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DISCRIMINANT BOUNDS

(description of tables)

i) Tables 1 and 3 assume the Generalized Riemann Hypothesis (GRH), while Tables 2 and 4 are unconditional. Tables 1 and 2 were derived from Tables 3 and 4, respectively.
ii) In Tables 1 and 2, an entry B in the totally complex $D^{1/n}$ column corresponding to $n = n_0$ means that for all fields of degrees $n \geq n_0$, the discriminant satisfies $D^{1/n} > B$. An entry A in the totally real $D^{1/n}$ column implies that for all totally real fields of degrees $n \geq n_0$, we have $D^{1/n} > A$. The b entries specify which inequalities in the other tables were used.
iii) In Tables 3 and 4, the notation is as follows. If K is an algebraic number field with r_1 real and $2r_2$ complex conjugate fields, and D denotes the absolute value of the discriminant of K, then for any b we have

$$D > A^{r_1} B^{2r_2} e^{f-E}$$

where A, B, and E are given in the table, and

$$f = 2 \sum_P \sum_{m=1}^{\infty} \frac{\log\ NP}{(NP)^{m/2}} F(\log\ NP^m)$$

where the outer sum is over all the prime ideals of K, N is the norm from K to Q, and

$$F(x) = G(x/b)$$

in the GRH case, and

$$F(x) = \frac{H(x/b)}{\cosh \frac{x}{2}}$$

in the unconditional case, where $G(x)$, $H(x)$ are even functions of x which vanish for $x > 2$, and for $0 \leq x \leq 2$ are given by

$$\begin{aligned} G(x) &= \left(1 - \frac{x}{2}\right) \cos \frac{\pi}{2} x + \frac{1}{\pi} \sin \frac{\pi}{2} x \\ H(x) &= \frac{1}{3}(2 - x) \left(1 + \frac{1}{2} \cos \pi x\right) + \frac{1}{2\pi} \sin \pi x\ . \end{aligned}$$

The values of A and B are lower estimates; the values of E have been rounded upwards from their true values, which are $8b/3$ in the unconditional case and

$$8\pi^2 b \left(\frac{e^{b/2} + e^{-b/2}}{\pi^2 + b^2} \right)^2$$

in the GRH case.
iv) Great care was taken to ensure that these bounds should be true lower bounds, rather than approximations. By selecting the parameter b more carefully, utilizing more precise estimates of integrals, and selecting better kernels, one can obtain improved lower bounds. For example, all fields of degrees ≥ 8 satisfy $D^{1/n} \geq 5.743$ on the GRH, and $D^{1/n} \geq 5.656$ unconditionally.