Turing and the Riemann zeta function

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The mysteries of prime numbers:

Mathematicians have tried in vain to discover some order in the sequence of prime numbers but we have every reason to believe that there are some mysteries which the human mind will never penetrate. To convince ourselves, we have only to cast a glance at tables of primes (which some have constructed to values beyond 100,000) and we should perceive that there reigns neither order nor rule.

Euler, 1751

Riemann, 1850s:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s \in \mathbb{C}, \ \mathrm{Re}(s) > 1\,.$$

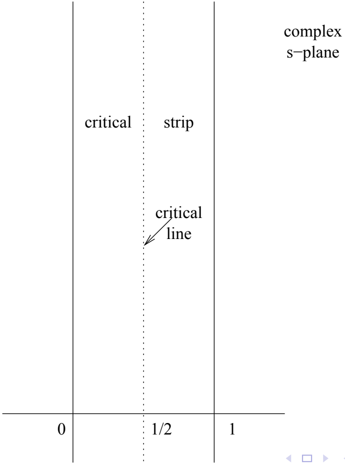
Showed $\zeta(s)$ can be continued analytically to $\mathbb{C} \setminus \{1\}$ and has a first order pole at $s = 1$ with residue 1. If

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)\ ,$$

then (functional equation)

$$\xi(s) = \xi(1-s)\ .$$

Critical strip:



Riemann explicit connection between primes and zeros:

$$\pi(x) = \mathrm{Li}(x) - \sum_{\rho} \mathrm{Li}(x^{\rho}) + O(x^{1/2} \log x)$$

where ρ runs over the nontrivial zeros of $\zeta(s)$

Hence size of $\pi(x) - \mathrm{Li}(x)$ depends on location of zeros

Riemann, 1859:

- almost all nontrivial zeros of the zeta function are on the critical line (positive assertion, no hint of proof)
- it is likely that all such zeros are on the critical line (now called the Riemann Hypothesis, RH)
- (ambiguous: cites computations of Gauss and others, not clear how strongly he believed in it) $\pi(x) < \mathrm{Li}(x)$

Hardy, Littlewood, Ingham, ..., and Skewes

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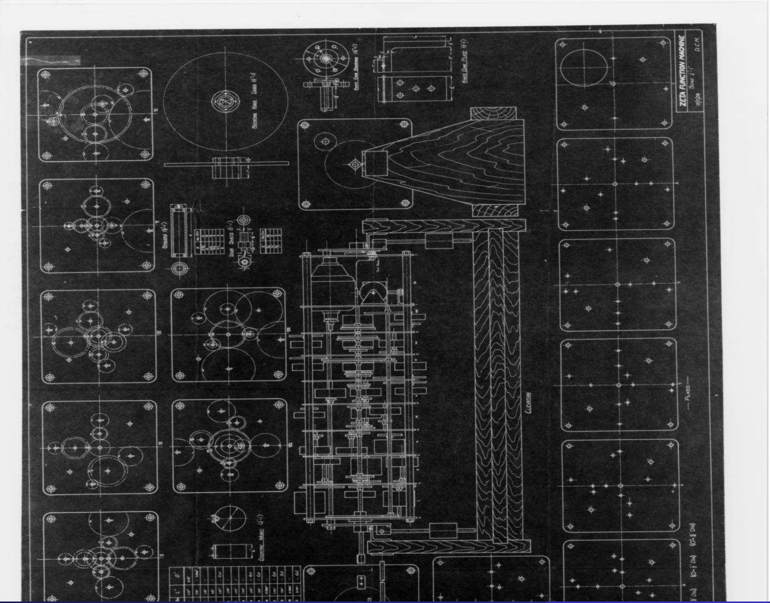
Turing zeta machine design, 1939:



Turing and the zeta function:

- persistent interest starting as undergraduate
- design of special zeta function machine, 1939
- first computation of zeta zeros on electronic digital computer, 1950
- new method for computing the zeta function
- studies of $\pi(x) - \mathrm{Li}(x)$ (with Skewes)
- "Turing method" for easy numerical verification of RH
- skepticism about validity of RH

Numerical verifications of RH for first n zeros:

Riemann 1859	?
...	
Hutchinson 1925	138
Titchmarsh et al. 1935/6	1,041
Turing 1950 (published 1953)	1,054
...	
te Riele et al. 1986	1,500,000,000
...	
Gourdon 2004	10,000,000,000,000

Interest in zeros of zeta function:

- numerical verification of RH
- $\pi(x) - \mathrm{Li}(x)$ and related functions
- (more recently) distribution questions related to hypothetical random matrix connections

Riemann and Ingham:

Riemann:

$$\pi(x) = \mathrm{Li}(x) - \sum_{\rho} \mathrm{Li}(x^{\rho}) + O(x^{1/2} \log x)$$

Ingham:

certain averages of $(\pi(x) - \mathrm{Li}(x))$ = nice sums over ρ

Riemann and Ingham:

- Littlewood (1914): Riemann "conjecture" that $\pi(x) < \mathrm{Li}(x)$ false
- Skewes (1933, assuming RH): first counterexample $< 10^{10^{10^{34}}}$
- Ingham approach (with extensive computations): $< 10^{317}$

“Turing method” for numerical verification of RH:

- $\xi(s) = \xi(1-s)$ means $\xi(s)$ takes real values on the critical line
- hence sign changes of $\xi(s)$ correspond to zeros of $\zeta(s)$ that are right on the critical line
- need to prove all zeros have been found
- traditionally done using the principle of the argument
- “Turing method” provides neat approach that uses just values of $\xi(s)$ on the critical line

"Turing method:"

$N(t)$ = number of zeros ρ with $0 < \mathrm{Im}(\rho) < t$

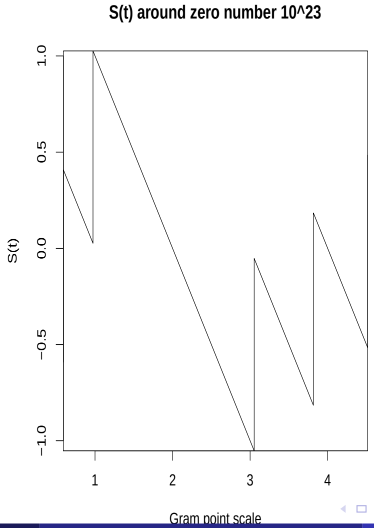
$$N(t) = 1 + \frac{1}{\pi}\theta(t) + S(t)$$

where $\theta(t)$ is a smooth function, and

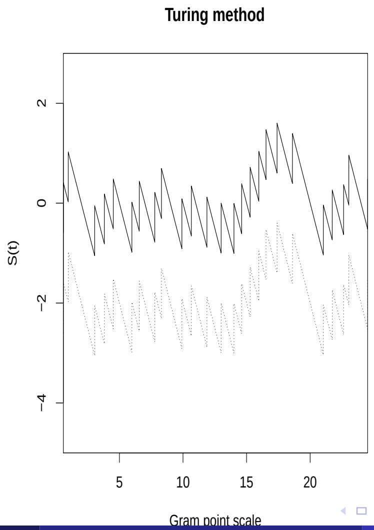
$$|S(t)| = O(\log t)$$

$$\int_{t_0}^{t_1} S(u)du = O(\log t_1)$$

Typical behavior of S(t):



Turing:



The validity of RH:

- Turing's skepticism grew with time
- many other famous number theorists were disbelievers (e.g., Littlewood)
- skepticism appears to have diminished, because of computations and various heuristics (many assisted by computations)
- but ...

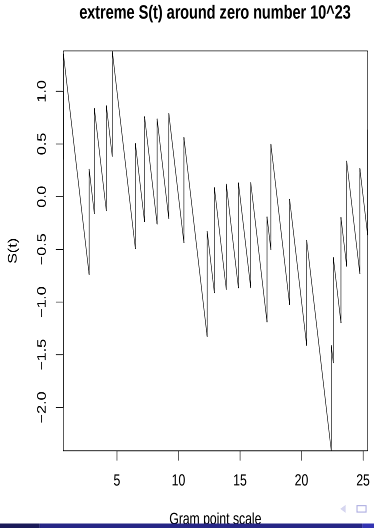
Beware the law of small numbers (especially in number theory):

$N(t) = 1 + \frac{1}{\pi}\theta(t) + S(t)$

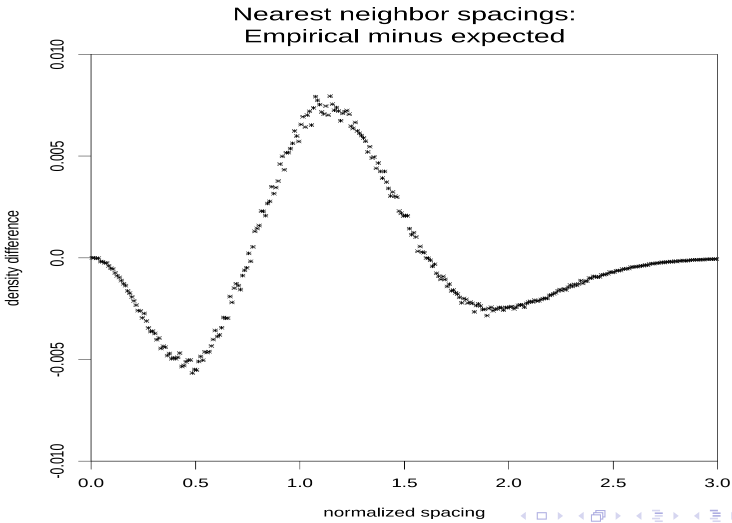
where $\theta(t)$ is a smooth function, and $S(t)$ is small:

- $|S(t)| = O(\log t)$
- $$\frac{1}{t}\int_{10}^{t} S(u)^2 du \sim c \log\log t$$
- $$|S(t)| < 1 \quad \text{for} \quad t < 280$$
- $$|S(t)| < 2 \quad \text{for} \quad t < 6.8 \times 10^6$$
- largest observed value of $|S(t)|$ only a bit over 3

extreme among 10^6 zeros near zero 10^{23}:



Zeta zeros and random matrices:



More information, papers and data sets:

http://www.dtc.umn.edu/∼odlyzko/

in particular, recent paper with Dennis Hejhal,
"Alan Turing and the Riemann zeta function"