho^\infty r^{-d-1}\int_{B_r} b^p\,dx\,dr \\ \leq N\rho^d A^p\int_\rho^\infty r^{-1-p\alpha}\,dr = N\rho^{d-p\alpha}A^p. \end{aligned}$$

This yields the result. □

Proof of Theorem 1. It suffices to concentrate on $f \geq 0$. Then first assume that $b^p \in A_1$, that is $\mathbb{M}(b^p) \leq N b^p$. Observe that for $v = R_\alpha f$ we have

$$I = \int_{\mathbb{R}^d} \big(b^r v^{r-1}\big) R_\alpha f \, dx = \int_{\mathbb{R}^d} R_\alpha \big(b^r v^{r-1}\big) f \, dx \leq \|f\|_{L_r} \big\| R_\alpha \big(b^r v^{r-1}\big) \big\|_{L_{r'}}, \tag{0.3}$$

where $r' = r/(r-1)$.

Next, take $\gamma > 0$, such that $(1+\gamma)r \leq p$, $1 + \gamma r' \leq p$, and $r \geq 1 + \gamma$. Note that

$$\begin{aligned} R_\alpha\big(b^r v^{r-1}\big) &= R_\alpha\big(b^{1+\gamma}\big(b^{r-1-\gamma} v^{r-1}\big)\big) \\ &\leq \Big(R_\alpha\big(b^{(1+\gamma)r}\big)\Big)^{1/r} \Big(R_\alpha\big(b^{r-\gamma r'} v^r\big)\Big)^{(r-1)/r}. \end{aligned}$$

It follows that

$$\big\| R_\alpha\big(b^r v^{r-1}\big)\big\|_{L_{r'}} \leq \Big(\int_{\mathbb{R}^d} b^{r-\gamma r'} v^r R_\alpha \Big[\Big(R_\alpha\big(b^{(1+\gamma)r}\big)\Big)^{1/(r-1)} \Big] \, dx \Big)^{(r-1)/r}.$$

Now in light of (0.3) we see that, to prove the theorem in our particular case, it only remains to show that

$$R_\alpha \Big[\Big(R_\alpha\big(b^{(1+\gamma)r}\big)\Big)^{1/(r-1)} \Big] \leq N b^{\gamma r'} A^{r'}. \tag{0.4}$$

By observing that $1 < (1+\gamma)r \leq p$ and using Lemma 4 we get that

$$R_\alpha\big(b^{(1+\gamma)r}\big) \leq NA\big(\mathbb{M}\big(b^{(1+\gamma)r}\big)\big)^{1-1/(r+\gamma r)},$$

where by assumption and Hölder's inequality

$$\begin{aligned} \big(\mathbb{M}\big(b^{(1+\gamma)r}\big)\big)^{1-1/(r+\gamma r)} &= \big[\big(\mathbb{M}\big(b^{(1+\gamma)r}\big)\big)^{1/(r+\gamma r)}\big]^{(1+\gamma)r-1} \\ &\leq N b^{(1+\gamma)r-1} = N b^{r-1+\gamma r}. \end{aligned}$$

Hence,

$$R_\alpha \Big[\Big(R_\alpha\big(b^{(1+\gamma)r}\big)\Big)^{1/(r-1)} \Big] \leq N A^{1/(r-1)} R_\alpha b^{1+\gamma r'}.$$

By Lemma 4

$$R_\alpha b^{1+\gamma r'} \leq NA(\mathbb{M}(b^{1+\gamma r'}))^{1-1/(1+\gamma r')} \leq NA b^{\gamma r'}.$$

This yields (0.4) and proves the lemma in our particular case.

We now get rid of the assumption that $\mathbb{M}(b^p) \leq N b^p$ as in [1]. For $p_0 = (r+p)/2$, $p_1 = (r+p_0)/2$ we have $b^{p_1} \leq (\mathbb{M}(b^{p_0}))^{p_1/p_0} := \tilde{b}^{p_1}$ and since $p_1/p_0 < 1$, $\tilde{b}^{p_1}$ is an A_1-weight with the A_1-constant depending only on p_1/p_0 (see, for instance, [3], p. 158). Therefore, (0.1) holds with $\tilde{b}$ in place of b and it only remains to show that for any x, ρ,

$$\int_{B_\rho(x)} \tilde{b}^{p_1} \, dx \leq N \rho^{d-p_1\alpha} A^{p_1}. \tag{0.5}$$

Of course, we may assume that $x = 0$.

Then by Hölder's inequality we see that the left-hand side of (0.5) is less than

$$N\rho^{d(p-p_1)/p}\Big(\int_{\mathbb{R}^d}(\mathbb{M}(b^{p_0}))^{p/p_0}I_{B_\rho}\,dx\Big)^{p_1/p},$$

where the integral by a Fefferman-Stein Lemma 1, p. 111 of [2] and the fact that $p/p_0>1$ is dominated by

$$N\int_{\mathbb{R}^d}b^p\mathbb{M}I_{B_\rho}\,dx\leq NA^p\rho^{d-p\alpha},$$

where we used Lemma 5. Hence,

$$\int_{B_\rho}\tilde{b}^{p_1}\,dx\leq N\rho^{d(p-p_1)/p}A^{p_1}\rho^{p_1d/p-p_1\alpha}$$

which is (0.5). The theorem is proved. □

References

[1] F. Chiarenza and M. Frasca, *A remark on a paper by C. Fefferman*, Proc. Amer. Math. Soc., Vol. 108 (1990), No. 2, 407–409.

[2] C. Fefferman and E. Stein, *Some maximal inequalities*, Amer. J. Math., Vol. 93 (1971), 107–115.

[3] J. Garcia-Guevra and J.L. Rubio de Francia, "Weighted norm inequalities and related topics", North-Holland Mat. Stud., Vol. 116, 1985.

[4] N.V. Krylov, *On parabolic Adams's, the Chiarenza-Frasca theorems, and some other results related to parabolic Morrey spaces*, Mathematics in Engineering, Vol. 5 (2023), No. 2, Paper No. 038, 20 pp. http://arxiv.org/abs/2110.09555, DOI:10.3934/ mine.2023038

[5] N.V. Krylov, *On strong solutions of time inhomogeneous Itô's equations with Morrey diffusion gradient and drift. A supercritical case*, http://arxiv.org/abs/2211.03719

Email address: nkrylov@umn.edu

127 Vincent Hall, University of Minnesota, Minneapolis, MN, 55455