1. **(45 points) You apply Newton's method to a function $f(x)$ starting with the initial guesses $p_0 = 0$ and $p_0 = 5$ and obtain the below sequences $\{p_n\}$. (The sequences are converging toward the roots $p = 1$ and $p = 4$, respectively).**

   (i) (20 points) What is the order of convergence of the sequence with $p_0 = 0$?

   (ii) (10 points) What can you conclude about the root of $f(x)$ at $p = 1$?

   (iii) (15 points) What is different about the root at $p = 4$? Why can you conclude that?

n	p_n
0	**0.00000000**
1	**0.30769231**
2	**0.52488688**
3	**0.67635507**
4	**0.78084509**
5	**0.85227576**

n	p_n
0	**5.00000000**
1	**4.42857143**
2	**4.11688312**
3	**4.01181964**
4	**4.00013754**
5	**4.00000002**

(45 points) (i) For sequence with $p_0 = 0$, since know $p = 1$, can calculate $e_n = p - p_n$ and test for linear convergence by looking to see if $|e_n|/|e_{n-1}|$ approach a constant

n	e_n	$\lvert e_n\rvert/\lvert e_{n-1}\rvert$
0	1	
1	0.6923	0.6923
2	0.4751	0.6863
3	0.3236	0.6823
4	0.2192	0.6771
5	0.1337	0.6284

It appears that $|e_n|/|e_{n-1}|$ approaches a constant, so we can conclude we have linear order of convergence with asymptotic error constant around 0.6 or 0.7.

(ii) Since Newton's method did not have quadratic convergence, we know that the root had multiplicity greater than 1. In fact, we know that if the multiplicity was m, the asymptotic error constant should be $1 - 1/m$. If $m = 3$, then this expression is $2/3$, which is close to the observed constant. So, if we really wanted to be precise, we could conclude that the function had a root of multiplicity $m = 3$ at $p = 1$.

(iii) The second sequence converges much more quickly. We suspect it could be quadratic convergence. Check $|e_n|/|e_{n-1}|$ and $|e_n|/|e_{n-1}|^2$ to see

n	e_n	$\lvert e_n\rvert/\lvert e_{n-1}\rvert$	$\lvert e_n\rvert/\lvert e_{n-1}\rvert^2$
0	1		
1	0.4286	0.4286	0.4286
2	0.1169	0.2727	0.6364
3	0.0118	0.1011	0.8652
4	0.0001	0.0116	0.9845
5	00000002	0.0001	1.0572

$|e_n|/|e_{n-1}|$ shrinks to zero but $|e_n|/|e_{n-1}|^2$ seems to approach a constant. Therefore, we conclude we have quadratic convergence. The function has a simple root at $p = 4$.

2. **(15 points) Consider solving a linear system $A\mathbf{x} = \mathbf{b}$ for the vector $\mathbf{x}$.**
   - **(i) (10 points) Why does one in general need to employ a pivoting strategy with Gaussian elimination? (Hint: What two things could go wrong without pivoting?)**
   - **(ii) (5 points) For which special classes of matrices A can one safely compute Gaussian elimination without pivoting?**

(15 points) (i) One reason to employ a pivoting strategy is to avoid having a zero pivot and dividing by zero. Another reason is to avoid large roundoff error for the case when a pivot is very small. (If the matrix is ill-conditioned, you still will have a problem even with pivoting.)

(ii) Strictly diagonally dominant and symmetric positive definite matrices do not require a pivoting strategy. (You might still need a pivoting strategy with tridiagonal systems, such as the simple example when the upper left entry is zero.)

3. **(40 points) For the below system of equations, set up the first iteration of Newton's method to calculate $\mathbf{x}^{(1)}$ from the initial guess $\mathbf{x}^{(0)} = [\pi \;\; 0]^T$. You need not solve the resulting linear system, but set up the linear system and explain how you would calculate $\mathbf{x}^{(1)}$ from the solution of the linear system.**

$$5\cos x + 6\cos(x+y) - 10 = 0$$
$$5\sin x + 6\sin(x+y) - 4 = 0$$

(40 points) The first iteration of Newton's method is

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} - [J(\mathbf{x}^{(0)})]^{-1}\mathbf{F}(\mathbf{x}^{(0)})$$

where

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} 5\cos x + 6\cos(x+y) - 10 \\ 5\sin x + 6\sin(x+y) - 4 \end{bmatrix}$$

so that

$$\mathbf{F}(\mathbf{x}^{(0)}) = \begin{bmatrix} 5\cos\pi + 6\cos\pi - 10 \\ 5\sin\pi + 6\sin\pi - 4 \end{bmatrix} = \begin{bmatrix} -21 \\ -4 \end{bmatrix}$$

The Jacobian is

$$J(\mathbf{x}) = \begin{bmatrix} -5\sin x - 6\sin(x+y) & -6\sin(x+y) \\ 5\cos x + 6\cos(x+y) & 6\cos(x+y) \end{bmatrix}$$

so that

$$J(\mathbf{x}^{(0)}) = \begin{bmatrix} 0 & 0 \\ -11 & -6 \end{bmatrix}$$

Rather than calculate J^{-1}, we solve the linear system

$$\begin{bmatrix} 0 & 0 \\ -11 & -6 \end{bmatrix}\mathbf{v}^{(0)} = -\begin{bmatrix} -21 \\ -4 \end{bmatrix} = \begin{bmatrix} 21 \\ 4 \end{bmatrix}$$

for $\mathbf{v}^{(0)}$ and let $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \mathbf{v}^{(0)}$.

Unfortunately $J(\mathbf{x}^{(0)})$ is singular so you can't actually do this. (My mistake; bad choice of initial guess $\mathbf{x}^{(0)}$.) I think only one or two people caught that, though.

4. **(30 points) Let A be the matrix**

$$A = \begin{bmatrix} 9 & 2 & 0 & 1 \\ 2 & 0 & -1 & 0 \\ 2 & 0 & -8 & -1 \\ 2 & 0 & 2 & 4 \end{bmatrix}$$

(i) (10 points) **Sketch the Gerschgorin circles for A.**

The Gerschgorin circles were

$$\begin{aligned} C_1 &= \{z \in \mathbb{C} : |z - 9| \leq 3\} \\ C_2 &= \{z \in \mathbb{C} : |z - 0| \leq 3\} \\ C_3 &= \{z \in \mathbb{C} : |z + 8| \leq 3\} \\ C_4 &= \{z \in \mathbb{C} : |z - 4| \leq 4\} \end{aligned}$$

(ii) (10 points) **Set up the inverse power method for approximating the one eigenvalue of A you can be certain is real and its associated eigenvector. The main point is to determine the matrix B to apply the power method to (though you don't compute B itself).**

In the isolated circle C_3 there must be a real eigenvalue in the interval $[-11, -5]$. To find it, we can set $q = -8$ to the middle of that interval and use the inverse power method. Let

$$B = (A + 8I)^{-1} = \begin{bmatrix} 17 & 2 & 0 & 1 \\ 2 & 8 & -1 & 0 \\ 2 & 0 & 0 & -1 \\ 2 & 0 & 2 & 12 \end{bmatrix}$$

and use the power method on B, setting $\mathbf{y}^{(m)} = B\mathbf{x}^{(m-1)}$, etc. (Solve linear system rather than computing inverse and multiplying, of course.)

(iii) (10 points) **Imagine that the inverse power method resulted in the sequence $\{\lambda^{(m)}\}$ converging to an eigenvalue of B and the sequence $\{\mathbf{x}^{(m)}\}$ converging to an eigenvector of B. If you stopped the iteration at $m = k$, what would be your estimate of the eigenvalue and eigenvector of A?**

The estimate of the eigenvector of A is simply $\mathbf{x}^{(k)}$ since it has the same eigenvectors as B. The estimate of the eigenvalue of A is $1/\lambda^{(k)} + q = 1/\lambda^{(k)} - 8$.

5. **(40 points) In running for student body president, if you spend x hours a day campaigning, your probability of winning the election is $x/10 + \sin(x)/4$, as shown in the below plot. (This formula is valid only for x less than 6.) Starting with the initial interval $(a, b) = (0, 6)$, use the method of false position to produce a sequence $\{p_n\}$ that approximates the time p you should spend campaigning so that your probability of winning is 0.4. You want your approximation to be accurate to within 0.1 hours, so use a stopping condition that your estimate of the error $|p_n - p|$ is less than the convergence tolerance $\epsilon = 0.1$. Be sure to show how you**

calculated the sequence $\{p_n\}$ and how you determined the stopping condition. Has your method determined an efficient campaign strategy?

(40 points) Let $f(x) = x/10 + \sin(x)/4 - 0.4$ so that need to solve for a root of f. Formula for false position is

$$p_n = b_n - f(b_n)\frac{b_n - a_n}{f(b_n) - f(a_n)}$$

Starting with $a_0 = 0$ and $b_0 = 6$, calculate $f(a_0) = -0.4$ and $f(b_0) = 0.1301$ so that $p_0 = 4.5271$.

To determine if a_1 or b_1 should be set to p_0, calculate $f(p_0) = -0.1930$. Since it is opposite sign of $f(b_1)$, set $a_1 = p_0$ and $b_1 = b_0$.

Then, $p_1 = 5.4068$, $f(p_1) = -0.0514$ so that $a_2 = p_1 = -0.0514$ and $b_2 = b_1 = 6$. Lastly, calculate $p_2 = 5.5748$.

After three points, can check stopping condition. Since false position is a first order method, we can estimate the error by

$$\left|\frac{\lambda}{\lambda - 1}\right| |p_n - p_{n-1}| \qquad \text{where} \qquad \lambda = \frac{p_n - p_{n-1}}{p_{n-1} - p_{n-2}}$$

For $n = 2$, calculate $\lambda = 0.1910$ and error estimate is 0.0397. Since $0.0397 < \epsilon = 0.1$, we can stop after these three iterations and use $p_2 = 5.5748$ as our estimate of the number of hours.

Clearly, this is not an efficient campaign strategy, as there are smaller values of x where $f(x) = 0$.

6. **(30 points) You are using an iterative method to solve for the vector $\mathbf{x}$ that satisfies $A\mathbf{x} = \mathbf{b}$ for given matrix A and vector $\mathbf{b}$. At each step m, the iterative method gives you the estimated answer $\mathbf{x}^{(m)}$. Your plan is to calculate the residual $\mathbf{r}^{(m)} = A\mathbf{x}^{(m)} - \mathbf{b}$ in order to estimate the size of the error $\mathbf{e} = \mathbf{x}^{(m)} - \mathbf{x}$. If the magnitude of $\mathbf{b}$ is $\|\mathbf{b}\|_\infty = 1$ and the condition number of A is $\kappa_\infty(A) = 1000$, how small must the residual $\mathbf{r}^{(m)}$ be to guarantee that the relative error $\|\mathbf{e}\|_\infty/\|\mathbf{x}\|_\infty$ is less than 0.1?**

(30 points) The relationship between condition number and relative error is

$$\frac{1}{\kappa_\infty(A)}\frac{\|\mathbf{r}\|_\infty}{\|\mathbf{b}\|_\infty} \leq \frac{\|\mathbf{e}\|_\infty}{\|\mathbf{x}\|_\infty} \leq \kappa_\infty(A)\frac{\|\mathbf{r}\|_\infty}{\|\mathbf{b}\|_\infty}$$

In this case, we only care about bounding the error from above, so we just care about the second inequality. If the residual is small enough to satisfy

$$\kappa_\infty(A)\frac{\|\mathbf{r}\|_\infty}{\|\mathbf{b}\|_\infty} < 0.1$$

then the relative error $\|\mathbf{e}\|_\infty/\|\mathbf{x}\|_\infty$ is also less than 0.1. In other words, we need

$$\|\mathbf{r}\|_\infty < 0.1\frac{\|\mathbf{b}\|_\infty}{\kappa_\infty(A)} = \frac{0.1}{1000} = 0.0001$$