p	$\langle t^p \rangle$	$\langle \delta_n'^p \rangle_1$	$\langle \delta_n'^p \rangle_2$	$\langle \delta_n'^p \rangle_3$
1	0.725227	0.731988	0.730706	0.725291
2	0.603251	0.606386	0.605762	0.602470
3	0.555775	0.551262	0.551956	0.553540
4	0.555527	0.540113	0.542599	0.551074
5	0.594314	0.563548	0.568467	0.586454
6	0.674002	0.620786	0.629172	0.660788
7	0.804518	0.717187	0.730735	0.782709
8	1.00515	0.864325	0.885824	0.969281
9	1.30870	1.08177	1.11583	1.24935
10	1.76924	1.40075	1.45504	1.67002

Table 1. Comparison of the moments of $nn(t)$ for the GUE (second column) and for 10^6 consecutive zeros of the Riemann zeta function (subsequent columns) on the critical line, starting near zero number 1, 10^6 and 10^{20} respectively. The mean spacing between consecutive (eigenvalues) zeros has been normalized to unity.

Figure 1 Comparison of $nn(t)$ for the GUE (solid line) and for 10^6 consecutive zeros of the Riemann zeta function on the critical line, starting near zero number 1 (open circles), 10^6 (asterisks) and 10^{20} (filled circles) respectively. The mean spacing between consecutive (eigenvalues) zeros has been normalized to unity.

To summarize, the exact evaluation of the p.d.f. $nn(t)$ for the spacing between nearest neighbor levels in the infinite GUE has been given in terms of a certain solution of the non-linear equation (6). This p.d.f. can be readily calculated from empirical eigenvalue data, so our exact evaluation provides a statistical test for the hypothesis that the data has the distribution of the eigenvalues of a random Hermitian matrix. Applying this test to the zeros of the Riemann zeta function on the critical line, we have found further evidence supporting the validity of the the GUE hypothesis.

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Using the facts that ϕ and ψ satisfy a pair of coupled first order differential equations, and that for b odd(even), $\phi(x)$ is even(odd) and $\psi(x)$ is odd(even), from the theory of [9] we can deduce that the following equations hold

$$tR = 2(-b + u - w)pq + t(p^2 + q^2) + 2(pq)^2 \tag{13}$$

$$tq' = (-b + u - w)q + tp \tag{14}$$

$$tp' = -tq - (-b + u - w)p \tag{15}$$

$$(tR)' = p^2 + q^2 \tag{16}$$

$$u' = 2q^2, \quad w' = 2p^2. \tag{17}$$

Also, it is easy to check from the definitions that

$$\frac{d}{dt}\log(1 - K_1) = -2R. \tag{18}$$

To derive (5a) we set

$$\sigma_1(2t) := -2tR \tag{19}$$

and integrate (18) (the factor $\pi\rho$ in the upper terminal of (5) results from changing the mean eigenvalue spacing from π to $1/\rho$). To derive (6) we multiply (14) by p, multiply (15) by q, add and use (17) to obtain

$$(pq)' = p^2 - q^2 = \frac{1}{2}(w' - u') \tag{20}$$

and consequently

$$pq = \frac{1}{2}(w - u). \tag{21}$$

Substituting (21) and (16) in (13) gives

$$tR = -2b(pq) - 2(pq)^2 + t(tR)' \tag{22}$$

which relates tR to pq. On the other hand, another equation relating these two quantities is obtained by squaring (16) and the first equality in (20) and subtracting:

$$((pq)')^2 - ((tR)')^2 = -4(pq)^2. \tag{23}$$

Solving (22) for pq (the negative square root is to be taken) and $(pq)'$, substituting in (23) and introducing the notation (19) gives (6). The boundary condition (7) follows from the fact that $R(s,s) \sim K_1(s,s)$ as $s \to 0$ and the corresponding behaviour of $K_1(s,s)$ deduced from (4).

matrix ensemble with unitary symmetry defined by the eigenvalue p.d.f.

$$\prod_{j=1}^{N} |x_j|^{2b} e^{-x_j^2} \prod_{1 \le j < k \le N} |x_k - x_j|^2, \qquad b > -1/2. \tag{8}$$

They prove that in the thermodynamic limit, with each x_j scaled $x_j \mapsto X_j/\sqrt{2N}$ so that the bulk density is $1/\pi$, the corresponding n-level distribution is given by

$$\rho_n(X_1, \dots, X_n) = \det[K_1(X_j, X_k)]_{j,k=1,\dots,n} \tag{9}$$

where $K_1(x,y)$ is given by (4) (for $b \notin Z_{\ge 0}$, $x(y) < 0$, $x(y)$ in the denominator needs to be replaced by $|x|(|y|)$, however below we will only consider the case $b \in Z_{\ge 0}$).

It follows from (9) (see e.g. ref. [1]) that the probability $E(0;(-t,t))$ of an interval $(-t,t)$ being free of eigenvalues in the ensemble (8) is given by $\det(1-K_1)$. Since in the case $b=1$ (8) is precisely the eigenvalue p.d.f. of the GUE with an eigenvalue fixed at the origin, the result (3) follows. In fact this interpretation of (8) suggests another derivation of (9) in the case $b=1$. Thus with $b=1$ (9) must be equal to the $(n+1)$-level distribution of the GUE (see e.g. Ref. [1])

$$\rho_{n+1}^{\rm GUE}(X_1, \dots, X_{n+1}) = \det[\frac{\sin(X_j - X_k)}{\pi(X_j - X_k)}]_{j,k=1,\dots,n+1} \tag{10}$$

with one of the levels, X_{n+1} say, fixed at the origin. Setting $X_{n+1} = 0$ in (10) and performing Gaussian elimination so that all entries below the first in the final column are zero gives (9) in the case $b=1$.

To derive (6) we introduce the quantities

$$(1-K_1)^{-1} \doteq \rho(x,y), \quad K_1(1-K_1)^{-1} \doteq R(x,y), \quad R(t,t) := R \tag{11}$$

(the symbol $\doteq$ denotes 'has kernel') and

$$Q(x) := (1-K_1)^{-1}\phi, \quad q := Q(t^-)$$

$$P(x) := (1-K_1)^{-1}\psi, \quad p := P(t^-)$$

$$u := \int_{-t}^{t} Q(y)\phi(y)dy, \quad w := \int_{-t}^{t} P(y)\psi(y)dy, \tag{12a}$$

where

$$\phi(x) = \sqrt{x/2}J_{b+1/2}(x), \quad \psi(x) = \sqrt{x/2}J_{b-1/2}(x) \tag{12b}$$

(note that $K_1(x,y) = (\phi(x)\psi(y) - \phi(y)\psi(x))/(x-y)$).

with $b=1$, subject to the boundary condition

$$\sigma_1(s) \sim -\frac{(s/2)^{2b+1}}{\Gamma(1/2+b)\Gamma(3/2+b)}, \quad \text{as} \quad s \to 0 \tag{7}$$

with $b=1$ (the parameter b is included above for later convenience). Note that with $b=0$ (6) reduces to (2).

We have computed many terms of the power series expansion of (6) about $s=0$ with $b=1$ and subject to (7). Comparison with the analogous expansion of (1) (see e.g. [1]) shows that $p(t)-(1/2)nn(t) = O(t^7)$, which in qualitative terms says that very small spacings between consecutive eigenvalues will most likely be nearest neighbor spacings (the factor of 1/2 accounts for the fact that the nearest neighbor occurs with equal probability to the left or to the right). The solution of (6) with $b=1$ was computed numerically (the power series solution to $O(s^{11})$ was used to compute $\sigma_1(1)$ and $\sigma_1'(1)$ which were used as initial conditions) and substituted in (5b) with $\rho=1$ to give the theoretical prediction for $nn(t)$ in the infinite GUE, which was then compared with $nn(t)$ determined empirically from the data of [6] for $\{\gamma_n\}$. Three sets of 10^6 consecutive zeros $1/2+i\gamma_n$ were analyzed, the data sets starting at zero number $N_1=1$, $N_2=10^6+1$ and $N_3=10^{20}+143,782,842$ respectively. The quantity $\delta_n' := \min(\delta_n, \delta_{n-1})$, where $\delta_n := (\gamma_{n+1}-\gamma_n)\rho_n$ with $\rho_n = (1/2\pi)\log(\gamma_n/2\pi)$ denoting the smoothed local density of zeros at $1/2+i\gamma_n$, was calculated and a histogram constructed for the number of values out of the 10^6 tested that fell into the intervals $((k-1)/20, k/20)$, $k=1,2\ldots$. In Figure 1 the corresponding empirical values of $nn(t)$ at the points $(k-1/2)/20$ are plotted and compared with the value of $nn(t)$ for the infinite GUE. The convergence towards the GUE value as the magnitude of the imaginary part increases is evident.

For further comparison the moments $\langle t^p\rangle := \int_0^\infty t^p nn(t)\,dt \quad (a=1,2,3)$, for $p=1,\ldots,10$ were calculated and compared with the empirical data according to the law of large numbers prediction $\langle t^p\rangle \approx \langle \delta_n'^p\rangle_a := 10^{-6}\sum_{n=N_a+1}^{N_a+10^6}\delta_n'^p$. The results are contained in Table 1. Again the trend is towards convergence to the GUE value. Note in particular the four figure agreement between $\langle t\rangle$ and $\langle\delta_n'\rangle_3$. The p.d.f. $nn(t)$ therefore provides quantative statistical evidence supporting the validity of the GUE hypothesis, thus adding to the statistical evidence obtained in Ref. [6] and the analytic arguments of Ref. [7].

Our derivation of (3)-(7) uses a recent result of Nagao and Slevin [8] to obtain (3), and the theory of Tracy and Widom [9] to obtain (6). Nagao and Slevin consider the random where $\sigma(s)$ satisfies the σ form of the Painlevé V equation:

$$(s\sigma'')^2 + 4(s\sigma' - \sigma)\Big(s\sigma' - \sigma + (\sigma')^2\Big) = 0 \tag{2}$$

subject to the boundary condition $\sigma(s) \sim -s/\pi - (s/\pi)^2$ as $s \to 0$.

The GUE is applicable to chaotic quantum systems with broken time reversal symmetry. The zeros of the Riemann zeta function $\zeta(z)$ for large imaginary part on the critical line $\mathrm{Re}(z) = 1/2$ are known to possess characteristics of such a system [5], and according to the so called GUE hypothesis (see e.g. Ref. [6]) in the limit of infinite imaginary part the joint distribution of the zeros is locally equal to the joint distribution of the eigenvalues of the GUE. The eigenvalues and zeros must be scaled so that their mean spacing takes on the same fixed value, $1/\rho$ say. In a large-scale numerical computation by one of the present authors [6], involving over 10^7 zeros $1/2 + i\gamma_n$ about $n = 10^{20}$ (here n labels the zeros along the critical line) the p.d.f. $p(s)$ has been determined empirically and compared with $p(s)$ for the GUE. Excellent agreement is found.

In this Letter a statistic for the infinite GUE which is very similar to the spacing between consecutive levels is calculated exactly, and compared to that obtained empirically from the data of [6] for $\{\gamma_n\}$. This statistic is the p.d.f. $nn(t)$ for the spacing between nearest neighbor levels (note that each eigenvalue has two neighbors but only one nearest neighbor).

Below the following results are established. The p.d.f. $nn(t)$ for the infinite GUE with mean eigenvalue spacing π is given in terms of a Fredholm determinant by

$$nn(t) = -\frac{d}{dt}\det(1 - K_1) \tag{3}$$

where K_1 is the integral operator on $(-t, t)$ with kernel

$$K_1(x,y) := \frac{\sqrt{xy}}{2(x-y)}\Big(J_{b+1/2}(x)J_{b-1/2}(y) - J_{b+1/2}(y)J_{b-1/2}(x)\Big), \tag{4}$$

($J_\alpha(x)$ denotes the Bessel function) and $b = 1$. Furthermore

$$\det(1 - K_1) = \exp\int_0^{\pi\rho t} \frac{\sigma_1(2t')}{t'}dt' \tag{5a}$$

and so

$$nn(t) = -\frac{\sigma_1(2\pi\rho t)}{t}\exp\int_0^{\pi\rho t}\frac{\sigma_1(2t')}{t'}dt' \tag{5b}$$

(here the mean eigenvalue spacing is $1/\rho$), where $\sigma_1(s)$ satisfies the non-linear equation

$$(s\sigma_1'')^2 + 4(-b^2 + s\sigma_1' - \sigma_1)\Big\{(\sigma_1')^2 + [b - (b^2 - s\sigma_1' + \sigma_1)^{1/2}]^2\Big\} = 0 \tag{6}$$

GUE EIGENVALUES AND RIEMANN ZETA FUNCTION ZEROS: A NON-LINEAR EQUATION FOR A NEW STATISTIC

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Abstract

For infinite GUE random matrices the probability density function $nn(t)$ for the nearest neighbor eigenvalue spacing (as distinct from the spacing between consecutive eigenvalues) is computed in terms of the solution of a certain non-linear equation, which generalizes the σ form of the Painlevé V equation. Comparison is made with the empirical value of $nn(t)$ for the zeros of the Riemann zeta function on the critical line, including data from 10^6 consecutive zeros near zero number 10^{20}.

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Random matrix theory successfully predicts many features of the statistical properties of the energy levels of classically chaotic quantum systems (see e.g. Refs. [1,2]). One such statistical property is the probability density function (p.d.f.), $p(s)$ say, for the spacing between consecutive energy levels. Jimbo et al. [3] (see Refs. [4] for subsequent derivations) proved that for the Gaussian Unitary Ensemble (GUE) of infinite dimensional random matrices, scaled so that the mean eigenvalue spacing is $1/\rho$, $p(s)$ is given by

$$p(s) = \frac{1}{\rho}\frac{d^2}{ds^2}\exp\int_0^{\pi\rho s}\frac{\sigma(s')}{s'}ds' \tag{1}$$

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