References

1. H. Aramata. Über die Teilbarkeit der Dekindschen Zetafunktionen. *Proc. Japan. Acad.*, 9:31–34, 1933.
2. R. Brauer. On the zeta-function of algebraic number fields. *Amer. J. Math.*, 69:243–250, 1947.
3. L. Goldstein. Relatively imaginary quadratic fields of class number 1 or 2. *Trans. Amer. Math. Soc.*, 165:353–364, 1972.
4. T. H. Gronwall. Sur les séries de Dirichlet correspondant à des caractères complexes. *Rend. Circ. Palermo*, 35:145–159, 1913.
5. B. Gross and D. Zagier. Heegner points and derivatives of L-functions. *Inventiones Math.*, 84:225–320, 1986.
6. H. Heilbronn. On real simple zeros of Dedekind ζ-functions. *Can. J. Math.*, 25:870–873, 1973.
7. J. Hoffstein and N. Jochnowitz. Artin's conjecture and the class number of certain CM fields. *Duke Math. J.*, 59:553–563, 1989.
8. J. Hoffstein and N. Jochnowitz. Artin's conjecture and the class number of certain CM fields-II. *Duke Math. J.*, 59:565–584, 1989.
9. C. Hooley. On Artin's conjecture. *J. reine angew. Math.*, 225:209–220, 1969.
10. C. Hooley. *Applications of Sieve Methods.* Cambridge University Press, 1976.
11. B. Huppert, *Endliche Gruppen I*, Springer, 1967.
12. I. M. Isaacs, *Character Theory of Finite Groups*, Academic Press, 1976.
13. I. M. Isaacs and D. S. Passman, A characterization of groups in terms of the degrees of their characters. II, *Pacific J. Math.*, 24:467–510, 1968.
14. J. C. Lagarias, H. L. Montgomery, and A. M. Odlyzko. A bound for the least prime ideal in the Chebotarev density theorem. *Inventiones math.*, 54:271–296, 1979.
15. E. Landau. Über die Klassenzahl imaginärquadratischer Zahlkörper. *Nachr. Ges. Wiss. Göttingen*, pages 265–296, 1918.
16. W. Narkiewicz. *Elementary and Analytic Theory of Algebraic Numbers.* Polish Scientific Publishers, 1973.
17. J. P. Serre. *Linear Representations of Finite Groups.* Springer-Verlag, 1977.
18. H. M. Stark. Some effective cases of the Brauer-Siegel theorem. *Inventiones math.*, 23:135–152, 1974.
19. J. S. Sunley. Class numbers of totally imaginary quadratic extensions of totally real fields. *Trans. Amer. Math. Soc.*, 175:209–232, 1973.
20. N. Tschebotaröw. *Grundzüge der Galois'schen Theorie.* Math. Inst. Univ. Amsterdam., 1950.
21. K. Uchida. On Artin L-functions. *Tohoku Math. J.*, 27:75–81, 1975.
22. I. M. Vinogradov. Artin L-series and his conjectures. *Proc. Steklov Inst. Math.*, 112:124–142, 1971.
23. R. W. van der Waall. On a conjecture of Dedekind on zeta-functions. *Indagationes Math.*, 37:83–86, 1975.
24. R. W. van der Waall. Holomorphy of quotients of zeta-functions, pp. 649–662 in *Algebraic Number Fields*, A. Fröhlich, ed., Academic Press, 1977.

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Proof: It suffices to prove that $\zeta_{F_i}(s)/\zeta_{F_{i-1}}(s)$ is analytic when F_i/F_{i-1} is either normal or an extension of the type considered in Theorem 3, for then we may "build up" in much the same way as Theorem 3 was proved. If F_i/F_{i-1} is normal, then the fundamental result of Aramata [1] tells us that $\zeta_{F_i}(s)/\zeta_{F_{i-1}}(s)$ is analytic. In the other cases, though, this follows from Proposition 1.

□

We also remark that our method sometimes yields stronger results than those given by the statements of our theorems. For example, if $L = Q[2^{1/p}]$, p an odd prime, and $K = Q$, Theorem 1 shows that $\zeta_L(s)$ has no Siegel zeros. If we let $M = L[\exp(2\pi i/p)]$ be the normal closure of L, then $\zeta_M(s)$ has no Siegel zeros by Heilbronn's result [6], say, and easy estimates of discriminants. This means that $\zeta_M(s)$ has no zeros within $(c_5 p^2 \log p)^{-1}$ of 1. Since we showed that $\zeta_L(s)$ has no Siegel zeros, it follows that $\zeta_M(s)$ has no zeros within $(c_6 p \log p)^{-1}$ of 1.

The proof of Theorem 3 relied on all the nonlinear irreducible characters of the group G that satisfy the conditions of case (ii) having the same degree f. I. M. Isaacs has pointed out that if f and $|G|/f$ are relatively prime, then it follows from [13] that G can be constructed as a semidirect product of a cyclic group H' acting regularly on an abelian group A'. In this case H' is necessarily a Frobenius complement in G. Thus our argument cannot be readily extended to other classes of groups. The basic approach can be used even if the degrees of the nonlinear irreducible characters of G differ, but we have not found any nice generalization.

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which is a contradiction. Hence β must be a zero of $\zeta_K(s)$. That it is a simple zero follows from $d_K \le d_L$. This establishes Theorem 3.

□

We next show how Theorem 1 can be derived from Theorem 2. Since any radical extension can be obtained as a tower of radical extensions of prime degrees, it suffices to prove Theorem 1 when all the degrees $[F_i : F_{i-1}]$ are odd primes. When $[F_i : F_{i-1}] = p$ is prime, there are no intermediate fields F_i', $F_{i-1} \subset F_i' \subset F_i$. Let $F_i = F_{i-1}[a^{1/p}]$, $a \in F_{i-1}$. Then it is easy to show (and well-known, cf. [20, 22]) that if F_i^* is the normal closure of F_i over F_{i-1}, then $\mathrm{Gal}(F_i^*/F_{i-1}) = A \cdot H$, where A is isomorphic to $\mathbb{Z}_p$ and H is isomorphic to a subgroup of $\mathbb{Z}_p^*$. Since $[F_i : F_{i-1}]$ is odd, Theorem 2 applies.

3. Concluding Remarks

We make some final remarks regarding Theorem 2 and some of its implications. First, one immediately notices that Theorem 2 and Heilbronn's results [6] can be combined to get

Theorem 4. Let $F_0 \subset F_1 \subset \cdots \subset F_m$ be a tower of extensions such that $[F_i : F_{i-1}]$ is odd and F_i/F_{i-1} is either normal or of the form of Theorem 2. Then any real zero of $\zeta_{F_m}(s)$ in the range
$$1 - \frac{c_0}{\log d_{F_m}} \le \beta < 1$$
is a simple real zero of $\zeta_{F_0}(s)$.

Also, there is the following application.

Corollary 2. Let $\mathbb{Q} = F_0 \subset F_1 \subset \cdots \subset F_m$ be a tower of fields as in Theorem 4. Let κ_m denote the residue of $\zeta_{F_m}(s)$ at $s = 1$. Then
$$\kappa_m > \frac{c_4}{\log d_{F_m}}$$
where c_4 is an effectively computable constant depending only on c_0.

Proof: This follows from Lemma 4 of [18] and Theorem 4.

□

We also make an observation about the analyticity of the ratios of the $\zeta_{F_i}(s)$. The result below follows already from the theorem of Uchida [21] and van der Waall [23, 24], which states that if M is a normal extension of a number field K with a solvable Galois group, then the quotient $\zeta_L(s)/\zeta_K(s)$ is entire for any number field L intermediate between K and M.

Theorem 5. Let $F_0 \subset F_1 \subset \cdots \subset F_m$ be a tower of fields as in Theorem 4 (or Theorem 2). Then, if $i > j$
$$\frac{\zeta_{F_i}(s)}{\zeta_{F_j}(s)} \quad \textit{is entire.}$$

and therefore

$$\sum_i s_i \chi_i(1) = |A| \ .$$

Next,

$$1 = (\chi_0\big|_H, \phi_0) = (\chi_0, \phi_0^*) \ ,$$

and so by (10) and (11)

$$\phi_0^*(1) = \chi_0(1) + fk_2 + \sum_{i \in X} s_i = |A| = 1 + fk_2 \ , \tag{12}$$

where X is the set of i such that χ_i is a linear character of G, $i \neq 0$. Since $s_i \geq 0$ for all i, (12) is possible only if $s_i = 0$ for all $i \in X$, which proves (B).

The claims (A) and (B), which were proved above, lead to the following result about zeta functions.

Proposition 1 *Under the conditions of case (ii) of Theorem* 3*,*

$$\zeta_L(s) = \zeta_K(s) L(s) \ , \tag{13}$$

$$\zeta_M(s) = \zeta_K(s) L_1(s) L(s)^f \ , \tag{14}$$

where $f = |H|$ and the functions $L_1(s)$ and $L(s)$ are entire.

Proof: Let Y be the set of nonlinear characters of G. We have

$$\zeta_L(s) = L(s, \phi_0) = L(s, \phi_0^*) = L(s, \chi_0) \prod_{\chi_i \in Y} L(s, \chi_i) \ .$$

If we let

$$L(s) = \prod_{\chi_i \in Y} L(s, \chi_i) \ ,$$

we obtain (13). By Lemma 4, each χ_i is induced by a linear character λ_j of A, so each $L(s, \chi_i) = L(s, \lambda_j)$ is entire, and so is $L(s)$.

Next,

$$\zeta_M(s) = \prod_{\chi_i} L(s, \chi_i)^{\chi_i(1)} = \zeta_K(s) L_1(s) L(s)^f \ ,$$

where $L_1(s)$ is the product of $L(s, \chi_i)$ over the nonprincipal linear irreducible characters χ_i of G. Since all these L-functions are abelian, they are entire, and so is $L_1(s)$.

□

Finally, we return to our original investigation of the possible real zeros of $\zeta_L(s)$. Suppose β is a real zero of $\zeta_L(s)$ in the range (2). If β is not a zero of $\zeta_K(s)$, then it must be a zero of $L(s)$, and so by Proposition 1, β is a zero of $\zeta_M(s)$ of multiplicity at least f. By Lemma 3,

$$\log d_M \leq 2f \log d_L .$$

Choose $c_0 = (4c_1)^{-1}$. Then for $r = c_0/(\log d_L)$, Lemma 1 applied to $\zeta_M(s)$ gives

$$f \leq n(r) < 1 + c_1 r \log d_M \leq 1 + \frac{f}{2} \leq f \ ,$$

(B) The only linear character χ_i of G with $s_i \neq 0$ is the identity character χ_0.

To prove (A), note that if $a \in A$, $h, h_1 \in H$, then since H is abelian (as we are in case (ii) of Theorem 3),

$$(ah_1)h(ah_1)^{-1} = aha^{-1} .$$

If $aha^{-1} = a'$ for some $a' \in A$, then $h = a^{-1}a'a \in A$, and so by Lemma 2, $h = 1$. Hence for $h \in H$,

$$\begin{aligned} \lambda_j^*(h) &= \frac{1}{|A|}\sum_{x\in G}\lambda_j(xhx^{-1}) = \frac{1}{|A|}\sum_{a\in A}\sum_{h_1\in H}\lambda_j((ah_1)h(ah_1)^{-1}) \\ &= \frac{|H|}{|A|}\sum_{a\in A}\lambda_j(aha^{-1}) = \begin{cases} |H| & \text{if } h = 1 , \\ 0 & \text{otherwise} , \end{cases} \end{aligned}$$

since $G = AH$ by Lemma 2. Therefore

$$\lambda_j^*\big|_H = \sum_i \phi_i . \tag{8}$$

Next, suppose χ_i is some nonlinear irreducible character of G. Then, by Lemma 4, $\chi_i = \lambda_j^*$ for some j, and

$$s_i = (\phi_0^*, \chi_i) = (\phi_0^*, \lambda_j^*) = (\phi_0, \lambda_j^*\big|_H) = 1 ,$$

and this proves (A).

To prove (B), note that since

$$|G| = \sum_i \chi_i(1)^2 ,$$

and $|G| = |A| \cdot f$, if we let k_1 be the number of linear characters of G and k_2 the number of nonlinear characters, then by Lemma 4,

$$|A|f = k_1 + f^2k_2 . \tag{9}$$

Now in general $k_1 = |G|/|C|$, where C is the commutator subgroup. In our case $C = A$, so $k_1 = |G|/|A| = f$, and so (9) yields

$$|A| = 1 + fk_2 . \tag{10}$$

Since A is normal, and $H \cap A = \{1\}$ by Lemma 2, we find that for $a \in A$,

$$\phi_0^*(a) = \frac{1}{|H|}\sum_{x\in G}\phi_0(xax^{-1}) = \begin{cases} |A| & \text{if } a = 1 , \\ 0 & \text{otherwise} . \end{cases}$$

Therefore

$$\phi_0^*\Big|_A = \sum_i \lambda_i .$$

Hence

$$\phi_0^*(1) = \sum_i \lambda_i(1) = |A| , \tag{11}$$

G, and so the fixed field of H_1 is a normal extension of K that lies between L and M. Since M is the minimal normal extension of L over K, we must have $H_1 = \{1\}$, which means that $h_1 = 1$, and this contradicts the basic assumption. Thus we must have $H \cap (gHg^{-1}) = \{1\}$ for all $g \in G \setminus H$.

□

Lemma 3. Under the conditions above,

$$d_M | d_L^{2f} \ , \tag{4}$$

where $f = |H|$.

Proof: Let $L' \neq L$ be a conjugate of L over K. We will prove below that $M = L \cdot L'$. Once this is established, we apply Lemma 7 of [18] and obtain

$$d_M | d_L^f d_{L'}^f \ ,$$

which yields (4).

It remains to show that $M = L \cdot L'$. The field L' is fixed by a conjugate of H, say gHg^{-1}, where $g \in G \setminus H$. By Lemma 2, $H \cap (gHg^{-1}) = \{1\}$. If $L \cdot L'$ were a proper subgroup of G, it would be fixed by a nontrivial subgroup of G, which would also fix L and L'. This would contradict $H \cap (gHg^{-1}) = \{1\}$.

□

We next investigate the characters of G. If η is a character of a subgroup of G, we let η^* denote the character of G induced by η.

Lemma 4. Under the above conditions on the group G, the nonlinear irreducible characters of G are of the form λ^*, where λ denotes an irreducible character of A, and are of degree $f = |H|$.

Proof: See p. 199 of [12].

□

Let $\{\chi_i\}$, $\{\phi_i\}$, and $\{\lambda_i\}$ be the sets of irreducible characters of G, H, and A, respectively. Let χ_0, ϕ_0, and λ_0 be the identity characters of G, H, and A. We then have

$$\zeta_M(s) = \prod_i L(s, \chi_i)^{\chi_i(1)} \tag{5}$$

and

$$\zeta_L(s) = L(s, \phi_o^*) \ , \tag{6}$$

where $L(s, \eta)$ is the Artin L-function of η. Let

$$\phi_0^* = \sum_i s_i \chi_i \ , \tag{7}$$

where the s_i are integers, $s_i \geq 0$. We will show that

(A) Every irreducible nonlinear character χ_i of G appears in (7) with $s_i = 1$.

Theorem 3. There is an effectively computable constant $c_0 > 0$ with the following property. Let $L \supset K$ be number fields with no intermediate field (i.e., no field L' such that $L \supset L' \supset K$). Let M be the normal closure of L over K, and assume that $G = \mathrm{Gal}(M/K)$ is solvable. Let A be the minimal normal subgroup of G with $|A| > 1$, and let H be the subgroup of G fixing L. If either (i) $G = A$ and $|G|$ is odd or (ii) H is abelian with $|H| > 1$, then any real zero of $\zeta_L(s)$ in the range

$$1 - \frac{c_0}{\log d_L} \leq \beta < 1 \tag{2}$$

is a simple zero of $\zeta_K(s)$.

Theorem 2 follows from repeated application of Theorem 3 to each stage of the tower of extensions.

From now on we concentrate on proving Theorem 3. We first note that if case (i) of Theorem 3 holds, then the conclusion of the theorem follows from Heilbronn's results [6], which say that if L/K is a normal extension of number fields, and K' is the compositum of quadratic subfields of L/K, then any real simple zero of $\zeta_L(s)$ is a zero of $\zeta_{K'}(s)$. Therefore from now on we assume that we are in case (ii).

Lemma 2. Under the above conditions, (i) $G = AH$, (ii) $A \cap H = \{1\}$, (iii) A is elementary abelian, and (iv) $H \cap (gHg^{-1}) = \{1\}$ for any $g \in G \setminus H$ (so that H is a Frobenius complement in G).

Proof: (i) Since L/K has no proper subfields, H must be a maximal subgroup of G. Therefore $AH = G$ or H. If $AH = H$, then $A \subseteq H$. Let $M' \subset M$ be the subfield fixed by A. Then $L \subseteq M'$ and M' is normal over K, contrary to the choice of M. This is a contradiction, so $AH = G$.

(ii) $A \cap H$ is normal in $AH = G$, so $A \cap H = \{1\}$ as A is minimal.

(iii) See Satz 9.13 on p. 52 of [11].

(iv) Suppose this claim is not true, and that $h_1 = gh_2g^{-1}$ for some $h_1, h_2 \in H$, $g \in G$, $h_1 \neq 1$. We can write $g = ah$, $a \in A$, $h \in H$. We consider a and h_1 to be fixed from now on. Since H is abelian,

$$h_1 = gh_2g^{-1} = ah_2a^{-1} .$$

Then

$$x = ah_2a^{-1}h_2^{-1} = h_1h_2^{-1} . \tag{3}$$

Since A is normal, $h_2a^{-1}h_2^{-1} \in A$, so $x = ah_2a^{-1}h_2^{-1} \in A$. But by (3), $x = h_1h_2^{-1} \in H$, and therefore, by part (ii) of this lemma, $x = 1$ and $h_1 = h_2$.

Let $H' = aHa^{-1}$, and let F be the subgroup of G generated by $\{hah^{-1} : h \in H\}$. F is a subgroup of A, since A is normal, and is fixed under conjugation by H. Since A is abelian, this means that F is a normal subgroup of G, and therefore $F = A$ or $\{1\}$. But $a \in F$, $a \neq 1$, so we must have $F = A$.

Since $F = A$, any element of A can be written as a product of elements of the form $h_3ah_3^{-1}$, $h_3 \in H$. But H is abelian, and h_1 commutes with a, so h_1 commutes with every element of A. Therefore the subgroup H_1 generated by h_1 is normal in near $s = 1$, more zeros than allowed by known bounds. This argument can be generalized to classes of fields other than those we consider, but its applicability is limited by the requirement that certain relations hold for the representations of the Galois group of the field being considered. What is essential for our method of proof is that in each stage of the tower of extensions, the normal closure M of an extension L of K is the compositum of only a few of the fields conjugate to L, so d_M is not large, and that $\zeta_M(s)/\zeta_K(s)$ is the product of a high power of $\zeta_L(s)/\zeta_K(s)$ and other well-behaved functions. The group representation arguments we use are similar to those employed by Uchida [21] and van der Waall [23, 24] in their proofs that if M is a normal extension of a number field K with a solvable Galois group, then the quotient $\zeta_L(s)/\zeta_K(s)$ is entire for any number field L with $K \subseteq L \subseteq M$. It would be desirable to prove a strengthening of Theorem 2, in which the only requirement would be that the groups $\mathrm{Gal}(F_i^*/F_{i-1})$ have to be solvable. Any such strengthening would have to allow for the possibility of a Siegel zero coming from a quadratic subfield. As we remark at the end of Section 3, there are indications that our arguments cannot be extended too far.

We prove Theorems 1 and 2 in Section 2. We conclude in Section 3 with some observations about the residue of $\zeta_{F_i}(s)$ at $s = 1$ and the analyticity of the functions $\zeta_{F_i}(s)/\zeta_{F_j}(s)$ $(i > j)$.

2. Proof of Theorem 2 and derivation of Theorem 1 from Theorem 2

In this section, we prove some results that imply Theorem 2. The first lemma presents an estimate of the number of zeros of $\zeta_K(s)$ in a region near 1 that is valid for all number fields K.

Lemma 1. Let $n(r)$ denote the number of zeros ρ of $\zeta_K(s)$ with $|1-\rho| \le r$. Then for all $r > 0$,

$$n(r) < 1 + c_1 r \log d_K \tag{1}$$

where c_1 is an effectively computable constant that is independent of K.

Proof: For $r \le (4 \log d_K)^{-1}$ the previously mentioned result of Stark implies the truth of (1). Suppose that $r > (4 \log d_K)^{-1}$. By Lemma 2.2 of [14], we know that

$$n(r) < c_2(1 + r \log d_K)$$

for some effectively computable constant c_2. (We have used $n_K < c_3 \log d_K$.) Put $c_1 = 5c_2$. Then

$$c_2(\frac{1}{r \log d_K} + 1) < 5c_2 = c_1.$$

Hence,

$$n(r) < c_2(1 + r \log d_K) \le 1 + c_1 r \log d_K.$$

□

Theorem 2 follows from the following result.

extensions of number fields. For radical extensions, their results give little information.

The restriction to odd degree extensions in Theorem 1 is necessary to exclude the possibility of a Siegel zero of $\zeta_{F_m}(s)$ coming from $\zeta_E(s)$ where E is a quadratic extension of some F_i. In the study of Siegel zeros, quadratic extensions are the hardest to deal with. As far as anyone knows, a quadratic number field K can have a Siegel zero within about $d_K^{-1/2}$ of 1. A standard result (cf. Lemma 11 of [18]) shows that the hypothetical Siegel zero is $< 1 - c'' d_K^{-1/2}$ for some $c'' > 0$, and the recent work of Gross and Zagier [5] allows one to increase the distance from 1 only by a power of $\log d_K$.

Theorem 1 is a consequence of the following more general result.

Theorem 2. There is an effectively computable constant $c_0 > 0$ with the following property. Let $F_0 \subset F \subset \cdots \subset F_m$ be a tower of extensions of number fields. Assume that for each i, $1 \leq i \leq m$, there is no number field F_i' such that $F_{i-1} \subset F_i' \subset F_i$. Let F_i^* be the normal closure of F_i over F_{i-1}, and assume that the Galois group $G_i = \mathrm{Gal}(F_i^*/F_{i-1})$ is solvable. Let A_i be the minimal normal subgroup of G_i with $|A_i| > 1$, and let H_i be the subgroup fixing F_i. If for each i, $1 \leq i \leq m$, either (i) $G_i = A_i$ and $|G_i|$ is odd, or (ii) H_i is abelian with $|H_i| > 1$, then any real zero β of $\zeta_{F_m}(s)$ in the range

$$1 - \frac{c_0}{\log d_{F_m}} \leq \beta < 1$$

is a simple zero of $\zeta_{F_0}(s)$.

The derivation of Theorem 1 from Theorem 2 will be sketched in Section 2. At this point we note that Theorem 2 is more general, as it shows, for example, that if L and K are number fields with L cubic over K, then any Siegel zero of $\zeta_L(s)$ is a Siegel zero of $\zeta_K(s)$.

We note here that we define a Siegel zero with respect to the field, and not to any of the L-functions it is a zero of. Thus, for example, if β is a Siegel zero of the ζ-function of the field $K = \mathbb{Q}[\sqrt{p}]$ for a prime p, then it is also a zero of the ζ-function of the field $L = \mathbb{Q}[\epsilon_p]$, where ϵ_p is a primitive pth root of unity. However, β is not a Siegel zero of $\zeta_L(s)$ since $\beta < 1 - c'' p^{-1/2}$, whereas $\log d_L$ is of order $p \log p$.

Zeros of zeta functions of radical extensions are especially interesting because of their connection with Artin's conjecture that any integer a not equal to $-1, 0$, or a perfect square is a primitive root for infinitely many primes (and even for a positive fraction of all primes). Hooley [9], [10] proved this conjecture under the assumption of the Generalized Riemann Hypothesis. Unfortunately, our results do not help to obtain an unconditional proof, because a much wider zero-free region appears to be needed (cf. [10]), and in any event, real zeros of $\zeta_K(s)$ for $K = \mathbb{Q}[\sqrt[q]{a}]$ would only help in bounding the error terms that arise in the proof.

The results of Sunley, Goldstein, Heilbronn, and Stark were proved by showing that a Siegel zero of $\zeta_K(s)$ had to come from $\zeta_F(s)$ for some subfield F of K, usually a quadratic subfield. Our proof relies on showing that for some normal extension L of K, a Siegel zero of $\zeta_K(s)$ would imply the existence of many zeros of $\zeta_L(s)$

only on c. The Brauer-Siegel Theorem [2] yields a lower bound for κ as K runs over certain sequences of normal extensions of $\mathbb{Q}$ by using the argument of Siegel that few fields can have Siegel zeros. However, the Brauer-Siegel result is ineffective.

Siegel zeros also play a role in determining the distribution of prime ideals. For example, the asymptotic size of the error in the prime ideal theorem, which counts prime ideals with norm $\leq x$, is determined by how close the zeros of $\zeta_K(s)$ come to the line $\mathrm{Re}(s) = 1$, with the influence of any single zero being negligible for x large. However, if there is a Siegel zero, x has to be extremely large before the influence of that zero becomes small. Therefore in statements of the prime ideal theorem that give explicit dependence on the parameters of the field, the influence of a possible Siegel zero is usually included separately from that of other zeros.

The best general results to date about Siegel zeros and the concomitant bounds for residues are due to Stark [18], who significantly generalized and strengthened earlier results of Sunley [19], Goldstein [3], and Heilbronn [6]. Other bounds on Siegel zeros of certain number fields have been obtained recently by Hoffstein and Jochnowitz [7], [8].

In what follows, we redefine a Siegel zero by altering the constant c. This is done to ensure that certain technical arguments go through. This paper is concerned with showing that for an appropriate choice of c, $\zeta_K(s)$ often has no Siegel zeros. We have not worried about getting the best possible constants, and it may be possible to prove that the results below hold with $c_0 = \frac{1}{4}$ or an even larger constant.

Recall that a radical extension of a number field K is a field $L = K[\alpha]$ such that $\alpha^n \in K$ for some integer n. We prove the following theorem about Siegel zeros of radical extensions.

Theorem 1. There is an effectively computable constant $c_0 > 0$ with the following property. Let $F_0 \subset F_1 \subset \cdots \subset F_m$ be a tower of extensions of number fields such that $[F_i : F_{i-1}]$ is odd and F_i is a radical extension of F_{i-1} for $i = 1, \ldots, m$. If β is a simple real zero of $\zeta_{F_m}(s)$ in the range

$$1 - \frac{c_0}{\log d_{F_m}} \leq \beta < 1,$$

then β is a simple real zero of $\zeta_{F_0}(s)$.

Since $\zeta_{\mathbb{Q}}(s)$ has no real zeros with $0 < \beta < 1$, we have the following corollary.

Corollary 1. Let $\mathbb{Q} = F_0 \subset F_1 \subset \cdots \subset F_m$ be a tower of extensions such that $[F_i : F_{i-1}]$ is odd and F_i is a radical extension of F_{i-1} for $i = 1, \ldots, m$. Then $\zeta_{F_m}(s)$ has no zeros in the region

$$\sigma \geq 1 - \frac{c_0}{\log d_{F_m}}, \quad |t| \leq \frac{c_0}{\log d_{F_m}}.$$

For towers of radical extensions, Corollary 1 is much stronger than the corresponding result of Stark, which has $n_{F_m} \log d_{F_m}$ in place of $\log d_{F_m}$ [18, Lemma 8]. The results of Sunley, Goldstein, Heilbronn, and Stark are strongest for normal

NON-EXISTENCE OF SIEGEL ZEROS IN TOWERS OF RADICAL EXTENSIONS

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Dedicated to the memory of Emil Grosswald

Abstract. Given a number field K, it is shown that for a tower of radical extensions of odd degree over K, a Siegel zero of any Dedekind ζ-function of a field in the tower must be a zero of the ζ-function of K. In particular, towers of radical extensions of odd degree over $\mathbb{Q}$ have no Siegel zeros. This result is derived from a more general theorem, covering wider classes of field extensions. The basic results are obtained by showing that a Siegel zero not coming from a lower field would be a zero of high multiplicity of some normal extension of the field under consideration, a multiplicity greater than allowed by simple bounds.

1. Introduction

Let K be an algebraic number field of degree n_K and let d_K be the absolute value of its discriminant. Let $\zeta_K(s)$ denote the Dedekind ζ-function of K. It is well known that there is an effectively computable constant c such that for any K, $\zeta_K(s)$ has at most one zero $\sigma + it$ in the range

$$\sigma \geq 1 - \frac{c}{\log d_K}, \quad |t| \leq \frac{c}{\log d_K} ;$$

if such a zero exists, it must be real and simple. Stark [18] has shown that one may take $c = \frac{1}{4}$. Exceptional zeros of this kind are called Siegel zeros. It is conjectured that there are no Siegel zeros, but this has not been proved in general.

The main motivation for studying Siegel zeros arises because they determine the size of the residue κ of $\zeta_K(s)$ at $s = 1$, and thus the product of the class number and the regulator of K. Good unconditional upper bounds for κ can be derived easily, but lower bounds are much harder to obtain. It was first observed by Gronwall [4] and Hecke (see Landau [15]) that for quadratic fields non-existence of Siegel zeros yields good lower bounds for κ. In general, it is known that if $\zeta_K(s)$ has no Siegel zero, then $\kappa \geq c'(\log d_K)^{-1}$ for an effectively computable constant c' that depends

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