Corrections and Comments on
Classical Invariant Theory, by Peter J. Olver
London Math. Soc. Student Texts, vol. 44
Cambridge University Press, Cambridge, 1999

Last modified October 7, 2018

****

On page 20, Exercise 2.8, there is a missing minus sign in the exponential:

$$Q(x,y) = \begin{cases} x^2 \exp(-x^2/y^2), & y \neq 0, \\ 0, & y = 0, \end{cases}$$

****

On page 25, formula (2.20), there should be no bar over J:

$$J(\mathbf{a},\mathbf{x}) = (\alpha\delta - \beta\gamma)^k \, J(\bar{\mathbf{a}},\bar{\mathbf{x}}).$$

****

On page 34, line 4 after (2.36), change:

$$\ldots \text{ process } R\,\partial/\partial Q \text{ to } J \qquad \text{to} \qquad \ldots \text{ process } R\,\partial/\partial Q \text{ to } K$$

****

On pages 36-37, the determinantal formula (2.42) for the resultant should be modified by reversing the order of the rows with the $\widetilde{b}_i$'s in them. This reordering only affects the sign of the resultant, but the construction of the subresultants in Exercise 2.38 requires that it be done.

$$\mathbf{R}[P,Q] \quad = \quad (-1)^k \det \begin{vmatrix} \widetilde{a}_m & \widetilde{a}_{m-1} & \ldots & \ldots & \widetilde{a}_0 & & & & \\ & \widetilde{a}_m & \widetilde{a}_{m-1} & \ldots & \ldots & \widetilde{a}_0 & & & \\ & & \ddots & & & & \ddots & & \\ & & & \widetilde{a}_m & \widetilde{a}_{m-1} & & \ldots & \ldots & \widetilde{a}_0 \\ & & & & \widetilde{b}_n & \widetilde{b}_{n-1} & & \ldots & \widetilde{b}_0 \\ & & & \widetilde{b}_n & \widetilde{b}_{n-1} & \ldots & \widetilde{b}_0 & & \\ & & \cdot^{\cdot^{\cdot}} & & & \cdot^{\cdot^{\cdot}} & & & \\ & \widetilde{b}_n & \widetilde{b}_{n-1} & \ldots & \widetilde{b}_0 & & & & \\ \widetilde{b}_n & \widetilde{b}_{n-1} & \ldots & \widetilde{b}_0 & & & & & \end{vmatrix},$$

where k is the integer quotient of $n/2$.

****

On page 38, in equation (2.45), there is a missing factor and the product should be over $\alpha < \beta$:

$$\Delta[Q] = \frac{(-1)^{n(n-1)/2}}{n^n} \prod_{\alpha<\beta} (x_\alpha y_\beta - y_\alpha x_\beta)^2. \tag{2.45}$$

Also, in the following displayed equation, there is a missing factor in the formula for $\partial Q/\partial x$:

$$Q(x,y) = \prod_{\alpha=1}^{n} (x y_\alpha - y x_\alpha), \qquad \text{so} \qquad \frac{\partial Q}{\partial x} = \sum_{\beta=1}^{n} y_\beta \prod_{\alpha\neq\beta} (x y_\alpha - y x_\alpha).$$

Thanks to Alexander Isaev for pointing this out.

****

As noted by Andries Brouwer, in the table on page 40, the number of independent invariants for a binary form of degree 10 computed by Sylvester is 99, not 104. Moreover, it is now known that Sylvester's counts of the number of invariants and covariants for forms of degrees 7,9,10 and 12 are all understated.

The corrected table, including all known results (as of June 2015) is the following:

degree	2	3	4	5	6	7	8	9	10	12
# invariants	1	1	2	4	5	30	9	92	106	$\geq$ 113
# covariants	2	4	5	23	26	147	69	$\geq$ 476	$\geq$ 510	$\geq$ 989

Details, as well as complete references to the literature, can be found on Brouwer's web page `http://www.win.tue.nl/~aeb/math/invar.html` that also includes bracket formulas for all the generating invariants of binary forms up to degree 9.

****

On page 49, in Example 3.24, the last sentence should read:

For n even, $\mathrm{PSL}(n,\mathbb{R})$ has two components. The group $\mathrm{SL}(n,\mathbb{R})$ is a two-fold covering of its connected component containing the identity, denoted $\mathrm{PSL}^+(n,\mathbb{R}) = \mathrm{SL}(n,\mathbb{R})/\{\pm\mathbb{1}\}$.

Thanks to Juha Pohjanpelto for pointing this out.

****

Footnote on p. 79. Delete "(written under Hilbert)". Junker was, in fact, a student of Brill in Tübingen.

Thanks to Emmanuel Briand for this remark.

****

On page 90, the third formula (5.11) should be

$$(Q,R)^{(3)} = m(m-1)(m-2)Q'''R - 3(m-1)(m-2)(n-2)Q''R' + \\ + 3(m-2)(n-1)(n-2)Q'R'' - n(n-1)(n-2)QR'''.$$

****

On page 92, formula (5.16) the last term should have -3 as a coefficient:

$$V = Q^3Q'''' - 4\,\frac{(n-3)}{n}\,Q^2Q'Q''' + 6\,\frac{(n-2)(n-3)}{n^2}\,QQ'^{\,2}Q'' - 3\,\frac{(n-1)(n-2)(n-3)}{n^3}\,(Q')^4$$

****

In a paper by D. Zagier, Modular forms and differential operators, *Proc. Indian Acad. Sci.* (*Math. Sci.*) **104** (1994) 57–75, the connection between transvectants and Rankin-Cohen brackets is presented as an open problem. Thus, the remark on p. 92 answers Zagier's question. The key is to note that the degree of a binary form or covariant is minus its weight when considered as a modular form. Therefore, the transvectants of negative weight covariants agree with the Rankin-Cohen brackets of positive weight modular forms. Note that the change in sign is in accordance with how the power of the multiplier appears in the transformation rule (2.8).

Further details and interesting new developments can be found in P.J. Olver and J. Sanders, Transvectants, modular forms, and the Heisenberg algebra, *Adv. Appl. Math.* **25** (2000) 252–283. See also Y.–J. Choie, B. Mourrain, and P. Solé, Rankin–Cohen brackets and invariant theory, *J. Alg. Combin.* **13** (2001) 5–13, for another perspective on this connection between classical invariant theory and modular forms.

Thanks to Patrick Solé and Jan Sanders for correspondence on this point.

****

The formula on the next to last line of page 98 should be $[\![P,Q]\!]_t = \{P,Q\} + \mathrm{O}(t^2)$

****

On p. 118, (6.40) should include the conditions $\alpha \neq \beta$, $\gamma \neq \delta$. Similarly, the Capelli identity (6.42) should include the condition $\alpha \neq \beta$.

****

On page 178, formula (8.32): change $(m-j)!$ to $(n-m)!$

****

On page 179, in (8.37), the formulas for γ and δ are each missing a term:

$$\gamma = \frac{1}{n} Q^{(1-n)/n}\, Q', \qquad\qquad \delta = Q^{1/n} - \frac{1}{n} p\, Q^{(1-n)/n}\, Q'.$$

****

On pages 192-193, the formulae for the polynomials A and B are not quite correct. One needs to cancel common factors in the rational covariants J and K before multiplying out. Details and further results on discrete symmetries of polynomials can be found in the paper: I. Berchenko and P.J. Olver, Symmetries of polynomials, *J. Symb. Comp.* **29** (2000) 485–514.

****

On page 193, part (a) of Corollary 8.68 should read only $k \leq 6n - 12$. Equality is not achieved if either the equation $A(p,q) = 0$ has multiple roots, or the covariants T and H have common factors. See the preceding paper for details.

****

On page 195, replace J by J^2 in equation (8.70), which should read

$$9j^2 \left(K + \tfrac{8}{3}\right)^3 = 2i^3 \left(K - \tfrac{1}{2} J^2 + \tfrac{16}{9}\right)^2 . \tag{8.70}$$

****

p. 210: In Theorem 9.28 change $X \subset \mathbb{R}^n$ to $X \subset \mathbb{R}^m$.

****

p. 210: Equation (9.21) is missing a centered dot: $I(e^{-tJ} \cdot x) = I(x)$.

****

The discussion of semi-invariants on pp. 218–221 is flawed. The basic (and classical) definition should state a semi-invariant is a relative invariant for the upper triangular subgroup U, meaning that

$$\mathbf{v}_-(F) = 0, \qquad \text{and} \qquad \mathbf{v}_0(F) = kF,$$

where $k = 2\,\mathrm{wt}\,F - \ \mathrm{ord}\,F \cdot \deg Q$ is the called the *index* of the semi-invariant.

An invariant is a semi-invariant of index 0. Proposition 9.45 should state that the leading coefficient $K(a)$, not the leading term, of a covariant (9.46) is a semi-invariant of index n. The proof should be modified accordingly. (The leading term $L(a,x) = K(a)x^n$ of (9.46) is invariant under $\mathbf{v}_0$, but not invariant at all under $\mathbf{v}_-$.)

Furthermore, in the proof of Theorem 9.49, one needs to modify the proof that $\mathbf{v}_+^{n+1}K = 0$, basing it on (9.49) as described in Hilbert's paper.

p. 219, last line: change $\widehat{Q} = a_0$ to $\widehat{Q} = a_3$.

p. 220, formulas (9.47–49): Several + and − signs need to be fixed:

$$\mathbf{v}_0\mathbf{v}_- = \mathbf{v}_-(\mathbf{v}_0 + 2), \qquad \mathbf{v}_0\mathbf{v}_+ = \mathbf{v}_+(\mathbf{v}_0 - 2), \qquad \mathbf{v}_+\mathbf{v}_- = \mathbf{v}_-\mathbf{v}_+ + \mathbf{v}_0. \tag{9.47}$$

$$\mathbf{v}_0(\mathbf{v}_-)^k = (\mathbf{v}_-)^k(\mathbf{v}_0 + 2k), \qquad \mathbf{v}_0(\mathbf{v}_+)^k = (\mathbf{v}_+)^k(\mathbf{v}_0 - 2k). \tag{9.48}$$

$$\begin{aligned} \mathbf{v}_+(\mathbf{v}_-)^k &= (\mathbf{v}_-)^k\mathbf{v}_+ + (\mathbf{v}_-)^{k-1}[-k\mathbf{v}_0 - k(k-1)], \\ \mathbf{v}_-(\mathbf{v}_+)^k &= (\mathbf{v}_+)^k\mathbf{v}_- + (\mathbf{v}_+)^{k-1}[k\mathbf{v}_0 - k(k-1)]. \end{aligned} \tag{9.49}$$

p. 221, line -5: change $\mathbf{v}_0(K) = -nK$ to $\mathbf{v}_0(K) = nK$.

Following equation:

$$\begin{aligned} &\mathbf{v}_0\big[\,\{(\mathbf{v}_+)^k K\}\,x^{n-k}y^k\,\big] = \\ &\qquad = \{(\mathbf{v}_+)^k(\mathbf{v}_0 - 2k)K\}\,x^{n-k}y^k + (2k-n)\{(\mathbf{v}_+)^k K\}\,x^{n-k}y^k = 0, \end{aligned}$$

p. 221, top displayed equation: Change sign in front of $k\mathbf{v}_0$ term:

$$\begin{aligned} &\mathbf{v}_-\big[\,\{(\mathbf{v}_+)^k K\}\,x^{n-k}y^k\,\big] = \\ &\quad = \{(\mathbf{v}_+)^k\mathbf{v}_-K + (\mathbf{v}_+)^{k-1}[k\mathbf{v}_0 - k(k-1)]K\}\,x^{n-k}y^k \\ &\qquad\quad - (n-k)\{(\mathbf{v}_+)^k K\}\,x^{n-k-1}y^{k+1} \\ &\quad = k(n-k+1)\{(\mathbf{v}_+)^{k-1}K\}\,x^{n-k}y^k - (n-k)\{(\mathbf{v}_+)^k K\}\,x^{n-k-1}y^{k+1}. \end{aligned}$$

Bottom displayed equation: Change sign of middle terms:

$$\begin{aligned} &\mathbf{v}_-(\mathbf{v}_+)^k(\mathbf{v}_-)^k = \\ &\quad = (\mathbf{v}_+)^k(\mathbf{v}_-)^{k+1} + k(\mathbf{v}_+)^{k-1}\mathbf{v}_0(\mathbf{v}_-)^k - k(k-1)(\mathbf{v}_+)^{k-1}(\mathbf{v}_-)^k \\ &\quad = (\mathbf{v}_+)^k(\mathbf{v}_-)^{k+1} + k(\mathbf{v}_+)^{k-1}(\mathbf{v}_-)^k\mathbf{v}_0 + k(k+1)(\mathbf{v}_+)^{k-1}(\mathbf{v}_-)^k. \end{aligned}$$

I would like to thank James Murdock and Frank Leitenberger for sharing their corrections to the material on semi-invariants.

****

pp. 252 & 261: change Hermite, M. to Hermite, Ch.

****

p. 263: change Reynolds, 74 to Reynolds, O., 74

****

I would like to thank also Ken Driessel for helpful comments and assistance preparing this TEX document.