

The n Body Matrix and its Determinant

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The primary purpose of this note is to prove two recent conjectures concerning the n body matrix that arose in recent papers of Escobar–Ruiz, Miller, and Turbiner on the classical and quantum n body problem in d -dimensional space. First, whenever the positions of the masses are in a nonsingular configuration, meaning that they do not lie on an affine subspace of dimension $\leq n - 2$, the n body matrix is positive definite and, hence, defines a Riemannian metric on the space coordinatized by their interpoint distances. Second, its determinant can be factored into the product of the order n Cayley–Menger determinant and a mass-dependent factor that is also of one sign on all nonsingular mass configurations. The factorization of the n body determinant is shown to be a special case of an intriguing general result proving the factorization of determinants of a certain form.

1. The n Body Matrix.

The n body problem, meaning the motion of n point masses (or point charges) in d -dimensional space under the influence of a potential that depends solely on pairwise distances, has a venerable history, capturing the attention of many prominent mathematicians, including Euler, Lagrange, Dirichlet, Poincaré, Sundman, etc., [19, 22]. The corresponding quantum mechanical system, obtained by quantizing the classical Hamiltonian to form a Schrödinger operator, has been of preeminent interest since the dawn of quantum mechanics, [12].

In three recent papers, [15, 20, 21], Escobar–Ruiz, Miller, and Turbiner made the following remarkable observation. Once the center of mass coordinates have been separated out, the quantum n body Schrödinger operator decomposes into a “radial” component that depends only upon the distances between the masses plus an “angular” component that involves the remaining coordinates and annihilates all functions of the interpoint distances. Moreover, the radial component is gauge equivalent to a second order differential operator which, as we will prove, is the Laplace–Beltrami operator on a certain Riemannian manifold coordinatized by the interpoint distances, whose geometry is as yet not well understood. This decomposition allows one to separate out the radial eigenstates that depend only upon the interpoint distances from the more general eigenstates that also involve the angular coordinates. A similar separation arises in the classical n body problem through the process of “dequantization”, i.e., reversion to the classical limit.

The primary goal of this paper is to prove several fundamental conjectures that were made in [15] concerning the algebraic structure of the underlying n body radial metric tensor. To be precise, suppose the point masses m_1, \dots, m_n occupy positions¹

$$\mathbf{p}_i = (p_i^1, \dots, p_i^d)^T \in \mathbb{R}^d, \quad i = 1, \dots, n.$$

We will refer to this as a *point configuration* consisting of n points (representing the positions of the n masses) in d -dimensional space. To streamline the presentation, we will allow one or more of the points to coincide, i.e., permit “collisions” of the masses. The subsequent formulas will slightly simplify if we express them in terms of the inverse masses

$$\alpha_i = \frac{1}{m_i}, \quad i = 1, \dots, n. \quad (1)$$

Definition 1. The point configuration $\mathbf{p}_1, \dots, \mathbf{p}_n$ will be called *singular* if and only if its elements lie on a common affine subspace of dimension $\leq n - 2$.

Thus, three points are singular if they are collinear; four points are singular if they are coplanar or collinear; etc. Note that nonsingularity requires that the underlying space be of sufficiently high dimension, namely $d \geq n - 1$.

¹We work with column vectors in \mathbb{R}^d throughout.

Using the usual dot product and Euclidean norm², let

$$r_{ij} = r_{ji} = \|\mathbf{p}_i - \mathbf{p}_j\| = \sqrt{(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}_i - \mathbf{p}_j)}, \quad i \neq j, \quad (2)$$

denote the $\binom{n}{2} = \frac{1}{2}n(n-1)$ interpoint distances $r = (\dots r_{ij} \dots)$. On occasion, formulas may also involve $r_{ii} = 0$. The subset traced out by the r_{ij} 's corresponding to all n point configurations (for any, or, equivalently, sufficiently large d) will be called the *Euclidean distance cone*, which forms a closed conical subset of the nonnegative orthant in $\mathbb{R}^{\binom{n}{2}}$, and denoted³

$$\mathcal{K}^{(n)} \subset \mathbb{R}_{\geq 0}^{\binom{n}{2}}. \quad (3)$$

We will explicitly characterize $\mathcal{K}^{(n)}$ in Theorem 9 below. As we will see, its boundary $\partial\mathcal{K}^{(n)}$ consists of the interpoint distances determined by singular point configurations while its interior, denoted $\mathcal{K}_0^{(n)}$, will correspond to the nonsingular configurations.

The n body matrix $B^{(n)} = B^{(n)}(\alpha, r)$ defined in [15] is the $\binom{n}{2} \times \binom{n}{2}$ matrix whose rows and columns are indexed by unordered pairs $\{i, j\} = \{j, i\}$ of distinct integers $1 \leq i < j \leq n$. Its diagonal entries are

$$b_{\{i,j\},\{i,j\}} = 2(\alpha_i + \alpha_j)r_{ij}^2 = 2(\alpha_i + \alpha_j)(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}_i - \mathbf{p}_j), \quad (4)$$

while its off diagonal entries are

$$\begin{aligned} b_{\{i,j\},\{i,k\}} &= \alpha_i(r_{ij}^2 + r_{ik}^2 - r_{jk}^2) = 2\alpha_i(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}_i - \mathbf{p}_k), & i, j, k \text{ distinct}, \\ b_{\{i,j\},\{k,l\}} &= 0, & i, j, k, l \text{ distinct}. \end{aligned} \quad (5)$$

For example, the 3 body matrix is

$$B^{(3)} = \begin{pmatrix} 2(\alpha_1 + \alpha_2)r_{12}^2 & \alpha_1(r_{12}^2 + r_{13}^2 - r_{23}^2) & \alpha_2(r_{12}^2 + r_{23}^2 - r_{13}^2) \\ \alpha_1(r_{12}^2 + r_{13}^2 - r_{23}^2) & 2(\alpha_1 + \alpha_3)r_{13}^2 & \alpha_3(r_{13}^2 + r_{23}^2 - r_{12}^2) \\ \alpha_2(r_{12}^2 + r_{23}^2 - r_{13}^2) & \alpha_3(r_{13}^2 + r_{23}^2 - r_{12}^2) & 2(\alpha_2 + \alpha_3)r_{23}^2 \end{pmatrix}, \quad (6)$$

where the rows and the columns are ordered as follows: $\{1, 2\}, \{1, 3\}, \{2, 3\}$. Our first main result concerns its positive definiteness.

Theorem 2. *The n body matrix $B^{(n)}(\alpha, r)$ is positive semi-definite for $\alpha \in \mathbb{R}_{\geq 0}^n$ and $r \in \mathcal{K}^{(n)}$, and is positive definite if and only if $\alpha \in \mathbb{R}_{> 0}^n$ and $r \in \mathcal{K}_0^{(n)}$, i.e., the masses are all positive and situated in a nonsingular point configuration.*

²It would be an interesting project to extend our results to other norms and, more generally, to Riemannian manifolds.

³In general, $\mathbb{R}_{\geq 0}^k = \{\mathbf{x} = (x_1, x_2, \dots, x_k)^T \mid x_i \geq 0, i = 1, \dots, k\}$ will denote the nonnegative orthant and $\mathbb{R}_{> 0}^k = \{\mathbf{x} = (x_1, x_2, \dots, x_k)^T \mid x_i > 0, i = 1, \dots, k\}$ the positive orthant of \mathbb{R}^k .

Consequently, for each fixed value of the mass parameters $\alpha \in \mathbb{R}_{>0}^n$, the n body matrix $B^{(n)}(\alpha, r)$ defines a Riemannian metric on the interior of the Euclidean distance cone. This result implies that the radial component of the quantum n body Schrödinger operator derived in [15] is the corresponding elliptic Laplace–Beltrami operator on the Riemannian manifold $\mathcal{K}_0^{(n)}$. The underlying Riemannian geometry of $\mathcal{K}_0^{(n)}$ is not well understood.

The determinant of the n body matrix

$$\Delta^{(n)} = \Delta^{(n)}(\alpha, r) = \det B^{(n)}(\alpha, r) \quad (7)$$

will be called the n body determinant. For example, a short computation based on (6) shows that the 3 body determinant can be written in the following factored form:

$$\begin{aligned} \Delta^{(3)} = \det B^{(3)} = & -2(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) (\alpha_3 r_{12}^2 + \alpha_2 r_{13}^2 + \alpha_1 r_{23}^2) \\ & (r_{12}^4 + r_{13}^4 + r_{23}^4 - 2r_{12}^2 r_{13}^2 - 2r_{12}^2 r_{23}^2 - 2r_{13}^2 r_{23}^2). \end{aligned} \quad (8)$$

Two important things to notice: ignoring the initial numerical factor, the first factor is the elementary symmetric polynomial of degree $n - 1 = 2$ in the mass parameters $\alpha_i = 1/m_i$; further, the final polynomial factor is purely *geometric*, meaning that it is independent of the mass parameters, and so only depends on the configuration of their locations through their interpoint distances. Positive definiteness of $B^{(3)}$ implies $\Delta^{(3)} > 0$ for nonsingular (i.e., non-collinear) configurations. In view of the sign of the initial numerical factor, this clearly implies the final geometrical factor is strictly negative on such configurations, a fact that is not immediately evident and in fact requires that the r_{ij} 's be interpoint distances satisfying the triangle inequalities, i.e., $r = (r_{12}, r_{13}, r_{23}) \in \mathcal{K}_0^{(3)}$; indeed, this factor is obviously positive for some non-geometrical values of the r_{ij} 's. Similar factorizations were found in [15] for the cases $n = 2, 3, 4$, and, in the case of equal masses, $n = 5, 6$, via symbolic calculations using both MATHEMATICA and MAPLE.

The geometrical factors that appear in each of these computed factorizations are, in fact, well known, and equal to the *Cayley–Menger determinant* of order n , a quantity that arises in the very first paper of Arthur Cayley, [2], written before he turned 20 and, apparently, inspired by reading Lagrange and Laplace! In this paper, Cayley uses the relatively new theorem that the determinant (a quantity he calls “tolerably known”) of the product of two matrices is the product of their determinants in order to solve the problem of finding the algebraic condition (or syzygy) relating the interpoint distances among singular configurations of 5 points in a three-dimensional space, as well as 4 coplanar points and 3 collinear points, each of which is characterized by the vanishing of their respective Cayley–Menger determinant. A century later, in the hands of Karl Menger, this determinantal quantity laid the foundation of the active contemporary field of distance geometry, [1, 13]; see also [16] for further results and extensions to other geometries.

The order n Cayley–Menger matrix is the symmetric matrix

$$C^{(n)} = C^{(n)}(r) = \begin{pmatrix} 0 & r_{12}^2 & r_{13}^2 & \cdots & r_{1n}^2 & 1 \\ r_{12}^2 & 0 & r_{23}^2 & \cdots & r_{2n}^2 & 1 \\ r_{13}^2 & r_{23}^2 & 0 & \cdots & r_{3n}^2 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ r_{1n}^2 & r_{2n}^2 & r_{3n}^2 & \cdots & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{pmatrix} \quad (9)$$

of size $(n + 1) \times (n + 1)$ involving the same interpoint distances (2). The Cayley–Menger matrix is a bordered version of the *Euclidean (squared) distance matrix*⁴

$$D^{(n)} = D^{(n)}(r) = \begin{pmatrix} 0 & r_{12}^2 & r_{13}^2 & \cdots & r_{1n}^2 \\ r_{12}^2 & 0 & r_{23}^2 & \cdots & r_{2n}^2 \\ r_{13}^2 & r_{23}^2 & 0 & \cdots & r_{3n}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{1n}^2 & r_{2n}^2 & r_{3n}^2 & \cdots & 0 \end{pmatrix} \quad (10)$$

of importance in a wide range of applications, including statistics, crystallography, protein structure, machine learning, sensor networks, acoustics, psychometrics, and elsewhere, [5, 13].

The order n Cayley–Menger determinant is defined as the determinant of the Cayley–Menger matrix:

$$\delta^{(n)}(r) = \det C^{(n)}(r). \quad (11)$$

For example, when $n = 3$,

$$C^{(3)} = \begin{pmatrix} 0 & r_{12}^2 & r_{13}^2 & 1 \\ r_{12}^2 & 0 & r_{23}^2 & 1 \\ r_{13}^2 & r_{23}^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad (12)$$

$$\delta^{(3)} = \det C^{(3)} = r_{12}^4 + r_{13}^4 + r_{23}^4 - 2r_{12}^2 r_{13}^2 - 2r_{12}^2 r_{23}^2 - 2r_{13}^2 r_{23}^2,$$

which coincides with the geometric polynomial factor in (8). Keep in mind that both the n body and Cayley–Menger determinants are homogeneous polynomials in the squared distances r_{ij}^2 . The general form of Cayley’s result can be stated as follows.

Theorem 3. *A collection of (squared) interpoint distances $r \in \mathcal{K}^{(n)}$ comes from a singular point configuration, so $r \in \partial\mathcal{K}^{(n)}$, if and only if the corresponding Cayley–Menger determinant vanishes: $\delta^{(n)}(r) = 0$.*

⁴Even though its entries are the *squared* Euclidean distances, the potentially confusing terminology “Euclidean distance matrix” is almost universal throughout the literature, and plays the preponderant role in all applications.

In other words, the boundary of the Euclidean distance cone is contained in the subvariety determined by the vanishing of a single polynomial — the Cayley–Menger determinant. Thus, Theorem 2 implies that the n body determinant, and hence the underlying metric, degenerates if and only if the Cayley–Menger determinant vanishes, and hence the masses are positioned on a lower dimensional affine subspace. See below for a modern version of Cayley’s original proof.

Remark: When $n = 3$, the Cayley–Menger determinant (12) factorizes:

$$\delta^{(3)}(r) = -(r_{12} + r_{13} + r_{23})(-r_{12} + r_{13} + r_{23})(r_{12} - r_{13} + r_{23})(r_{12} + r_{13} - r_{23}), \quad (13)$$

which, except for the sign and a factor of $\frac{1}{16}$, is *Heron’s formula* for the squared area of a triangle. On the other hand, when $n \geq 4$, the Cayley–Menger determinant is an irreducible polynomial in the distance variables r_{ij} ; see [3, 9], keeping in mind that their n is our $n - 1$.

We further note that the relation to Heron’s formula is no accident. Indeed, if r_{ij} are the interpoint distances between n points $\mathbf{p}_1, \dots, \mathbf{p}_n \in \mathbb{R}^d$, then, by a theorem of Menger, [14] — see also (**) in Section 40 of [1] — the Cayley–Menger determinant $\delta^{(n)}(r)$ equals $(-1)^n 2^{n-1} (n - 1)!^2$ times the squared volume of the n -simplex formed by these points.

Based on their above-mentioned symbolic calculations, Miller, Turbiner, and Escobar–Ruiz, [15], conjectured the following result.

Theorem 4. *The n body determinant factors,*

$$\Delta^{(n)}(\alpha, r) = e_{n-1}(\alpha) \delta^{(n)}(r) \sigma^{(n)}(\alpha, r), \quad (14)$$

into the product of the elementary symmetric polynomial $e_{n-1}(\alpha)$ of order $n - 1$ in the mass parameters $\alpha = (\alpha_1, \dots, \alpha_n)$ times the Cayley–Menger determinant $\delta^{(n)}(r)$ of order n depending only on the interpoint distances $r = (\dots r_{ij} \dots)$ times a polynomial $\sigma^{(n)}$ that depends upon both the α_i and the r_{ij} .

Unfortunately, our proof of Theorem 4 is purely existential; it does not yield an independent formula for the non-geometrical factor, other than the obvious $\sigma^{(n)}(\alpha, r) = \Delta^{(n)}(\alpha, r) / (e_{n-1}(\alpha) \delta^{(n)}(r))$. Thus, the problem of characterizing and understanding the non-geometric factor $\sigma^{(n)}$ remains open, although interesting formulas involving geometric quantities — a polynomial involving sums of the squared volumes of the subsimplices determined by the point configuration weighted by the mass parameters — are known when n is small, [15], where it is conjectured this will hold in general.

We shall, in fact, prove Theorem 4 as a special case of a much more general determinantal factorization, Theorem 10, which replaces the squared distances r_{ij}^2 by n^2 arbitrary elements $s_{i,j}$, not necessarily satisfying either $s_{i,j} = s_{j,i}$ or $s_{i,i} = 0$. We shall also generalize the dependence on the inverse mass parameters α_i using the following elementary observation.

Lemma 5. Given the parameters $\alpha_1, \dots, \alpha_n$, consider the following $(n+1) \times (n+1)$ matrix

$$C_A = \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 & 1 \\ 0 & \alpha_2 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \alpha_n & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix}. \quad (15)$$

Then,

$$\det C_A = -e_{n-1}(\alpha). \quad (16)$$

To establish this formula, one can simply expand the determinant along its last row. Thus, the two initial factors in the n body determinant factorization formula (14) are both realized by determinants of $(n+1) \times (n+1)$ -matrices whose final row and column are of a very particular form. In our further generalization of the n body determinant factorization formula (14), we will replace the upper left $n \times n$ block in (9) by a general matrix depending on n^2 arbitrary elements $s_{i,j}$ and the upper left $n \times n$ block in (15) by a general matrix depending on an additional n^2 arbitrary elements $t_{k,l}$. See below for details.

Combining Theorems 2 and 4 allows us to resolve another conjecture in [15], that for nonsingular point configurations, the mass-dependent factor $\sigma^{(n)}(\alpha, r)$ is of one sign.

Theorem 6. All factors in the n body determinant factorization (14) are of one sign, namely

$$\Delta^{(n)}(\alpha, r) > 0, \quad e_{n-1}(\alpha) > 0, \quad (-1)^n \delta^{(n)}(r) > 0, \quad (-1)^n \sigma^{(n)}(\alpha, r) > 0, \quad (17)$$

provided the mass parameters $\alpha_i = 1/m_i$ are positive, so $\alpha \in \mathbb{R}_{>0}^n$, and their positions \mathbf{p}_i do not all lie in an affine subspace of dimension $\leq n-2$, so $r \in \mathcal{K}_0^{(n)}$.

Proof: Since the determinant of a positive definite matrix is positive, [11, 17], Theorem 2 immediately implies the first inequality in (17). The positivity of the elementary symmetric polynomial is trivial. The sign of the Cayley–Menger determinant $\delta^{(n)}(r)$ on nonsingular point configurations is well known; see (30) below for a proof. The final inequality follows immediately from the factorization (14). Q.E.D.

2. Positive Definiteness.

In this section, we present a proof of Theorem 2, as well as the known results concerning the vanishing and the sign of the Cayley–Menger determinant. These results will, modulo the proof of the Factorization Theorem 4, establish the Sign Theorem 6. We begin by introducing an important collection of matrices that are closely related to the Cayley–Menger matrices.

Given a point configuration $\mathbf{p}_1, \dots, \mathbf{p}_n \in \mathbb{R}^d$, for each $k = 1, \dots, n$, consider the $(n-1) \times (n-1)$ matrix $M_k^{(n)} = M_k^{(n)}(r)$ with entries

$$\begin{aligned} m_{ij} &= 2(\mathbf{p}_i - \mathbf{p}_k) \cdot (\mathbf{p}_j - \mathbf{p}_k) = \|\mathbf{p}_i - \mathbf{p}_k\|^2 + \|\mathbf{p}_j - \mathbf{p}_k\|^2 - \|\mathbf{p}_i - \mathbf{p}_j\|^2 \\ &= r_{ik}^2 + r_{jk}^2 - r_{ij}^2, \quad i, j \neq k, \end{aligned} \quad (18)$$

where the indices i, j run from 1 to n omitting k , and where $r_{ii} = 0$. Note that its diagonal entries are $m_{ii} = 2r_{ik}^2$. Thus, in particular, $M_n^{(n)}(r)$ is explicitly given by

$$\begin{pmatrix} 2r_{1n}^2 & r_{1n}^2 + r_{2n}^2 - r_{12}^2 & r_{1n}^2 + r_{3n}^2 - r_{13}^2 & \dots & r_{1n}^2 + r_{n-1,n}^2 - r_{1,n-1}^2 \\ r_{1n}^2 + r_{2n}^2 - r_{12}^2 & 2r_{2n}^2 & r_{2n}^2 + r_{3n}^2 - r_{23}^2 & \dots & r_{2n}^2 + r_{n-1,n}^2 - r_{2,n-1}^2 \\ r_{1n}^2 + r_{3n}^2 - r_{13}^2 & r_{2n}^2 + r_{3n}^2 - r_{23}^2 & 2r_{3n}^2 & \dots & r_{3n}^2 + r_{n-1,n}^2 - r_{3,n-1}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{1n}^2 + r_{n-1,n}^2 - r_{1,n-1}^2 & r_{2n}^2 + r_{n-1,n}^2 - r_{2,n-1}^2 & r_{3n}^2 + r_{n-1,n}^2 - r_{3,n-1}^2 & \dots & 2r_{n-1,n}^2 \end{pmatrix}, \quad (19)$$

with evident modifications for the general case $M_k^{(n)}(r)$. For example, when $n = 3$,

$$\begin{aligned} M_1^{(3)} &= \begin{pmatrix} 2r_{12}^2 & r_{12}^2 + r_{13}^2 - r_{23}^2 \\ r_{12}^2 + r_{13}^2 - r_{23}^2 & 2r_{13}^2 \end{pmatrix}, \\ M_2^{(3)} &= \begin{pmatrix} 2r_{12}^2 & r_{12}^2 + r_{23}^2 - r_{13}^2 \\ r_{12}^2 + r_{23}^2 - r_{13}^2 & 2r_{23}^2 \end{pmatrix}, \\ M_3^{(3)} &= \begin{pmatrix} 2r_{13}^2 & r_{13}^2 + r_{23}^2 - r_{12}^2 \\ r_{13}^2 + r_{23}^2 - r_{12}^2 & 2r_{23}^2 \end{pmatrix}. \end{aligned} \quad (20)$$

The associated quadratic forms

$$q_k(\mathbf{x}) = \mathbf{x}_k^T M_k^{(n)} \mathbf{x}_k = \sum_{i,j \neq k} (r_{ik}^2 + r_{jk}^2 - r_{ij}^2) x_i x_j, \quad (21)$$

where \mathbf{x}_k is obtained from $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ by omitting the entry x_k , first appear in a paper by Fréchet, [7], in the special case $n = 4$, and, in general, in a supplement written by Schoenberg, [18], who uses them to prove the fundamental Theorem 9 below. Further, as shown by Schoenberg, the quadratic form (21) equals the negative of the quadratic form

$$q(\mathbf{x}) = \mathbf{x}^T D^{(n)} \mathbf{x} = \sum_{i,j=1}^n r_{ij}^2 x_i x_j \quad (22)$$

associated with the Euclidean distance matrix (10) when restricted to the hyperplane

$$\mathcal{H} = \{\mathbf{x} \in \mathbb{R}^n \mid x_1 + \dots + x_n = 0\},$$

so that

$$q(\mathbf{x}) = -q_k(\mathbf{x}) \quad \text{whenever} \quad x_1 + \cdots + x_n = 0, \quad k = 1, \dots, n. \quad (23)$$

The matrices $M_k^{(n)}$ explicitly appear in a slightly later but independently written paper by Gale Young and Alston Householder, [23], who were motivated by questions arising in psychometrics; their paper contains a much simplified proof of Schoenberg's Theorem 9 characterizing the Euclidean distance cone, as well as establishing their explicit connection with the Cayley–Menger matrices; see Theorem 7 below. Although they appear often in the distance geometry literature, we have been unable to find an actual name for them. Schoenberg's quadratic form identity (23) suggests calling them *reduced Euclidean distance matrices*; alternatively, one could name them after Fréchet, Schoenberg, Young, and Householder, giving the appealing (slightly permuted) acronym FYSH matrix. However, in view of later developments, we choose to follow the former more descriptive nomenclature here.

The following result shows that the reduced Euclidean distance determinants of order n (i.e. the determinants of the reduced Euclidean distance matrices) are independent of k and, up to sign, are equal to the order n Cayley–Menger determinant.

Theorem 7. *The Cayley–Menger determinant coincides, up to sign, with every reduced Euclidean distance determinant of the same order:*

$$\delta^{(n)}(r) = (-1)^n \det M_k^{(n)}(r), \quad (24)$$

for any value of $k = 1, \dots, n$.

Proof: Our proof follows [23]. Let us concentrate on the case $k = n$, noting that all formulas are invariant under permutations of the points and, hence, it suffices to establish this particular case. We perform the following elementary row and column operations on the Cayley–Menger matrix $C^{(n)}$, cf. (9), that do not affect its determinant. We subtract its n -th row from the first through $(n - 1)$ -st rows, and then subtract its n -th column, which has not changed, from the resulting first through $(n - 1)$ -st columns. The result is the $(n + 1) \times (n + 1)$ -matrix⁵

$$\tilde{C}^{(n)} = \begin{pmatrix} -M_n^{(n)} & * & \mathbf{0} \\ * & 0 & 1 \\ \mathbf{0} & 1 & 0 \end{pmatrix}$$

where the upper left $(n - 1) \times (n - 1)$ block is $-M_n^{(n)}$, the n -th row and column of $\tilde{C}^{(n)}$ are the same as the n -th row and column of $C^{(n)}$ (the stars indicate the entries, whose precise values are not needed), and the last row and column have all zeros except for their n -th entry. We can further subtract suitable multiples of the last row

⁵The bold face $\mathbf{0}$'s indicate row or column vectors of 0's, while the ordinary 0's are scalars.

and column from the first $n - 1$ rows and columns in order to annihilate their n -th entries, leading to

$$\widehat{C}^{(n)} = \begin{pmatrix} -M_n^{(n)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 1 \\ \mathbf{0} & 1 & 0 \end{pmatrix}.$$

It is then easy to see that

$$\delta^{(n)} = \det C^{(n)} = \det \widetilde{C}^{(n)} = \det \widehat{C}^{(n)} = (-1)^n \det M_n^{(n)}. \quad \text{Q.E.D.}$$

Now, dropping the $^{(n)}$ superscript and $_n$ subscript from here on to avoid cluttering the formulas, the first formula in (18) implies that, up to a factor of 2, the reduced Euclidean distance matrix $M = M_n^{(n)}$ is a *Gram matrix*, [13, 17], namely,

$$M = 2A^T A, \quad \text{where} \quad A = (\mathbf{p}_1 - \mathbf{p}_n, \dots, \mathbf{p}_{n-1} - \mathbf{p}_n) \quad (25)$$

is the $d \times (n - 1)$ matrix with the indicated columns. In accordance with our choice of nomenclature, we will refer to A as the *reduced point configuration matrix* since it agrees with the $d \times n$ *point configuration matrix* $P = (\mathbf{p}_1, \dots, \mathbf{p}_n)$, whose columns are the points or mass locations, when restricted to the hyperplane \mathcal{H} :

$$A\widehat{\mathbf{x}} = P\mathbf{x} \quad \text{when} \quad x_1 + \dots + x_n = 0, \quad \text{where} \quad \begin{aligned} \widehat{\mathbf{x}} &= (x_1, x_2, \dots, x_{n-1})^T, \\ \mathbf{x} &= (x_1, x_2, \dots, x_n)^T. \end{aligned} \quad (26)$$

We know that $\delta^{(n)} = (-1)^n \det M = 0$ if and only if $\ker M \neq \{\mathbf{0}\}$, meaning there exists $\mathbf{0} \neq \widehat{\mathbf{x}} \in \mathbb{R}^{n-1}$ such that

$$M\widehat{\mathbf{x}} = \mathbf{0}. \quad (27)$$

Multiplying the left hand side by $\widehat{\mathbf{x}}^T$ and using (25), we find

$$\widehat{\mathbf{x}}^T M \widehat{\mathbf{x}} = 2\widehat{\mathbf{x}}^T A^T A \widehat{\mathbf{x}} = 2\|A\widehat{\mathbf{x}}\|^2 \geq 0 \quad \text{for all} \quad \widehat{\mathbf{x}} \in \mathbb{R}^{n-1}. \quad (28)$$

This identity implies that the reduced Euclidean distance matrix M is positive semi-definite, and is positive definite if and only if the reduced point configuration matrix has trivial kernel: $\ker A = \{\mathbf{0}\}$. Consequently, (27) holds if and only if

$$A\widehat{\mathbf{x}} = \mathbf{0}. \quad (29)$$

Since we assumed $\widehat{\mathbf{x}} \neq \mathbf{0}$, this is equivalent to the linear dependence of the columns of A , meaning the vectors $\mathbf{p}_1 - \mathbf{p}_n, \dots, \mathbf{p}_{n-1} - \mathbf{p}_n$ span a subspace of dimension $\leq n - 2$, which requires that $\mathbf{p}_1, \dots, \mathbf{p}_n$ lie in an affine subspace of dimension $\leq n - 2$, i.e., they form a singular point configuration. We conclude that this occurs if and only if the Cayley–Menger determinant vanishes, which thus establishes Cayley’s Theorem 3.

Moreover, positive (semi-)definiteness of M implies non-negativity of its determinant and, hence, by (24),

$$(-1)^n \delta^{(n)}(r) \geq 0 \quad \text{with equality if and only if} \quad \ker A \neq \{\mathbf{0}\}, \quad (30)$$

thus establishing the second to last inequality in (17). Replacing \mathbf{p}_n by \mathbf{p}_k does not change the argument, and hence we have established the following known result.

Theorem 8. *Given a point configuration $\mathbf{p}_1, \dots, \mathbf{p}_n \subset \mathbb{R}^d$, and any $1 \leq k \leq n$, the corresponding reduced Euclidean distance matrix $M_k^{(n)}(r)$ depending on the interpoint distances $r_{ij} = \|\mathbf{p}_i - \mathbf{p}_j\|$ is positive semi-definite, and is positive definite if and only if the point configuration is nonsingular.*

The condition that the reduced Euclidean distance matrix $M_k^{(n)}$ be positive semi-definite is, in fact, both necessary and sufficient in order that a given collection of nonnegative numbers $r_{ij} \geq 0$ be the interpoint distances of a bona fide point configuration, i.e. satisfy (2) for some $\mathbf{p}_1, \dots, \mathbf{p}_n \in \mathbb{R}^d$. This theorem is due to Schoenberg, [18], and, independently, to Young and Householder, [23], whose simple proof is, for completeness, included here. This key result, in addition to its foundational role in distance geometry, [13], directly inspired the powerful and widely used method of contemporary statistical analysis known as *multidimensional scaling*, [6].

Theorem 9. *Given $r = (\dots r_{ij} \dots) \in \mathbb{R}_{\geq 0}^{\binom{n}{2}}$, there exists $\mathbf{p}_1, \dots, \mathbf{p}_n \subset \mathbb{R}^d$ for some $d > 0$ such that $r_{ij} = \|\mathbf{p}_i - \mathbf{p}_j\|$ if and only if for any (and hence all) $1 \leq k \leq n$, the corresponding reduced Euclidean distance matrix $M_k^{(n)}(r)$ is positive semi-definite. The point configuration $\mathbf{p}_1, \dots, \mathbf{p}_n$ is unique up to a Euclidean transformation of \mathbb{R}^d . The minimal d for which this occurs is given by the rank of $M_k^{(n)}(r)$.*

Proof: Let us without loss of generality set $k = n$. In view of Theorem 8, we need only prove that, given $r \in \mathbb{R}_{\geq 0}^{\binom{n}{2}}$, if the corresponding reduced Euclidean distance matrix $M_n^{(n)}(r)$ is positive semi-definite, then we can find a point configuration for which the r_{ij} are the interpoint distances. Consider its spectral factorization

$$M_n^{(n)} = Q D Q^T, \quad (31)$$

in which Q is an $(n-1) \times (n-1)$ orthogonal matrix and D is a diagonal matrix of the same size with the eigenvalues λ_i , $i = 1, \dots, n-1$, on its diagonal and which, by positive semi-definiteness, are all ≥ 0 ; see [17] for linear-algebraic details. Then set

$$A = \frac{1}{\sqrt{2}} \sqrt{D} Q^T, \quad (32)$$

where \sqrt{D} is the diagonal matrix whose entries are $\sigma_i = \sqrt{\lambda_i}$. Equation (31) immediately implies that A satisfies the Gram equation (25) and, hence, defines a corresponding reduced point configuration matrix, namely, its i -th column gives $\mathbf{p}_i - \mathbf{p}_n$, where the final point \mathbf{p}_n can be arbitrarily specified. Note also that the (nonzero) σ_i are the singular values of the reduced point configuration matrix A . The proof of the remaining statements is left to the reader. Q.E.D.

Theorem 9 identifies the set of positive semi-definite reduced Euclidean distance matrices with the Euclidean distance cone $\mathcal{K}^{(n)} \subset \mathbb{R}^{\binom{n}{2}}$. When $n = 3$, in view of Heron's formula (13), positive semi-definiteness reduces to the triangle inequalities among the three interpoint distances. However, when $n > 3$, positive semi-definiteness imposes additional constraints on the distances beyond those required by the triangle inequalities among each triple of points in the configuration.

Remark: As a corollary of Schoenberg's identity (23), one sees that a Euclidean distance matrix $D^{(n)} = D^{(n)}(r)$, as in (10), arises from a bona fide point configuration if and only if

$$\mathbf{x}^T D^{(n)} \mathbf{x} \leq 0 \quad \text{for all } \mathbf{x} = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n \quad \text{such that } x_1 + \dots + x_n = 0. \quad (33)$$

Let us next prove Theorem 2 establishing the positive definiteness of the n body matrix for nonsingular point configurations. Observe that the n body matrix is linear in the inverse mass parameters α_i . Moreover, if we let the mass parameter $m_k = 1$ and send all other $m_j \rightarrow \infty$, or, equivalently, $\alpha_k = 1$ and $\alpha_j = 0$ for $j \neq k$, then the n body matrix $B = B^{(n)}$ reduces to the matrix $B_k^{(n)} = \widehat{M}_k^{(n)}$ obtained by placing the (i, j) -th entry of the k -th reduced Euclidean distance matrix $M_k^{(n)}$ based at the point \mathbf{p}_k in the position labelled by the unordered index pairs $\{i, k\}$ and $\{j, k\}$, and setting all other entries, i.e., those with one or both labels not containing k , to zero. We have thus shown that the n body matrix decomposes into a linear combination thereof:

$$B^{(n)}(\alpha, r) = \sum_{k=1}^n \alpha_k \widehat{M}_k^{(n)}(r), \quad (34)$$

since each entry of $B^{(n)}$ is linear in the α_i 's. For example when $n = 3$, we write (6) as

$$B^{(3)} = \alpha_1 \widehat{M}_1^{(3)} + \alpha_2 \widehat{M}_2^{(3)} + \alpha_3 \widehat{M}_3^{(3)} = \alpha_1 \begin{pmatrix} 2r_{12}^2 & r_{12}^2 + r_{13}^2 - r_{23}^2 & 0 \\ r_{12}^2 + r_{13}^2 - r_{23}^2 & 2r_{13}^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} +$$

$$\alpha_2 \begin{pmatrix} 2r_{12}^2 & 0 & r_{12}^2 + r_{23}^2 - r_{13}^2 \\ 0 & 0 & 0 \\ r_{12}^2 + r_{23}^2 - r_{13}^2 & 0 & 2r_{23}^2 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2r_{13}^2 & r_{13}^2 + r_{23}^2 - r_{12}^2 \\ 0 & r_{13}^2 + r_{23}^2 - r_{12}^2 & 2r_{23}^2 \end{pmatrix},$$

and recognize the nonzero entries of its three matrix summands as forming order 3 reduced Euclidean distance matrices (20).

Now, to prove positive definiteness of $B = B^{(n)}$, we need to show positivity of the associated quadratic form:

$$\mathbf{z}^T B \mathbf{z} > 0 \quad \text{for all} \quad \mathbf{0} \neq \mathbf{z} = (\dots z_{\{i,j\}} \dots)^T \in \mathbb{R}^{\binom{n}{2}}. \quad (35)$$

Using (34), we can similarly expand this quadratic form:

$$\mathbf{z}^T B \mathbf{z} = \sum_{k=1}^n \alpha_k \mathbf{z}^T \widehat{M}_k^{(n)} \mathbf{z} = \sum_{k=1}^n \alpha_k \mathbf{z}_k^T M_k^{(n)} \mathbf{z}_k, \quad (36)$$

where

$$\mathbf{z}_k = (z_{\{1,k\}}, \dots, z_{\{k-1,k\}}, z_{\{k+1,k\}}, \dots, z_{\{n,k\}})^T \in \mathbb{R}^{n-1} \quad \text{for} \quad k = 1, \dots, n,$$

so $z_{\{k,k\}}$ is omitted from the vector, keeping in mind that the indices are symmetric, so $z_{\{i,j\}} = z_{\{j,i\}}$. The final identity in (36) comes from eliminating all the terms involving the zero entries in $\widehat{M}_k^{(n)}$. Now, Theorem 8 implies positive semi-definiteness of the reduced Euclidean distance matrices $M_k^{(n)}$ and, hence

$$\mathbf{z}_k^T M_k^{(n)} \mathbf{z}_k \geq 0, \quad (37)$$

which, by (36), establishes positive semi-definiteness of the n body matrix. Moreover, if the point configuration $\mathbf{p}_1, \dots, \mathbf{p}_n$ is nonsingular, Theorem 8 implies positive definiteness of the reduced Euclidean distance matrices, and hence (37) becomes an equality if and only if $\mathbf{z}_k = \mathbf{0}$. Moreover, if $\mathbf{z} \neq \mathbf{0} \in \mathbb{R}^{n(n-1)/2}$, then at least one $\mathbf{z}_k \neq \mathbf{0} \in \mathbb{R}^{n-1}$, and hence at least one of the summands on the right hand side of (36) is strictly positive, which establishes the desired inequality (35), thus proving positive definiteness of the n body matrix. On the other hand, if the configuration is singular, the corresponding Cayley–Menger determinant vanishes, and so the Factorization Theorem 4, to be proved below, implies that the n body determinant also vanishes, which means that the n body matrix cannot be positive definite. *Q.E.D.*

3. Factorization of Certain Determinants.

In order to prove the Factorization Theorem 4, we will, in fact, significantly generalize it. A proof of the generalization will establish the desired result as a special case.

Notation: For each nonnegative $m \in \mathbb{Z}$, we let $[m]$ be the set $\{1, 2, \dots, m\}$.

Let us define a class of matrices that includes the Cayley–Menger matrix $C^{(n)}$ in (9) and the matrix C_A in (15).

Let \mathcal{R} be a ring. Fix an integer $n \geq 1$. If $H = (h_{i,j})_{1 \leq i \leq n, 1 \leq j \leq n} \in \mathcal{R}^{n \times n}$ is an $n \times n$ -matrix over \mathcal{R} , then we define an $(n+1) \times (n+1)$ -matrix $C_H \in \mathcal{R}^{(n+1) \times (n+1)}$

by

$$C_H = \left(\begin{array}{l} h_{i,j}, \quad \text{if } i, j \in [n]; \\ 1, \quad \text{if exactly one of } i, j \text{ belongs to } [n]; \\ 0, \quad \text{if } i = j = n + 1 \end{array} \right)_{1 \leq i \leq n+1, 1 \leq j \leq n+1}$$

$$= \begin{pmatrix} h_{1,1} & h_{1,2} & \cdots & h_{1,n} & 1 \\ h_{2,1} & h_{2,2} & \cdots & h_{2,n} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{n,1} & h_{n,2} & \cdots & h_{n,n} & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix}. \quad (38)$$

Observe that our earlier matrices $C^{(n)} = C_R$, as in (9), and C_A , as in (15), are both of this form based, respectively, on the $n \times n$ matrices

$$R = \begin{pmatrix} 0 & r_{12}^2 & r_{13}^2 & \cdots & r_{1n}^2 \\ r_{12}^2 & 0 & r_{23}^2 & \cdots & r_{2n}^2 \\ r_{13}^2 & r_{23}^2 & 0 & \cdots & r_{3n}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{1n}^2 & r_{2n}^2 & r_{3n}^2 & \cdots & 0 \end{pmatrix}, \quad A = \begin{pmatrix} \alpha_1 & 0 & 0 & \cdots & 0 \\ 0 & \alpha_2 & 0 & \cdots & 0 \\ 0 & 0 & \alpha_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_n \end{pmatrix}, \quad (39)$$

the former being the Euclidean distance matrix (10).

We will work in the polynomial ring

$$\mathcal{R} = \mathbb{Z} [\{s_{i,j} \mid i, j \in [n]\} \cup \{t_{k,l} \mid k, l \in [n]\}], \quad (40)$$

consisting of polynomials with integer coefficients depending on the $n^2 + n^2 = 2n^2$ independent variables $s_{i,j}, t_{k,l}$. Define the corresponding pair of $n \times n$ -matrices

$$S = (s_{i,j})_{1 \leq i \leq n, 1 \leq j \leq n} \in \mathcal{R}^{n \times n}, \quad T = (t_{i,j})_{1 \leq i \leq n, 1 \leq j \leq n} \in \mathcal{R}^{n \times n}, \quad (41)$$

which we use to construct the $(n + 1) \times (n + 1)$ matrices C_S and C_T via (38).

Next, let E be the set of all 2-element subsets of $[n]$; we regard these subsets as unordered pairs of distinct elements of $[n]$. Note that $|E| = n(n - 1) / 2$. Our generalization of the n body matrix will be the matrix $W_{S,T} \in \mathcal{R}^{E \times E}$ — that is, a matrix whose rows and columns are indexed by elements of E — whose entries are given by

$$w_{\{i,j\},\{k,l\}} = (t_{j,k} + t_{i,l} - t_{i,k} - t_{j,l}) (s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l}). \quad (42)$$

It is easy to see that (42) is well defined for any $\{i, j\}, \{k, l\} \in E$, since the right hand side is unchanged when i is switched with j , and is also unchanged when k is switched with l . We also remark that the factor $s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l}$ on the right hand side of (42) can be rewritten as

$$\det \begin{pmatrix} s_{j,l} & s_{j,k} & 1 \\ s_{i,l} & s_{i,k} & 1 \\ 1 & 1 & 0 \end{pmatrix} = \det (C_{S[j,i|l,k]}), \quad \text{where} \quad S[j, i \mid l, k] = \begin{pmatrix} s_{j,l} & s_{j,k} \\ s_{i,l} & s_{i,k} \end{pmatrix},$$

and similarly for the first factor $t_{j,k} + t_{i,l} - t_{i,k} - t_{j,l}$. Thus, each entry of $W_{S,T}$ is the product of the determinants of a pair of 3×3 matrices that also have our basic form (38).

Since $W_{S,T}$ is a square matrix of size $|E| \times |E|$, it has a determinant $\det W_{S,T} \in \mathcal{R}$. The main result of this sections is its divisibility:

Theorem 10. *We have $(\det C_S) (\det C_T) \mid \det W_{S,T}$ in \mathcal{R} .*

Notice that Theorem 10 is a divisibility in \mathcal{R} . Thus, the quotient is a polynomial $Z_{S,T} = \det W_{S,T} / (\det C_S \det C_T)$, with integer coefficients, in the independent variables $s_{i,j}, t_{i,j}$. Thus,

$$\det W_{S,T} = (\det C_S) (\det C_T) Z_{S,T}. \quad (43)$$

Observe that, whereas the left hand side of (43) depends on all $2n^2$ variables, the first factor depends only on the $s_{i,j}$ and the second factor only on the $t_{k,l}$, while the final factor is, in general, a “mixed” function of both sets of variables. As in Theorem 4, the factorization (43) is existential, and we do not have a direct formula for the mixed factor $Z_{S,T}$. Finding such a formula and giving it an algebraic or geometric interpretation is an outstanding and very interesting problem.

If we now specialize $S \mapsto R$ and $T \mapsto A$, where R, A are the matrices (39), then $W_{S,T}$ reduces to

$$W_{R,A} = -B^{(n)}, \quad (44)$$

where $B^{(n)}$ is the n body matrix defined by (4), (5), and thus the left hand side of formula (43) reduces to the n body determinant

$$\det W_{R,A} = (-1)^{\binom{n}{2}} \det B^{(n)} = (-1)^{\binom{n}{2}} \Delta^{(n)}. \quad (45)$$

On the other hand, we use (11) to identify $\det C_R$ with the Cayley–Menger determinant $\delta^{(n)}$, and (16) to identify $\det C_A$ with the negative of the elementary symmetric polynomial $-e_{n-1}(\alpha)$. Thus, the general factorization formula (43) reduces to the n body determinant factorization formula (14) where the mass-dependent factor

$$\sigma^{(n)} = (-1)^{\binom{n}{2}+1} Z_{R,A} \quad (46)$$

is identified with the corresponding reduction of the mixed factor in (43). Thus, Theorem 10 immediately implies the Factorization Theorem 4 upon specialization. Again, we do not have a direct formula for constructing either factor $Z_{S,T}$ or $Z_{R,A}$.

Our proof of Theorem 10 will rely on basic properties of UFDs (unique factorization domains), which are found in most texts on abstract algebra, e.g., [10, Section VIII.4]. We shall also use the fact that any polynomial ring (in finitely many variables) over \mathbb{Z} is a UFD. (This follows, e.g., from [10, Corollary 8.21] by induction on the number of variables.) Moreover, we shall use the fact (obvious from degree considerations) that the only units (i.e., invertible elements) of a polynomial ring are constant polynomials. Hence, a polynomial p in a polynomial ring $\mathbb{Z}[x_1, x_2, \dots, x_k]$ is irreducible if its *content*,

i.e., the gcd of its coefficients, is 1 and p is irreducible in the ring $\mathbb{Q}[x_1, x_2, \dots, x_k]$ (since any constant factor of p in $\mathbb{Z}[x_1, x_2, \dots, x_k]$ must divide the content of p).

Before we prove Theorem 10, we require a technical lemma:

Lemma 11. *Assume that $n > 1$. Then, $\det C_S$ is a prime element of the UFD \mathcal{R} .*

Proof of Lemma 11. Expanding $\det C_S$ as a sum over all $(n+1)!$ permutations π of $[n+1]$, we observe that the permutations π satisfying $\pi(n+1) = n+1$ give rise to summands that equal 0, whereas all the other permutations π contribute *pairwise distinct monomials* to the sum⁶. This shows that the polynomial $\det C_S$ has content 1; indeed, each of its nonzero coefficients is 1 or -1 . Moreover, it shows that $\det C_S$ is a polynomial of degree 1 in each of the indeterminates $s_{i,j}$ (not 0 because $n > 1$). Furthermore, in the expansion of $\det C_S$ into monomials, each monomial contains at most one variable from each row and at most one from each column. Thus, the same argument that is used in [4, proof of Lemma 5.12] to prove the irreducibility of $\det S$ can be used to see that $\det C_S$ is an irreducible element of the ring $\mathbb{Q}[s_{i,j} \mid i, j \in [n]]$. Hence, since $\det C_S$ has content 1, it is an irreducible element of the ring $\mathcal{R}_0 = \mathbb{Z}[s_{i,j} \mid i, j \in [n]]$ as well. Hence, $\det C_S$ is also an irreducible element of the ring \mathcal{R} (which differs from \mathcal{R}_0 merely in the introduction of n^2 new variables $t_{i,j}$, which clearly do not contribute any possible divisors to $\det C_S$). Since \mathcal{R} is a UFD, we thus conclude that $\det C_S$ is a prime element of \mathcal{R} . *Q.E.D.*

Proof of Theorem 10. If $n = 1$, then Theorem 10 is clear, since $\det C_S = -1$ and $\det C_T = -1$ in this case. Thus, without loss of generality, assume that $n > 1$.

Since \mathcal{R} is a polynomial ring over \mathbb{Z} , it is a UFD. Moreover, Lemma 11 yields that $\det C_S$ is a prime element of \mathcal{R} . Similarly, $\det C_T$ is a prime element of \mathcal{R} .

Let $\mathcal{Q} = \mathcal{R}/\det C_S$ be the quotient ring, which is an integral domain since $\det C_S$ is a prime element of \mathcal{R} . Since, by construction, $\det C_S = 0$ in \mathcal{Q} , the matrix C_S is singular over \mathcal{Q} and hence has a nontrivial kernel because \mathcal{Q} is an integral domain. In other words, there exists a nonzero vector $\mathbf{0} \neq \mathbf{x}^* = (x_1, x_2, \dots, x_n, v)^T \in \mathcal{Q}^{n+1}$ such that

$$C_S \mathbf{x}^* = \mathbf{0}. \quad (47)$$

The entries of the vector identity (47) imply⁷

$$\sum_l s_{i,l} x_l + v = 0 \quad \text{for all } i \in [n], \quad \text{and} \quad \sum_l x_l = 0. \quad (48)$$

⁶Why pairwise distinct? The monomial corresponding to such a permutation π is $\prod_{i \in [n]; \pi(i) \neq n+1} s_{i, \pi(i)}$. Knowing this monomial, we can reconstruct the value of π at the unique i satisfying $\pi(i) = n+1$ (namely, this value is the unique $k \in [n]$ for which no entry from the k -th row of S appears in the monomial), as well as the remaining values of π on $[n]$ (by inspecting the corresponding $s_{i,j}$ in the monomial), and finally the value of π at $n+1$ (as the remaining element of $[n+1]$). Thus, we can reconstruct π uniquely from this monomial.

⁷Here and in the following, “ \sum_l ” always means “ $\sum_{l=1}^n$ ”.

Given such an \mathbf{x}^* , let $\mathbf{x} = (x_1, x_2, \dots, x_n)^T \in \mathcal{Q}^n$ be the vector obtained by omitting the last entry. If $\mathbf{x} = \mathbf{0}$, then, according to the first equations in (48), this would require $v = 0$, which would contradict the fact that $\mathbf{x}^* \neq \mathbf{0}$. Thus, $\mathbf{x} \neq \mathbf{0}$, which, by the last equation in (48), implies that \mathbf{x} has at least two nonzero entries, so $x_i x_j \neq 0$ for some $i \neq j$, since \mathcal{Q} is an integral domain.

Define the vector $\mathbf{z} \in \mathcal{Q}^E$ whose entries are indexed by unordered pairs $\{i, j\} \in E$ and given by the products of distinct entries of \mathbf{x} , so

$$z_{\{i,j\}} = x_i x_j, \quad \{i, j\} \in E.$$

Hence, $\mathbf{z} \neq \mathbf{0}$ (since $x_i x_j \neq 0$ for some $i \neq j$).

Let us abbreviate $W = W_{S,T}$. We shall prove that $\mathbf{0} \neq \mathbf{z} \in \ker W$. To this end, for any $1 \leq i < j \leq n$, the $\{i, j\}$ -th entry of the vector $W\mathbf{z}$ is

$$\begin{aligned} & \sum_{\{k,l\} \in E} w_{\{i,j\},\{k,l\}} x_k x_l \\ &= \sum_{k < l} (t_{j,k} + t_{i,l} - t_{i,k} - t_{j,l}) \underbrace{(s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l})}_{\text{(by (42))}} x_k x_l \\ &= \sum_{k < l} \underbrace{(t_{j,k} + t_{i,l} - t_{i,k} - t_{j,l})}_{=(t_{j,k} - t_{i,k}) - (t_{j,l} - t_{i,l})} (s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l}) x_k x_l \\ &= \sum_{k < l} ((t_{j,k} - t_{i,k}) - (t_{j,l} - t_{i,l})) (s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l}) x_k x_l \\ &= \sum_{k < l} (t_{j,k} - t_{i,k}) (s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l}) x_k x_l - \sum_{k < l} (t_{j,l} - t_{i,l}) \underbrace{(s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l})}_{=-(s_{j,l} + s_{i,k} - s_{i,l} - s_{j,k})} \underbrace{x_k x_l}_{=x_l x_k} \\ &= \sum_{k < l} (t_{j,k} - t_{i,k}) (s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l}) x_k x_l + \sum_{k < l} (t_{j,l} - t_{i,l}) (s_{j,l} + s_{i,k} - s_{i,l} - s_{j,k}) x_l x_k \\ &= \sum_{k < l} (t_{j,k} - t_{i,k}) (s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l}) x_k x_l + \sum_{k > l} (t_{j,k} - t_{i,k}) (s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l}) x_k x_l \\ & \quad \text{(here, we switched the roles of } k \text{ and } l \text{ in the second sum)} \\ &= \sum_{k,l \in [n]} (t_{j,k} - t_{i,k}) (s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l}) x_k x_l \\ & \quad \left(\text{here, we have combined the two sums, while also including extra} \right. \\ & \quad \left. \text{terms for } k = l \text{ (which don't change the sum since they are 0)} \right) \\ &= \sum_{k \in [n]} (t_{j,k} - t_{i,k}) x_k \sum_{l \in [n]} (s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l}) x_l \\ &= \sum_{k \in [n]} (t_{j,k} - t_{i,k}) x_k \left(\underbrace{s_{j,k} \sum_{l \in [n]} x_l}_{=0} + \underbrace{\sum_{l \in [n]} s_{i,l} x_l}_{=-v} - s_{i,k} \underbrace{\sum_{l \in [n]} x_l}_{=0} - \underbrace{\sum_{l \in [n]} s_{j,l} x_l}_{=-v} \right) \\ & \quad \left(\text{here, we have used the equations in (48) on each set of terms} \right) \end{aligned}$$

$$= \sum_{k \in [n]} (t_{j,k} - t_{i,k}) x_k \underbrace{(s_{j,k}0 + (-v) - s_{i,k}0 - (-v))}_{=0} = 0.$$

Hence, $W\mathbf{z} = \mathbf{0}$. Since \mathbf{z} is a nonzero vector, this shows that W has a nontrivial kernel over \mathcal{Q} . Since \mathcal{Q} is an integral domain, we thus conclude that $\det W = 0$ in \mathcal{Q} . In other words, $\det C_S \mid \det W$. The same argument shows that $\det C_T \mid \det W$ also, since S and T play symmetric roles in the definition of the matrix W .

Finally, we note that the two prime elements $\det C_S$ and $\det C_T$ of \mathcal{R} are distinct — indeed, they are polynomials in disjoint sets of indeterminates $s_{i,j}$ and $t_{k,l}$, respectively, so they could only be equal if they were both constant, which they are not. Thus, they are coprime. Hence, an element of the UFD \mathcal{R} divisible both by $\det C_S$ and by $\det C_T$ must also be divisible by their product $\det C_S \det C_T$. Applying this to the element $\det W$ completes the proof of the General Factorization Theorem 10. Q.E.D.

A further generalization of Theorem 10 will be given in the upcoming work [8].

4. A Biquadratic Form Identity.

In this section we establish a striking identity involving the matrix $W_{S,T}$, which naturally defines a biquadratic form that factorizes over a particular pair of hyperplanes. The reduction of this formula to the n body matrix is also of note.

As above, let $W = W_{S,T} \in \mathcal{R}^{E \times E}$ be the $|E| \times |E|$ matrix whose entries are given by (42). Let \mathcal{A} be a commutative \mathcal{R} -algebra. Define the biquadratic form

$$Q_W(\mathbf{x}, \mathbf{y}) = \sum_{\{i,j\} \in E} \sum_{\{k,l\} \in E} w_{\{i,j\},\{k,l\}} x_i x_j y_k y_l, \quad (49)$$

where $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)^T$ are vectors in \mathcal{A}^n .

Theorem 12. *When $x_1 + x_2 + \dots + x_n = 0$ and $y_1 + y_2 + \dots + y_n = 0$, the biquadratic form (49) factors into a product of two elementary bilinear forms⁸ based on the matrices S, T given in (41):*

$$Q_W(\mathbf{x}, \mathbf{y}) = (\mathbf{x}^T S \mathbf{y}) (\mathbf{x}^T T \mathbf{y}). \quad (50)$$

⁸The T superscripts are transposes of column vectors, and have nothing to do with the matrix T .

Proof. We calculate

$$\begin{aligned}
 Q_W(\mathbf{x}, \mathbf{y}) &= \sum_{\underbrace{\{i,j\} \in E}_{=\sum_{i < j}} \sum_{\underbrace{\{k,l\} \in E}_{=\sum_{k < l}}} \underbrace{w_{\{i,j\},\{k,l\}}}_{(by (42))} x_i x_j y_k y_l \\
 &= \sum_{i < j} \sum_{k < l} (t_{j,k} + t_{i,l} - t_{i,k} - t_{j,l}) (s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l}) x_i x_j y_k y_l \\
 &= \sum_{i < j} \sum_{k < l} t_{j,k} (s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l}) x_i x_j y_k y_l + \underbrace{\sum_{i < j} \sum_{k < l} t_{i,l} (s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l}) x_i x_j y_k y_l}_{=\sum_{i > j} \sum_{k > l} t_{j,k} (s_{i,l} + s_{j,k} - s_{j,l} - s_{i,k}) x_j x_i y_l y_k} \\
 &\quad \underbrace{- \sum_{i < j} \sum_{k < l} t_{i,k} (s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l}) x_i x_j y_k y_l}_{=\sum_{i > j} \sum_{k < l} t_{j,k} (s_{i,k} + s_{j,l} - s_{j,k} - s_{i,l}) x_j x_i y_k y_l} - \underbrace{\sum_{i < j} \sum_{k < l} t_{j,l} (s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l}) x_i x_j y_k y_l}_{=\sum_{i < j} \sum_{k > l} t_{j,k} (s_{j,l} + s_{i,k} - s_{i,l} - s_{j,k}) x_i x_j y_l y_k} \\
 &= \sum_{i < j} \sum_{k < l} t_{j,k} (s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l}) x_i x_j y_k y_l + \sum_{i > j} \sum_{k > l} t_{j,k} \underbrace{(s_{i,l} + s_{j,k} - s_{j,l} - s_{i,k})}_{=s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l}} \underbrace{x_j x_i y_l y_k}_{=x_i x_j y_l y_k} \\
 &\quad - \sum_{i > j} \sum_{k < l} t_{j,k} \underbrace{(s_{i,k} + s_{j,l} - s_{j,k} - s_{i,l})}_{=-(s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l})} \underbrace{x_j x_i y_k y_l}_{=x_i x_j y_k y_l} - \sum_{i < j} \sum_{k > l} t_{j,k} \underbrace{(s_{j,l} + s_{i,k} - s_{i,l} - s_{j,k})}_{=-(s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l})} \underbrace{x_i x_j y_l y_k}_{=y_k y_l} \\
 &= \sum_{i < j} \sum_{k < l} t_{j,k} (s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l}) x_i x_j y_k y_l + \sum_{i > j} \sum_{k > l} t_{j,k} (s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l}) x_i x_j y_k y_l \\
 &\quad + \sum_{i > j} \sum_{k < l} t_{j,k} (s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l}) x_i x_j y_k y_l + \sum_{i < j} \sum_{k > l} t_{j,k} (s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l}) x_i x_j y_k y_l \\
 &= \sum_{i \neq j} \sum_{k \neq l} t_{j,k} (s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l}) x_i x_j y_k y_l \\
 &\quad \text{(here, we have combined all four sums into a single one)} \\
 &= \sum_{i,j} \sum_{k,l} t_{j,k} (s_{j,k} + s_{i,l} - s_{i,k} - s_{j,l}) x_i x_j y_k y_l \\
 &\quad \left(\text{here, we have inserted extraneous addends for } i = j \text{ and for } k = l, \right. \\
 &\quad \left. \text{which are 0 and therefore don't change our sum} \right) \\
 &= \underbrace{\sum_{i,j} \sum_{k,l} t_{j,k} s_{j,k} x_i x_j y_k y_l}_{=(\sum_i x_i) (\sum_{j,k,l} t_{j,k} s_{j,k} x_j y_k y_l)} + \underbrace{\sum_{i,j} \sum_{k,l} t_{j,k} s_{i,l} x_i x_j y_k y_l}_{=(\sum_{i,l} x_i s_{i,l} y_l) (\sum_{j,k} x_j t_{j,k} y_k)} \\
 &\quad - \underbrace{\sum_{i,j} \sum_{k,l} t_{j,k} s_{i,k} x_i x_j y_k y_l}_{=(\sum_l y_l) (\sum_{i,j,k} t_{j,k} s_{i,k} x_i x_j y_k)} - \underbrace{\sum_{i,j} \sum_{k,l} t_{j,k} s_{j,l} x_i x_j y_k y_l}_{=(\sum_i x_i) (\sum_{j,k,l} t_{j,k} s_{j,l} x_j y_k y_l)}
 \end{aligned}$$

$$\begin{aligned}
 &= \underbrace{\left(\sum_i x_i \right)}_{=x_1+x_2+\dots+x_n=0} \left(\sum_{j,k,l} t_{j,k} s_{j,k} x_j y_k y_l \right) + \left(\sum_{i,l} x_i s_{i,l} y_l \right) \left(\sum_{j,k} x_j t_{j,k} y_k \right) \\
 &\quad - \underbrace{\left(\sum_l y_l \right)}_{=y_1+y_2+\dots+y_n=0} \left(\sum_{i,j,k} t_{j,k} s_{i,k} x_i x_j y_k \right) - \underbrace{\left(\sum_i x_i \right)}_{=x_1+x_2+\dots+x_n=0} \left(\sum_{j,k,l} t_{j,k} s_{j,l} x_j y_k y_l \right) \\
 &= \underbrace{\left(\sum_{i,l} x_i s_{i,l} y_l \right)}_{=\mathbf{x}^T \mathbf{S} \mathbf{y}} \underbrace{\left(\sum_{j,k} x_j t_{j,k} y_k \right)}_{=\mathbf{x}^T \mathbf{T} \mathbf{y}} = (\mathbf{x}^T \mathbf{S} \mathbf{y}) \cdot (\mathbf{x}^T \mathbf{T} \mathbf{y}).
 \end{aligned}$$

Q.E.D.

Let us now specialize the identity in Theorem 12 to the n body case, so that C_S, C_T reduce, respectively, to C_A, C_R . We further set $\mathbf{x} = \mathbf{y}$ and then use (21), (22), (23), (25), (26), and (44) to obtain the following intriguing result, reminiscent of Schoenberg's formula (23) for the quadratic forms based on the (reduced) Euclidean distance matrices.

Theorem 13. *Define the quadratic forms*

$$a(\mathbf{x}) = \sum_{i=1}^n \alpha_i x_i^2 = \sum_{i=1}^n \frac{x_i^2}{m_i}, \quad p(\mathbf{x}) = \sum_{i,j} (\mathbf{p}_i \cdot \mathbf{p}_j) x_i x_j = \|P \mathbf{x}\|^2, \quad (51)$$

where $P = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n)$ is the point configuration matrix, as in (26). Given the n body matrix $B = B^{(n)}(\alpha, r)$ with entries (4), (5), define the corresponding homogeneous quartic form⁹

$$Q_B(\mathbf{x}) = \sum_{\substack{\{i,j\}, \{k,l\} \\ \{i,j\}, \{k,l\}}} b_{\{i,j\}, \{k,l\}} x_i x_j x_k x_l, \quad \mathbf{x} = (x_1, \dots, x_n)^T. \quad (52)$$

Then, when restricted to the hyperplane \mathcal{H} , the n body quartic $Q_B(\mathbf{x})$ factors as the product of the preceding quadratic forms (51):

$$Q_B(\mathbf{x}) = 2a(\mathbf{x}) p(\mathbf{x}) \quad \text{when} \quad x_1 + x_2 + \dots + x_n = 0. \quad (53)$$

Note that because $b_{\{i,j\}, \{k,l\}} = 0$ when i, j, k, l are distinct, the n body matrix $B^{(n)}$ is, in fact uniquely determined by its associated quartic form $Q_B(\mathbf{x})$.

If the point configuration is nonsingular, then, in view of (26), the right hand side of (53) is positive whenever $\mathbf{0} \neq \mathbf{x} \in \mathcal{H}$, which implies $Q_B(\mathbf{x}) > 0$ under the same conditions. However, this result does not lead to the conclusion that the n body matrix, which forms the coefficients of $Q_B(\mathbf{x})$, is itself positive definite, and hence we needed a different approach to establish this result.

⁹The sum is over all ordered pairs $(\{i, j\}, \{k, l\})$ of unordered pairs $\{i, j\}$ and $\{k, l\}$.

5. Future Directions.

As noted above, the challenge now is to determine an explicit geometrical formula for the mass-dependent factor $\sigma^{(n)}(\alpha, r)$ in the n body determinant factorization formula (14) or, more generally, the mixed factor $Z_{S,T}$ in our generalized factorization formula (43), and to ascertain its algebraic and/or geometric significance. Is there some as yet undetected interesting determinantal identity or algebraic structure, perhaps representation-theoretic, that underlies this factorization and its compound determinantal generalization found in [8]? Do the biquadratic and quartic form identities we found in (50), (53) provide any additional insight into these issues?

Another important open problem is to understand the geometric structure of the associated Riemannian manifold $\mathcal{K}_0^{(n)}$ that prescribes the radial n body Laplace–Beltrami operator constructed in [15] on the Euclidean distance cone, and its consequences for the classical and quantum n body problems.

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