

# Study guide for the third exam

Math 1241, Fall 2012

The third exam is based on textbook section 3.7, most of chapter 4 (except for section 4.7), sections 5.1-3.3, and Math Insight parts 18-25. Using the book sections as a guide, the following highlights what is and what is not good potential material for the third exam.

## 1. Taylor polynomials (section 3.7)

Be able to compute linear and quadratic approximations of a function around a point.

The linear approximation should match the function and its derivative at the point (i.e., be a tangent line approximation).

The quadratic approximation should match the function, its first derivative, and its second derivative at the point.

We won't test you on secant line approximations.

## 2. Differential equations (section 4.1)

In pure-time differential equations, we are given the rate of change as a function of time (such as a measured rate of change).

In autonomous differential equations, we are given a rule that gives the rate of change as a function of the value of the state variable.

To determine a specific solution to a differential equation, we also need an initial condition.

We can solve a pure-time differential equation graphically. It must increase when the rate of change is positive and decrease when the rate of change is negative.

We can approximate the solution to a pure-time differential equation with Euler's method. Just look up the rate of change at the initial condition, then take a step while moving at that rate change. Look up a new rate of change before taking the next step. Continue in this stepwise fashion.

## 3. Solving pure-time differential equations (section 4.2)

We solve by finding antiderivatives.

The whole family of antiderivatives, with arbitrary constant, is the indefinite integral.

The arbitrary constant can be used to match initial conditions.

By using the power rule, constant product rule, and sum rule, you can solve polynomial differential equations.

## 4. Integration of special functions (section 4.3)

Be able to integrate  $e^x$  and  $1/x$ .

## 5. The chain rule and integration by substitution (section 4.3)

Be able to integrate a function like  $p'(x)e^{p(x)}$  where  $p(x)$  is a polynomial.

We didn't cover the more general algorithm for integration by substitution.

6. Integration by parts and partial fractions (section 4.3)

These topics are not part of this course.

7. Integrals and sums (section 4.4)

The left Riemann sum is the same as Forward Euler. The right Riemann sum is the same as Backward Euler.

For either the Riemann sum perspective or the Euler algorithm perspective, you replace an integral by a sum, where you pretend that the function is constant along each little interval.

8. Definite and indefinite integrals (section 4.5)

The indefinite integral is a function, an antiderivative. One method to solving a pure-time differential equation  $\frac{dF}{dt} = f(t)$  is to calculate the indefinite integral  $\int f(t)dt = F(t) + c$  and compute  $c$  from the initial condition.

The definite integral is a number, defined by a limit of Riemann sums (or Euler's method). One way to solve the pure-time differential equation  $\frac{dF}{dt} = f(t)$  is to use a Riemann sum (Euler's method) to estimate the total change in  $F$  between any two times, as  $t = a$  and  $t = b$ . When the time steps become tiny, the Riemann sum becomes the definite integral  $\int_a^b f(t)dt$ .

The Fundamental Theorem of Calculus says these two methods are the same. The change in  $F(t)$  from  $t = a$  to  $t = b$  using the antiderivative method is  $F(b) - F(a)$ . The change using the definite integral is  $\int_a^b f(t)dt$ . Equating the two is the Fundamental Theorem of Calculus:

$$\int_a^b f(t)dt = F(b) - F(a)$$

where  $F(t)$  is an antiderivative of  $f(t)$ .

9. Applications of integrals (section 4.6)

The area underneath the graph of a positive function  $f(x)$  over the interval  $[a, b]$  is the definite integral  $\int_a^b f(x)dx$ .

The average value of  $f$  over the interval  $[a, b]$  is

$$\frac{1}{b-a} \int_a^b f(x)dx.$$

10. Improper integrals (section 4.7)

This topic are not part of this course.

11. Basic differential equations (section 5.1)

Pure-time and autonomous differential equations.

Two methods to solve autonomous differential equations are guessing and Foward Euler.

12. Equilibria and display of autonomous differential equations (section 5.2)

For the autonomous equation  $\frac{du}{dt} = g(u)$ , the equilibria are the points where the rate of change is zero, i.e., the values of  $u$  where  $g(u) = 0$ . (Here we use  $u$  for the state variable, but of course, the actual letter will be different for different problems.)

From a plot of  $g(u)$ , determine where  $u(t)$  is increasing, decreasing, or constant. Use this information to sketch the solution  $u(t)$ . This is the graphical method to solve autonomous differential equations.

One can summarize where  $u(t)$  is increasing, decreasing, or constant using a phase-line diagram, with arrows (a vector field) showing the direction of movement.

Be able to go between a plot of  $g$  versus  $u$  (where  $u$  is the horizontal axis) and plot of  $u$  versus  $t$  (where  $u$  is the vertical axis). In the phase line,  $u$  is also the horizontal axis.

From the graph of  $g(u)$ , or the vector field on the phase-line, determine the stability of equilibria.

13. Stable and unstable equilibria (section 5.3)

For the autonomous equation  $\frac{dv}{dt} = f(v)$ , we can use the stability theorem to determine stability of the equilibria without looking at the graph of  $f(v)$ .

The stability theorem says for an equilibrium  $v^*$  (with  $f(v^*) = 0$ ), the equilibrium is stable if  $f'(v^*) < 0$  and unstable if  $f'(v^*) > 0$ .