REVIEWS

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Experimental Mathematics in Action. By D. H. Bailey, J. M. Borwein, N. J. Calkin, R. Girgensohn, D. R. Luke, and V. H. Moll. A K Peters, Wellesley, MA, 2007. xii + 322 pp., ISBN 978-1-56881-271-7. $65.

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That computers are revolutionizing mathematics (and almost everything else) is now a cliché. But clichés are often true, and sometimes bear re-examining to gain a deeper understanding. *Experimental Mathematics in Action* by David Bailey, Jonathan Borwein, Neil Calkin, Roland Girgensohn, Russell Luke, and Victor Moll provides an opportunity to do so. It is a very nice and useful book. However, one can hope it will become obsolete in a few years. We are in a transitional period in terms of acceptance of computers into mainstream mathematics, and the lessons of this book should soon become fully absorbed into standard books.

Revolution or evolution? Practically every new development has enthusiasts who proclaim it changes everything, while others claim it is just a minor extension of known techniques (and yet others decry it for destroying the supposedly wonderfully ordered life of old). Whether something is truly revolutionary or not is a subjective judgment, and generally appears to depend on the scale and speed of change. The electric telegraph, made practical about 170 years ago, was the first technology that could truly claim to "annihilate space and time," in a phrase popular at that time. It allowed correspondents on opposite ends of the globe to be in essentially instantaneous communication with each other. (The semaphore telegraph and the railroad were earlier perceived to have similar almost-magical effects.) So the telegraph could be said to be revolutionary, in some ways a direct forerunner of today's Internet. But its effects were limited. It was expensive and had low transmission capacity. Its influence on mathematics was barely noticeable, especially when compared to that of the Internet. The latter allows us to distribute our papers all over the world as soon as we write them, and to engage in collaborative research using wikis. Still, the volume and intensity of interaction have been growing steadily over time, although at an uneven pace. We can view the telegraph and the Internet as contributors to a steady growth in volume and intensity of communication and interaction.

The latest wave of hype in science is over a supposedly new paradigm of research, namely data-driven science. It is proclaimed to join the more established approaches, namely theory, experiment, and computation. A skeptical view says that much of ancient research was data-driven. Just consider the work of Gauss on predicting the orbits of Ceres and other celestial bodies, or, earlier, the work of Tycho Brahe and Kepler. These efforts were driven by growth in astronomical data. They influenced mathematics by leading, for example, to the invention of the method of least squares. Still, there

http://dx.doi.org/10.4169/amer.math.monthly.118.10.946

has been a dramatic change recently in both the volume of data and in the methods available for dealing with it. Just about a decade ago, data stored on magnetic disks surpassed in extent data in all libraries, film archives, and other analog repositories, and it is doubling about every two years or so. Hence very little of it will ever be processed directly by the human senses of sight and hearing. An increasingly dominant share will only ever be accessed by various automated systems. The goals set for these systems will come from people, and the fundamental task will continue to be to provide insight for people. But we can talk of a data revolution. Many, perhaps most, areas of research will be affected by the needs and opportunities offered by the growing flood of data. This will happen in the sciences and mathematics, as well as in the humanities. There will surely be a growth in the ranks of experts in many disciplines who work primarily with large data sets. Furthermore, the methods for dealing with such data collections, and the habits of using them whenever appropriate, will be incorporated into the toolkits of most researchers, including those in mathematics. Will that lead to the perception that data-driven science is a distinct branch, comparable to theory, experiment, and computation? Only time will tell. However, we can already see data-driven mathematics, as attention shifts to methods of analyzing and modeling large data sets.

While data-driven science and mathematics are rising in prominence, computation is still struggling to establish itself in the same rank as theory and experiment. *Experimental Mathematics in Action* is best viewed as a collection of demonstrations of serious applications of computing in mathematics, applications designed to illustrate the wide range of techniques and results that are playing a growing but still somewhat controversial role.

In general, we should expect information and communications technologies (ICT) to keep invading mathematics, and thereby inevitably also to keep changing it. In George Gilder's memorable phrase, "you waste that which is plentiful," and progress is making ICT more plentiful, and far more capable. One can certainly sympathize with those who complain about the loss of the deliberate pace of earlier days. But such complaints are not new. Earlier technologies also faced opposition, as in the passage in Plato's *Phaedrus* addressed to the reputed inventor of writing:

> [Writing] will create forgetfulness in the learners' souls, because they will not use their memories; they will trust to the external written characters and not remember of themselves. [Writing] is an aid not to memory, but to reminiscence, and you give your disciples not truth, but only the semblance of truth; they will be hearers of many things and will have learned nothing; they will appear to be omniscient and will generally know nothing; they will be tiresome company, having the show of wisdom without the reality.

It is hard to deny the wisdom in this passage, just as it is hard to deny the advantages of occasionally cutting oneself off from all modern technologies in order to be able to think deeply without distractions. But on balance we should expect ICT to play a growing role in mathematicians' lives and work. Writing may "create forgetfulness" in our souls, but it was crucial in enabling us to deal with the growth in data, information, and knowledge. ICT should similarly be seen as crucial to enabling us to deal with the growing complexity in mathematics and other disciplines. Much of the change in mathematical research that is ascribed to computers is caused by internal dynamics of the field, and not by the new technology. (It is often facilitated and amplified by that technology, though.) The fraction of mathematical papers covered by *Mathematical Reviews* that have a single author has declined from well over 90% in

1940 to under half today. The process has been gradual, predating the Internet, and almost surely reflects the greater need for collaboration as individuals can truly master a decreasing fraction of all relevant information. Similar phenomena may be behind the frequent complaints that the peer review system is breaking down. To the extent these complaints are true (and unfortunately we do not have reliable measures of this phenomenon), the problem may also be caused by growing complexity and the related inability of referees to understand and verify all that they are asked to read.

Some of the most intense discussions about the nature of mathematical proof occurred in the late 1970s, prompted by the proof of the four color theorem (FCT) by Appel, Haken, and Koch [**2**]. There was widespread discomfort at having to depend on extensive computer calculations to establish the validity of a mathematical result that was simple to state. Yet at about the same time similar discomfort was caused by the classification of the finite simple groups. That project did not rely on computer calculation to any great extent. However, this classification did depend on the correctness of tens of thousands of pages of journal articles. Which was more believable, the large body of conventional papers, or the computer calculation? In principle, both proofs are completely rigorous by the usual standards, as both consist of clear logical steps that can be verified one-by-one by a competent reader. The trouble is the number and complexity of those steps, which no single mathematician can reasonably be expected to carry out completely.

Dissatisfaction with the FCT and with the classification of finite simple groups has stimulated searches for better solutions. It appears that progress has been faster on the FCT. This is almost surely because it can more easily benefit from "wasting that which is plentiful," namely computer power and sophistication of computer algorithms, as opposed to that which is scarce, namely mathematicians' time. Both the FCT and the classification of finite simple groups have seen significant improvements in the basic mathematical approaches. However, the FCT has also benefited from (i) the size of the computation, which required heroic efforts in the 1970s, now being almost trivial, as well as from (ii) the development of formal methods that reduce the correctness of the computer proof to that of a more manageable (but still not fully verified) program [**6**]. As a result, the validity of the proof of the FCT is still debated, but it does appear to be far more widely accepted than before. (A similar process is in progress with regard to the Hales and Ferguson proof of the Kepler conjecture on the densest packing of spheres in three dimensions [**7**]. The extensive and controversial computations of the original demonstration are now in the process of being formalized; see [**8**].)

More instances of "wasting that which is plentiful" leading to changes in mathematics can be seen in other areas. Back just a few decades ago, a mathematician (or engineer, or physicist) who wished to identify an integer sequence that arose in some investigation had to go to the library and attempt some (hopefully not completely random) searches, or else would have to ask around among friends and colleagues. After 1973, such a person could consult Neil Sloane's *A Handbook of Integer Sequences* [**11**], whose utility is certified by the myriad citations in the scientific literature. Today, such a person is far more likely to resort to the On-Line Encyclopedia of Integer Sequences, OEIS (http://www.oeis.org/), which was created by Sloane and his collaborators. The OEIS provides access to almost two orders of magnitude more sequences than the couple of thousand in [**11**]. Further, it provides software tools (many of which rely on deep mathematical algorithms) that automatically transform a given sequence into a variety of forms and tell if those match any of the stored patterns. Thus OEIS is another example of using ICT to extend the scale and scope of mathematical research. It does not change its fundamental nature, but does make it far more effective and efficient.

There is no doubt that there will continue to be simple and elegant surprises, such as the Agrawal, Kayal, and Saxena proof that testing integers for primality can be done in polynomial time [**1**]. Contrary to the consensus of experts, no new tools were needed to prove this expected result, just some cleverness. It would be presumptuous and contrary to all the evidence to conclude that this was the last such instance, and that all the "low-hanging fruit" have been picked. (Perhaps even the FCT and the classification of finite simple groups will see short proofs!) And there will surely continue to be plenty of complicated and sophisticated proofs that do not require computers, such as Perelman's proof of the Poincaré conjecture. But ICT will inevitably further permeate mathematics research and mathematics teaching. It will continue to spur collaboration on many levels, for, as Licklider and Taylor divined in the 1960s, the computer is, more than anything else, a communication device [**9, 10**]. And it will be used in a variety of ways in mathematical research.

Experimental Mathematics in Action is based on lectures given at a Mathematical Association of America short course on experimental mathematics given at the 2006 annual AMS–MAA meeting. Its aim is to facilitate the adoption of nontrivial computational techniques in mathematics. Chapter 1 has a brief philosophical introduction to the role of computing in mathematics, and then lists eight roles for it (pp. 13–15, based on [**4, 5**]):

- Gaining insight and intuition, or just knowledge.
- Discovering new facts, patterns, and relationships.
- Graphing to expose mathematical facts, structures, or principles.
- Rigorously testing and especially falsifying conjectures.
- Exploring a possible result to see if it *merits* formal proof.
- Suggesting approaches for formal proof.
- Computing replacing lengthy hand derivations.
- Confirming analytically derived results.

There follow brief sketches of examples for each. Many of the examples illustrate, as might be expected, several of these eight roles simultaneously. Chapters 2–8 then present a variety of other applications in more detail. It is impossible to give a fair picture of the immense variety of these applications in a few words. They range over prime number computations, numerical integration, nowhere differentiable functions, inverse scattering, and closed-form evaluations of definite integrals. As just one concrete example, Chapter 2 presents the now-classic result, along with generalizations, that

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} + \frac{1}{8k+5} - \frac{1}{8k+6} \right). \tag{1}$$

This identity was found "by a combination of inspired guessing and extensive searching using the PSLQ integer relations algorithm" [3, p. 905], but has been proved completely rigorously by standard methods. It is of general interest because it enables the computation of the nth digit of π in base 16 very fast, without having to compute preceding digits. (Only a simple argument is needed to show that the rational function in k for the kth term in the identity above is innocuous, and does not materially affect the running time of the algorithm.) The identity of equation (1) fits well in the book, since it illustrates several of the roles that computation can play, involving the interplay of number crunching, sophisticated algorithms, and intuition.

Experimental Mathematics in Action fulfills well the goal of the MAA short course on which it was based, namely "to present a coherent variety of accessible examples of modern mathematics where intelligent computing plays a significant role and in doing so to highlight some of the key algorithms and to teach some of the key experimental approaches" (p. xi). It is not a textbook. The examples are usually not presented in full detail, but are sketched, and references are given to the appropriate research papers. Further, the contexts in which these examples arise are seldom explained. Hence this volume is likely to be of greatest interest to professional mathematicians who are looking to broaden their research horizons or are looking for interesting material to enrich their classes. The best course for them might be to browse and concentrate on the chapters or sections that catch their attention.

Experimental Mathematics in Action covers a wide variety of topics, but that breadth is still just a minute fraction of all interactions of computers and mathematics. Experts in areas that have long relied heavily on computation, such as fluid dynamics, combinatorial optimization, or statistics of large data sets, are not likely to find much here of direct relevance to their work. (But one should never discount serendipity!) The intended audience appears to be people in areas such as number theory, algebra, discrete mathematics, or dynamical systems who have not used computers much in their research, and insist on strict rigor. For many of them, this book will be a great resource. It is well written, with extensive references.

There is substantial overlap between this book and the two books [**4, 5**], whose authors are subsets of those of the volume under review. What sets *Experimental Mathematics in Action* apart is Chapter 9, which makes up almost a quarter of the 320+ pages of the book. It contains a wealth of exercises for Chapters 2–8, as well as some additional ones. Some come from the problem section of the MONTHLY, others from the Putnam exam, and yet others from research papers. An important and interesting example from the last category is the Dyson conjecture (p. 296) that for nonnegative integers $a_1, a_2, \ldots, a_n$, the constant term of

$$\prod_{1 \le i \ne j \le n} \left(1 - \frac{x_i}{x_j}\right)^{a_i} \tag{2}$$

equals

$$\frac{(a_1 + a_2 + \cdots + a_n)!}{a_1! a_2! \cdots a_n!}. \tag{3}$$

This result, initially proved by Wilson and Gunson half a century ago, is now just one instance of more general results that can be found and proved in an automated way using computers. For someone looking for problems to assign to a class, problems that are well thought out and relevant to the topic of the book, Chapter 9 is likely to be a very valuable resource.

One can hope that in another decade the results, methods, and philosophy of experimental mathematics will become absorbed fully into all disciplines. In the meantime, though, the authors have performed a valuable service to the community by writing this volume.

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Mathematics and Eternal Truths

"Eternal truths are ultimately invisible, and you won't find them in material things or natural phenomena, or even in human emotions. Mathematics, however, can illuminate them, can give them expression—in fact, nothing can prevent it from doing so."

Yoko Ogawa, *The Housekeeper and the Professor*,
trans. Stephen Snyder, Picador, New York, 2009, p. 116

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