

The 10^{21} -st zero of the Riemann zeta function

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This is an extremely brief report on computation of large numbers of high zeros of the Riemann zeta functions, with pointers to sources of more detailed information. There have been many calculations that verified the Riemann Hypothesis (RH) for initial sets of zeros of the zeta function. The largest of these was that of van de Lune, te Riele, and Winter [LRW]. They checked that the first 1.5×10^9 nontrivial zeros all lie on the critical line. Their computations used about 1500 hours on one of the most powerful computers in existence at that time. Since then, better algorithms have been developed, and much more computing power has become available. Therefore it would be easy to extend their verification of the RH, should someone wish to do so.

Starting in the late 1970s, I carried out a series of computations that not only verified that nontrivial zeros lie on the critical line, but in addition obtained accurate values of those zeros. These calculations were designed to check the Montgomery pair-correlation conjecture [Mon], as well as further conjectures that predict that zeros of the zeta function behave like eigenvalues of matrices from the GUE. Instead of starting from the lowest zeros, these computations obtained values of blocks of consecutive zeros high up in the critical strip. The motivation for jumping high up was to come closer to observing the true asymptotic behavior of the zeta function, which is often approached slowly.

The first computations, done on a Cray supercomputer using the standard Riemann-Siegel formula, were described in [Od1]. The highest zeros they included were around zero # 10^{12} . Those calculations stimulated the invention, jointly with Arnold Schönhage [Od2, OS], of an improved algorithm for computing large sets of zeros. This algorithm, with some technical improvements, was implemented and used to compute several hundred million zeros at large heights, many near zero # 10^{20} , and some near zero # 2×10^{20} . Implementation details and results are described in the manuscripts [Od3, Od4] that have not been published, but have

circulated widely.

During the last couple of years, the algorithms of [Od3, Od4] have been ported from Cray supercomputers to Silicon Graphics workstations, and have been used to compute several billion high zeros of the zeta function. Some of those zeros are near zero $\# 10^{21}$, and it has been established (not entirely rigorously, though, as is explained in [Od3, Od4]) that zeros number $10^{21} - 1$, 10^{21} , and $10^{21} + 1$ are

144176897509546973538.2256529...
144176897509546973538.2912188...
144176897509546973538.4980696...

These values and many others can be found at

http://www.research.att.com/~amo/zeta_tables/index.html).

Further computations are continuing, and it is likely that some zeros near zero $\# 10^{22}$ will be computed. A revision of [Od3, Od4] that describes them will be prepared and published in the future. Results will be available through my home page,

<http://www.research.att.com/~amo>).

Finally, let me mention that many other computations of zeros of various zeta and L-functions have been done. Many are referenced in [Od5]. There are also interesting new results in the recent Ph.D. thesis of Michael Rubinstein [Ru].

References

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