Corrections to the First Printing of

Olver, P.J., *Applications of Lie Groups to Differential Equations*,
Second Edition, Springer–Verlag, New York, 1993.

Last updated: February 21, 2023

⋆⋆⋆ *On page 5, lines 24–27, change*

Thus T^2 can be covered by two coordinate charts

$$U_1 = \{(\theta,\rho) : 0 < \theta < 2\pi,\ 0 < \rho < 2\pi\},$$
$$U_2 = \{(\theta,\rho) : \pi < \theta < 3\pi,\ \pi < \rho < 3\pi\},$$

with overlap function ...
to

Thus T^2 can be covered by three coordinate charts, e.g.

$$U_1 = \{(\theta,\rho) : 0 < \theta < 2\pi,\ 0 < \rho < 2\pi\},$$
$$U_2 = \{(\theta,\rho) : \pi < \theta < 3\pi,\ \pi < \rho < 3\pi\},$$
$$U_3 = \left\{\, (\theta,\rho) : \tfrac{1}{2}\pi < \theta < \tfrac{5}{2}\pi,\ \tfrac{1}{2}\pi < \rho < \tfrac{5}{2}\pi \,\right\}.$$

The first overlap function is ...

⋆⋆⋆ *On page 10, line 6, change*

$\phi \circ \tilde{\phi}^{-1} \colon \mathbb{R} \to \mathbb{R}$

to

$\phi^{-1} \circ \tilde{\phi} \colon \mathbb{R} \to \mathbb{R}$

⋆⋆⋆ *On page 19, line 17, change*

$x \in V_0 = \left\{\, x : |\, x \,| < \tfrac{1}{2} \,\right\}$

to

$x \in V_0 = \left\{\, -1 < x < \tfrac{1}{3} \,\right\}$

⋆⋆⋆ *On page 36, line 9, change*

for all $\varepsilon, \theta \in \mathbb{R}$, $x \in M$, such that both sides are defined, if and only if

to

for all $x \in M$, and $(\varepsilon,\theta) \in V$, where $V \subset \mathbb{R}^2$ is a connected open subset containing $(0,0)$ such that both sides of (1.34) are defined at all points therein, if and only if

$\star\star\star$ *On page 37, lines 7–9, change*

... plane:

$$V = \{(\theta, \varepsilon) : \text{both sides of (1.34) are defined at } (\theta, \varepsilon)\}$$

and

$$U = \{(\theta, \varepsilon) : \text{both sides of (1.34) are defined and equal at } (\theta, \varepsilon)\}$$

to

... plane: first V is the connected component of

$$\widehat{V} = \{(\theta, \varepsilon) : \text{both sides of (1.34) are defined at} (\theta, \varepsilon)\}$$

containing the origin; second $U = \widehat{U} \cap V$, where

$$\widehat{U} = \{(\theta, \varepsilon) : \text{both sides of (1.34) are defined and equal at} (\theta, \varepsilon)\}$$

$\star\star\star$ *On page 37, line 10, delete the sentence*

Note that $U \subset V$, and that V is a connected subset of the (θ, ε) plane.

$\star\star\star$ *On page 37, line 14, add the following sentence after the final $U = V$.*

Warning: It is not, in general, true that $\widehat{U} = \widehat{V}$!

Remark: The preceding corrections are because Theorem 1.34 had a subtle flaw in it, first pointed out to me by James Devlin. The following exercise gives a counterexample to the original version. More details can be found in my paper: Olver, P.J., Non-associative local Lie groups, *J. Lie Theory* **6** (1996) 23–51. Thanks also to Hans Lundmark for comments.

Exercise: Let $M = \{\, (r, \theta) \mid r > 0 \,\}$. Prove that the two vector fields

$$\mathbf{v} = \cos\theta \, \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \, \frac{\partial}{\partial \theta}\,, \qquad \mathbf{w} = \sin\theta \, \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \, \frac{\partial}{\partial \theta}\,,$$

commute, $[\mathbf{v}, \mathbf{w}] = 0$ on M, but their flows do not globally commute. *Hint*: Consider r, θ as polar coordinates.

$\star\star\star$ *On page 43, line 23, change*

smoth vector fields

to

smooth vector fields

$\star\star\star$ *On page 52, Example 1.58, change*

Lie proved that

to

Lie proved, [4], that

⋆⋆⋆ *On page 64, line 10, delete the middle terms between the two = signs. Thus, the equation should read*

$$exp(\varepsilon \mathbf{v}_0)^* \big[\, \omega|_{exp(\varepsilon \mathbf{v}_0)x} \big] = \sum_I \alpha_I(e^{\varepsilon}x)e^{k\varepsilon} dx^I,$$

⋆⋆⋆ *On page 64, line 18, change*

$$\int_{\log \varepsilon}^{1}$$

to

$$\int_{\exp \varepsilon}^{1}$$

⋆⋆⋆ *On page 64, line 19, change*

$$\lambda = \log \widetilde{\varepsilon}$$

to

$$\lambda = e^{\tilde{\varepsilon}}$$

⋆⋆⋆ *On page 67, line 15, change*

sort

to

short

⋆⋆⋆ *On page 70, Exercise 1.8, change*

in polar coordinates,

to

in polar coordinates with $r > 0$,

⋆⋆⋆ *On page 72, Exercise 1.24(b), change*

$\mathfrak{h} \subset \mathfrak{g}$ has the property

to

$\mathfrak{h} \subset \mathfrak{g}$ is a *normal subalgebra* (or *ideal*), meaning that it has the property

⋆⋆⋆ *On pages 79–81, the ends of the proofs of Proposition 2.6 and Theorem 2.8 that refer back to* (1.40) *are subtly flawed, in that, under our definition of a local group action, even when $g = g_1 \cdot g_2$ with $g, g_1, g_2 \in G_x$, there is no guarantee that $g_1 \in G_{g_2 \cdot x}$, i.e., that $g_1 \cdot (g_2 \cdot x)$ is defined even though $g \cdot x = (g_1 \cdot g_2) \cdot x$ is defined.*

However, a more direct argument, that avoids this difficulty, is to replace the end of the proof of Proposition 2.6 by the following:

... Conversely, if (2.1) holds everywhere, then

$$\frac{d}{d\varepsilon}\, \zeta(\exp(\varepsilon \mathbf{v})\, x) = 0$$

where defined, and hence $\zeta(\exp(\varepsilon\mathbf{v})\,x) = \zeta(x) = c$ is constant for ε in the connected interval containing 0 in $\{\,\varepsilon \in \mathbb{R} \mid \exp(\varepsilon\mathbf{v}) \in G_x\,\}$. Using the fact that the exponential map is a local diffeomorphism from a neighborhood of $0 \in \mathfrak{g}$ to a neighborhood of $e \in G_x$, we conclude that $\zeta(g \cdot x) = c$ for all g in an open neighborhood of the identity in G_x. Now, set $\widetilde{G}_x = \{\,g \in G_x \mid \zeta(g \cdot x) = c\,\}$. Applying the preceding argument at the point $g \cdot x$ for any $g \in \widetilde{G}_x$ proves that $\widetilde{G}_x$ is open, while continuity proves that it is closed in G_x. Thus, by connectivity, $\widetilde{G}_x = G_x$, and the result follows. $\square$

Similarly, replace the end of the proof of Theorem 2.8 by:

... We have thus shown that if x_0 is a solution to $F(x) = 0$, and $\mathbf{v}$ is an infinitesimal generator of G, and ε is sufficiently small, then $\exp(\varepsilon\mathbf{v})\,x_0$ is also a solution. As in the proof of Proposition 2.6, one then shows that $\widetilde{G}_x = \{\,g \in G_x \mid F(g \cdot x_0) = 0\,\}$ is both open and closed in G_x, and hence, by connectivity, $\widetilde{G}_x = G_x$. $\square$

- *Thanks to Colin James Stockdale Klaus for correspondence on this point.*

$\star\star\star$ *On page 83, line 14, change*

folowing

to

following

$\star\star\star$ *On page 93, line 8, change*

functon

to

function

$\star\star\star$ *On page 106, line -5, the term inside the parentheses should be $\Xi_\varepsilon(x)$. Thus, the equation should read*

$$\sum_l \frac{\partial^2 \Xi^k_{-\varepsilon}}{\partial \tilde{x}^j\,\partial \tilde{x}^l}\,\big(\,\Xi_\varepsilon(x)\,\big)\,\frac{d\,\Xi^l_\varepsilon}{d\varepsilon}(x) = 0,$$

$\star\star\star$ *On page 111, line 3, change*

$\mathrm{J}\tilde{f}_\varepsilon(x)$

to

$\mathrm{J}\tilde{f}_\varepsilon(\tilde{x})$

$\star\star\star$ *On page 113, line 4, change*

$\xi = u$

to

$\xi = -u$

$\star\star\star$ *On page 123, line 5, change*

$$\tau \frac{\partial}{\partial \tau}$$

to

$$\tau \frac{\partial}{\partial t}$$

$\star\star\star$ *On page 151, line after* (2.110)*, change*

normal subalgebra of $\mathfrak{g}$

to

normal subalgebra (ideal) of $\mathfrak{g}$

$\star\star\star$ *On page 163, line -3, change*

differential equatons

to

differential equations

$\star\star\star$ *On page 167, line 2, change*

$\binom{p+k-1}{k} \cdot l$

to

$\binom{p+k}{k} \cdot l$

$\star\star\star$ *On page 167, Definition 2.83, change*

differential equatons

to

differential equations

$\star\star\star$ *On page 169, line -9, change*

Exercise 2.32

to

Exercise 2.33

- *Thanks to Mariano Hermida de La Rica for the preceding four corrections.*

$\star\star\star$ *On page 187, line 8, change*

with respect y

to

with respect to y

⋆⋆⋆ *On page 187, line 14, change*

$$\frac{\partial \delta}{\partial \hat{x}} + \frac{\partial \delta}{\partial y}\frac{\partial \eta}{\partial x} + \frac{\partial \delta}{\partial v}\frac{\partial v}{\partial y}\frac{\partial \eta}{\partial x}$$

to

$$\frac{\partial \delta}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial x} + \frac{\partial \delta}{\partial y}\frac{\partial \eta}{\partial x} + \frac{\partial \delta}{\partial v}\frac{\partial v}{\partial y}\frac{\partial \eta}{\partial x}$$

- *Thanks to Wen-xiu Ma and his class for catching this and several other errors.*

⋆⋆⋆ *On page 197, line -10, change*

SO(3)–invariant solutions exist.

to

SO(3)–invariant solutions can be constructed by this technique.

⋆⋆⋆ *On page 202, lines 13–14, change*

linear system of ordinary differential equations,

to

linear, constant coefficient system of ordinary differential equations,

⋆⋆⋆ *On page 207, line -2, change*

Example 2.64

to

Example 2.44

⋆⋆⋆ *On page 212, on both line 12 and line 13, change*

M/G

to

M/G^2

⋆⋆⋆ *On page 214, line 13, change*

$\widetilde{F}(x) = F[\pi(x)]$

to

$F(x) = \widetilde{F}[\pi(x)]$

⋆⋆⋆ *On page 215, line -1, change*

$\widetilde{F} = \widetilde{F}(\pi_1, \ldots, \pi_{m-s})$

to

$\widetilde{F} = \widehat{F}(\pi_1, \ldots, \pi_{m-s})$

$\star\star\star$ *On page 217, line 6, change*

$F(R,K) = 0$

to

$\widehat{F}(R,K) = 0$

$\star\star\star$ *On page 238, Exercise 3.7, change the formula to*

$$R = \left(\frac{t^2 E}{p_0} \right)^{1/5} \; h \left[p_0 \left(\frac{t^6}{\rho_0^3 \, E^2} \right)^{1/5} \right]$$

- *Thanks to Kameron Decker Harris for spotting this.*

$\star\star\star$ *On page 273, line -7, change*

tecnique

to

technique

$\star\star\star$ *On page 280, in the table, change*

$I_x = x\,D - y\,A + \frac{1}{2}\,x\,u\,u_t + t\,M_x$

to

$I_x = x\,D + y\,A + \frac{1}{2}\,x\,u\,u_t + t\,M_x$

- *Thanks to Gehrt Hartjen for checking through this table and the table on page 340 in his Mathematics Diplomarbeit in Aachen, 2001.*

$\star\star\star$ *On page 285, line 5, change*

4.13. (a)

to

** 4.13. (a)

$\star\star\star$ *On page 285, line 7, change*

Prove that the reduced system Δ/G for the G-invariant solutions of Δ is also the Euler-Lagrange equations for some variational problem on the quotient manifold M/G. Does this generalize to nonvariational symmetry groups?

to

Is the reduced system Δ/G for the G-invariant solutions of Δ necessarily the Euler-Lagrange equations for some variational problem on the quotient manifold M/G? See I.M. Anderson and M. Fels, Symmetry reduction of variational bicomplexes and the principle of symmetric criticality, *Amer. J. Math.* **119** (1997) 609–670, for details.

$\star\star\star$ *On page 290, line -4, change*

Exercise 2.33

to

Exercise 2.35

$\star\star\star$ *On page 323, line -8 change*

a third order evolution equation is integrable

to

a third order evolution equation in which u_{xxx} occurs linearly is integrable

$\star\star\star$ *On page 328, line 10, change*

Bluman and Kumei, [3],

to

Bluman and Kumei, [2],

$\star\star\star$ *On page 331, insert minus sign after equals sign in second displayed equation:*

$$D_\Delta^* Q = -\, q_t - q_{xx} + u\, q_x$$

$\star\star\star$ *On page 340, in line 4 of the table, change*

$-y\, u_{xxx} + x\, u_{xyy} + u_{xy}$

to

$-y\, u_{xxx} + x\, u_{xxy} + u_{xy}$

$\star\star\star$ *On page 340, in line 5 of the table, change*

$u_{xx}(y\, u_{yt} + \frac{1}{2}\, u_t) - u_{yy}(x\, u_{xt} + \frac{1}{2}\, u_t)$

to

$-\, u_{xx}(y\, u_{yt} + \frac{1}{2}\, u_t) + u_{yy}(x\, u_{xt} + \frac{1}{2}\, u_t)$

$\star\star\star$ *On page 350, lines -8 to -7, change*

the their Fréchet

to

their Fréchet

$\star\star\star$ *On page 364, Theorem 5.92, first line, change*

$P[u] \in \mathcal{A}^p$

to

$P[u] \in \mathcal{A}^q$

- *Thanks to Thomas von Schroeter for spotting this.*

$\star\star\star$ *On page 366, line -7, change*

$J \setminus I$

to

$I \setminus J$

- *Thanks to Rob Thompson for catching this and several other errors.*

$\star\star\star$ *On page 381, Exercise 3.16a*:

The system does not, in fact have a recursion operator, although there is a recursive formula for generating the higher order symmetries. On the other hand, the related system

$$u_t = u_{xx} + v^2, \qquad v_t = v_{xx},$$

does admit a recursion operator. Details can be found in: Beukers, F., Sanders, J.A., and Wang, J.P., On integrability of systems of evolution equations, *J. Diff. Eq.* **172** (2001), 396-408.

$\star\star\star$ *On page 381, Exercise 3.16b*:

A proof that the Bakirov system has only one generalized symmetry can now be found in: Beukers, F., Sanders, J.A., and Wang, J.P., One symmetry does not imply integrability, *J. Diff. Eq.* **146** (1998), 251–260.

$\star\star\star$ *On page 387, Exercise 5.51, change*

homogeneous system of differential equations

to

homogeneous, totally nondegenerate system of differential equations

Also, delete the first sentence in the Hint, which is not valid when the conservation law is not polynomial.

- *Thanks to Kostya Druzhkov for correspondence on this exercise.*

$\star\star\star$ *On page 397, equation* (6.17)*, change closing* $\}$ *to* $\rangle$:

$$\{F, H\}(x) = \langle\, x\,;[\,\nabla F(x), \nabla H(x)\,]\,\rangle, \qquad x \in \mathfrak{g}^*, \tag{6.17}$$

$\star\star\star$ *On page 420, line -8, change*

rocedure.

to

procedure.

⋆⋆⋆ *On page 427, lines 23–24, change*

Recently, Weinstein, [3], proposed the less historically motivated term "Casimir function" for these objects, . . .

to

Later, R. Littlejohn, *AIP Conference Proc.* **88** (1982), 47–66, and A. Weinstein, *AIP Conference Proc.* **88** (1982), 1–11, and [3], proposed the less historically motivated term "Casimir function" for these objects, . . .

- *Thanks to Phil Morrison for alerting me to these earlier references.*

⋆⋆⋆ *On page 430, Exercise 6.16:*

One needs to assume that $\partial^2 L/\partial u_n^2 \neq 0$ in order to (locally) solve the final equation for u_n in terms of p, q.

- *Thanks to Thomas von Schroeter for alerting me to this omission.*

⋆⋆⋆ *On page 469, change title of second reference by H. Bateman to*

On dissipative systems and related variational principles

⋆⋆⋆ *On page 480, line 2 change*

Leipz. Berich. **1** (1895)

to

Leipz. Berichte **47** (1895)

⋆⋆⋆ *On page 480, line 5 change*

Leipz. Berich. **3** (1897)

to

Leipz. Berichte **49** (1897)

⋆⋆⋆ *On page 501, change*

adjoint 314, 465

to

adjoint 314, 329, 343, 465