Canonical Anisotropic Elastic Moduli

PETER J. OLVER

Abstract. We discuss the determination of canonical elastic moduli in elasticity. In the linear case, complete results are known for planar bodies and some preliminary results on planar displacements of three-dimensional bodies are indicated. Applications to conservation laws are also presented.

The detailed investigation of complex mathematical objects can often be simplified through the use of specially adapted coordinate systems in which the object takes a simple "canonical form". Elementary examples include the Jordan canonical form of a square matrix, Sylvester's Theorem on the representation of a quadratic form as a sum of squares, and Darboux' Theorem on the canonical form of Hamiltonian structures. Use of a canonical form invariably results in a great simplification of what might otherwise be impossibly complicated calculations, and often provides extra geometric insight which might be difficult to extract in a general coordinate frame.

In elasticity, the determination of canonical forms for elastic materials, either linear or nonlinear, does not appear to have been investigated in the literature until recently. The basic mathematical problem is to find a specially adapted coordinate system in which the elastic material has as simple expression as possible. In this paper, I will review earlier work, [1], [2], on canonical forms in linear planar elasticity. Very recent unpublished extensions to planar displacements of a three dimensional body, the case described by the Stroh formalism, [3], [4], discussed elsewhere in these proceedings, will be presented in some detail, including some new results on materials with planes of symmetry. Also, a few remarks on the possible use of canonical forms in nonlinear elasticity, currently under

* School of Mathematics, University of Minnesota, Minneapolis, MN.
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investigation, will be provided. Finally, the paper will review applications of the resulting canonical forms to the classification of conservation laws (path-independent integrals), [5], [6], which are of crucial importance in crack and dislocation theory.

Changes of Variables and Elasticity. The equations of hyper-elasticity constitute a self-adjoint, strongly elliptic quasi-linear system of second-order partial differential equations for the deformation (or, in the linear case, displacement) $\mathbf{u} = \mathbf{f}(\mathbf{x})$, where $\mathbf{u} = (u^1, \dots, u^q) \in \mathbb{R}^q$, and $\mathbf{x} = (x_1, \dots, x_p)$ are the material coordinates in the elastic body $\Omega \subset \mathbb{R}^p$. For planar elasticity, $p = q = 2$, while $p = q = 3$ for fully three-dimensional elastic media. The Stroh formalism applies to a hybrid case, that of planar displacements of three-dimensional bodies, where $p = 2$, while $q = 3$. The equilibrium equations are the Euler-Lagrange equations for the stored energy functional

$$\mathcal{W}[\mathbf{u}] = \int_\Omega W(\mathbf{x}, \nabla\mathbf{u})\, d\mathbf{x} . \tag{1}$$

The physical conditions of frame indifference, strong ellipticity, etc., will restrict the class of stored energy functions which are of relevance to elasticity, although our initial remarks apply to quite general variational problems. The stored energy is not uniquely determined by its Euler-Lagrange equations, since we can add any *null Lagrangian* or total divergence, replacing W by $W + \mathrm{Div}\, P$, although this will, in general, alter the natural boundary conditions associated with the problem.

At a fixed material point $\mathbf{x} = \mathbf{a}$ and a fixed value of deformation gradient $\nabla\mathbf{u} = F$, we define the *symbol* of the variational problem (1) to be the "biquadratic" polynomial

$$Q_{\mathbf{a},F}(\mathbf{x}, \mathbf{u}) = \sum_{i,j,k,\ell} \frac{\partial^2 W}{\partial u^i_j \, \partial u^k_\ell}(\mathbf{a}, F)\; u^i u^k x_j x_\ell , \quad \mathbf{x} \in \mathbb{R}^p , \quad \mathbf{u} \in \mathbb{R}^q . \tag{2}$$

Note especially that the symbol is unaffected by the addition of a null Lagrangian to W owing to the general result, [7], that each first order null Lagrangian is a linear combination, with coefficients depending on $\mathbf{x}, \mathbf{u}$, of the minors of the deformation gradient $\nabla\mathbf{u}$. The Legendre-Hadamard condition of strong ellipticity requires that the symbol Q be *positive definite* in the sense that

$$Q_{\mathbf{a},F}(\mathbf{x}, \mathbf{u}) > 0 \qquad \text{whenever} \quad \mathbf{x} \neq \mathbf{0} \text{ and } \mathbf{u} \neq \mathbf{0} , \tag{3}$$

for all $\mathbf{a} \in \Omega$, and F such that $\det F > 0$.

Now, consider the effect of a general change of variables

$$\tilde{\mathbf{x}} = \varphi(\mathbf{x}, \mathbf{u}) , \qquad \tilde{\mathbf{u}} = \psi(\mathbf{x}, \mathbf{u}) , \tag{4}$$

on the variational problem (1). Physically, one might wish to restrict to transformations which do not mix up the independent and dependent variables, but the following remarks hold for the more general class (4). According to the chain rule, the deformation gradient transforms according to $\widetilde{\nabla}\tilde{\mathbf{u}} = \chi(\mathbf{x}, \mathbf{u}, \nabla\mathbf{u})$, with components

$$\chi^i_j(\mathbf{x}, \mathbf{u}, \nabla\mathbf{u}) = \sum_{k=1}^{q} (J^{-1})_{kj}\, D_k \psi^i ,$$

where J is the $p \times p$ matrix of total derivatives

$$J_{ij} = D_j \varphi^i, \quad \text{with} \quad D_j = \frac{\partial}{\partial x_j} + \sum_{k=1}^{q} \frac{\partial u^k}{\partial x_j} \frac{\partial}{\partial u^k}.$$

A straightforward (but long) calculation proves that the symbol of the new stored energy function is related to that of the old by the basic formula

$$\widetilde{Q}_{\tilde{a},\tilde{F}}(\mathbf{x}, \mathbf{u}) = |\det J|\, Q_{a,F}(J^{-T}\mathbf{x}, K\mathbf{u}), \tag{5}$$

where

$$K_{ij} = \frac{\partial \psi^i}{\partial u^j} - \sum_{k=1}^{p} \chi^i_k(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u}) \frac{\partial \varphi^k}{\partial u^j}.$$

The problem of determining canonical forms for nonlinear elasticity is extremely complicated, and requires rather sophisticated mathematical machinery, of which the powerful Cartan equivalence method, [8], seems particularly apt. In principle, through the algorithmic determination of a complete set of invariants for the equivalence problem, the Cartan method can provide explicit necessary and sufficient conditions for two stored energy functions to be equivalent under a general nonlinear change of variables. The main complication is that the intervening calculations can become extremely complicated, and have only been pursued to completion in very simple cases. It can be shown that the first of the invariants arising from the Cartan method is the symbol (2), and that we must understand canonical forms of biquadratic polynomials in order to make further progress on the general nonlinear problem. However, as we will soon see, the problem of canonical forms for biquadratic polynomials under the change of variables (5) is essentially the same as that of canonical forms for quadratic variational problems, i.e. the canonical form problem of linear elasticity. In conclusion, one must fully understand the linear equivalence problem before any assault can be made on the nonlinear problem.

In order to simplify the subsequent analysis, we will restrict to homogeneous materials, whereby the stored energy function $W(\nabla \mathbf{u})$ depends only on the deformation gradient. In the linear case, W is a symmetric quadratic function

$$W(\nabla \mathbf{u}) = \sum_{i,k=1}^{q} \sum_{j,\ell=1}^{p} a_{ijk\ell} \frac{\partial u^i}{\partial x_j} \frac{\partial u^k}{\partial x_\ell}, \tag{6}$$

of the displacement gradient, where the constants $a_{ijk\ell}$, which satisfy $a_{ijk\ell} = a_{k\ell ij}$, are called the *variational moduli* for the given problem. The equilibrium equations are

$$\sum_{j,\ell=1}^{p} \sum_{k=1}^{q} a_{ijk\ell} \frac{\partial^2 u^k}{\partial x_j \partial x_\ell} = 0, \qquad i = 1, \ldots, q. \tag{7}$$

For a general quadratic variational problem (6), the symbol Q is independent of the value of the displacement gradient F, and also the material point $\mathbf{a}$ provided the body is homogeneous. It can be found directly by replacing $\nabla \mathbf{u}$ in W by the rank one tensor $\mathbf{u} \otimes \mathbf{x} = \mathbf{u}\,\mathbf{x}^T$:

$$Q(\mathbf{x}, \mathbf{u}) \;=\; W(\mathbf{x} \otimes \mathbf{u}) \;=\; \sum_{i,\,j,\,k,\,\ell} a_{ijk\ell}\, u^i u^k x_j x_\ell\,, \qquad \mathbf{x} \in \mathbb{R}^p\,, \quad \mathbf{u} \in \mathbb{R}^q\,. \tag{8}$$

Since every quadratic null Lagrangian is a linear combination of the 2×2 Jacobian determinants $\partial(u^i, u^k) / \partial(x_j, x_\ell)$, in the linear case, a homogeneous quadratic stored energy function is uniquely determined by its symbol up to a divergence.

The assumption of frame indifference in linear elasticity requires that the stored energy function depends only on the strain tensor $\varepsilon = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$, so

$$W(\nabla \mathbf{u}) \;=\; \sum_{i,\,j} \sigma_{ij}\, \varepsilon_{ij} \;=\; \sum_{i,\,j,\,k,\,\ell} c_{ijk\ell}\, \varepsilon_{ij}\, \varepsilon_{k\ell}\,, \tag{9}$$

where $\sigma = C[\varepsilon]$ is the associated stress tensor, and the constants $c_{ijk\ell}$ are the *elastic moduli*, which obey the symmetry restrictions

$$c_{ijk\ell} \;=\; c_{jik\ell} \;=\; c_{ij\ell k}\,, \qquad\qquad c_{ijk\ell} \;=\; c_{k\ell ij}\,. \tag{10}$$

(Note that the variational moduli $a_{ijk\ell}$ are certain linear combinations of the elastic moduli.) The elastic moduli must also satisfy certain inequalities stemming from the Legendre-Hadamard strong ellipticity condition $Q(\mathbf{x}, \mathbf{u}) > 0$. The symmetry restrictions (10) placed on the elastic moduli are easily seen to be equivalent to the condition that the symbol Q be *symmetric*, i.e. $Q(\mathbf{x}, \mathbf{u}) = Q(\mathbf{u}, \mathbf{x})$.

In the case linear elasticity, we may restrict our attention to linear changes of variables in both the material coordinates $\mathbf{x}$ and the displacement $\mathbf{u}$:

$$\mathbf{x} \longmapsto A\mathbf{x}\,, \qquad\qquad \mathbf{u} \longmapsto B\mathbf{u}\,. \tag{11}$$

Our fundamental problem, then, is to determine matrices A and B which will simplify the elastic moduli $c_{ijk\ell}$ (or variational moduli $a_{ijk\ell}$ in the general case) as much as possible. Stated in this form, the question appears to be quite natural from a mathematical point of view, even though it may not have an immediate physical motivation. Indeed, the linear maps determined by the matrices A and B will not in general have any direct physical interpretation, except in the special case of orthogonal transformations (rotations), when they represent a physical change of frame. See Lekhnitskii, [9], and Ting, [10], for a discussion of the problem of determining canonical forms and invariants for elastic moduli under the more restrictive class of rotations. Of course, one difficulty with the admission of general linear change of variables is that the boundary conditions are not necessarily respected, since non-orthogonal coordinate changes can alter the normal direction for the boundary; however, we will ignore this complication here.

Under the change of variables (11), the stored energy gets transformed according to the usual change of variables formula for multiple integrals:

$$W \longmapsto \widetilde{W}(\nabla \mathbf{u}) \;=\; W(B\, \nabla \mathbf{u}\, A^{-1})\ |\det A|\,.$$

Thus, given a stored energy function W, the goal is to find matrices A and B such that, up to a null Lagrangian, the new stored energy function $\tilde{W}$ is as simple as possible; the associated elastic (variational) moduli will then be termed canonical Since the minima of the two variational problems $\mathcal{W}$ and $\tilde{\mathcal{W}}$ are in one-to-one correspondence, from a coordinate-free standpoint they are essentially the same problem. The symbol of the new stored energy is related to that of the old by the earlier formula (5), with A, B being as in (11). It helps to simplify matters by replacing the matrix A by the matrix $\sqrt{|\det A|}\cdot A^{-T}$. Thus we are led to the problem of determining canonical forms for biquadratic polynomials under the change of variables $Q(\mathbf{x}, \mathbf{u}) \longmapsto Q(A\,\mathbf{x}, B\,\mathbf{u})$. Also, as mentioned above, this same problem must be properly understood before any significant progress on the nonlinear case can be made.

Canonical Forms and Symbols. The number of canonical moduli can be determined directly by a simple dimension count. A general biquadratic polynomial or symbol $Q(\mathbf{x}, \mathbf{u})$ depending on $\mathbf{x} \in \mathbb{R}^p$ and $\mathbf{u} \in \mathbb{R}^q$ has a total of $\frac{1}{4}p\,(p+1)\,q\,(q+1)$ independent variational moduli. The possible changes of variables (11) will involve $p^2 + q^2$ arbitrary parameters, but the transformation just rescaling $\mathbf{x}$ (where A is a multiple of the identity) has the same effect as that rescaling $\mathbf{u}$, so there are $p^2 + q^2 - 1$ independent parameters at our disposal. Thus, in general, we expect the canonical quadratic variational problem to depend on

$$\frac{p\,(p+1)\,q\,(q+1)}{4} - p^2 - q^2 + 1$$

canonical moduli. For planar elasticity, $p = q = 2$, so we will find just 2 canonical elastic moduli. In three dimensions, we should obtain 19 canonical elastic moduli; however, imposing the symmetry conditions (10) reduces the count to 12. In the case $p = 2, q = 3$ covered by the Stroh formalism, we expect 6 independent canonical elastic moduli. However, these naïve dimension counts provide us with no indication of the precise canonical forms a linear elastic stored energy function can take, and one must use much more powerful algebraic tools in order to make progress on the determination of explicit canonical forms for quadratic variational problems.

We now discuss the relevant algebraic properties of biquadratic symbols, concentrating on the cases $p = q = 2$ and $p = 2, q = 3$. (Note that if either $p = 1$ or $q = 1$, the symbol is an ordinary quadratic polynomial, whose canonical forms, determined by Sylvester's law of inertia, are well known, [11]. In particular, only the rank and signature are invariants, and there are no canonical variational moduli in these special cases.) First write the symbol in the matrix form

$$Q(\mathbf{x}, \mathbf{u}) = \mathbf{u}^T A(\mathbf{x})\,\mathbf{u}\,, \qquad (12)$$

where, assuming strong ellipticity, $A(\mathbf{x})$ is a real $q \times q$ symmetric positive definite matrix of homogeneous quadratic polynomials of the variables $\mathbf{x}$. Just as the analysis of ordinary real polynomials requires an understanding of their complex roots, and so we may regard $\mathbf{x}$ and $\mathbf{u}$ as complex vectors, and Q as a complex-valued biquadratic polynomial. By the strong ellipticity assumption (which is a special case of nondegeneracy), for generic vectors $\mathbf{x} \in \mathbb{C}^p$, the matrix $A(\mathbf{x})$ has full rank. (In general, we define the *rank* of a homogeneous quadratic polynomial $P(\mathbf{u}) = \mathbf{u}^T A\,\mathbf{u}$ to be the rank of the associated symmetric matrix A;

in particular, P has full rank if and only if $\det A \neq 0$.) It is useful to distinguish the exceptional points where Q has less than maximal rank. Define the *discriminant*

$$\Delta_{\mathbf{u}}(\mathbf{x}) = \det A(\mathbf{x}), \tag{13}$$

which is a homogeneous polynomial of degree $2q$ of the p complex variables $\mathbf{x}$. A *root* of $\Delta_{\mathbf{u}}$ is a nonzero vector $0 \neq \mathbf{x} \in \mathbb{C}^p$ satisfying $\Delta_{\mathbf{u}}(\mathbf{x}) = 0$. Since $\Delta_{\mathbf{u}}(\mathbf{x})$ is homogeneous, any complex scalar multiple $\lambda\,\mathbf{x}$ of a root $\mathbf{x}$ is also a root, so we will only distinguish roots if they are not scalar multiples of each other. (Equivalently we view $\Delta_{\mathbf{u}}(\mathbf{x})$ as a polynomial on the complex projective space $\mathbb{CP}^{q-1}$.) The roots of the discriminant play a crucial role in the classification of these biquadratic polynomials, and hence of quadratic variational problems. Note that strong ellipticity implies that the discriminant has no real roots, and so the roots always come in complex conjugate pairs.

Clearly, one can interchange the roles of $\mathbf{x}$ and $\mathbf{u}$ in the above discussion, producing a corresponding discriminant $\Delta_{\mathbf{x}}(\mathbf{u})$. Except in the symmetric elastic case with $p = q$, these two polynomials are not the same (indeed, if $p \neq q$, they do not even depend on the same variables), nor are their roots easily compared. Nevertheless, there are subtle and remarkable relations between the roots of the two discriminants. For example, in the planar case $p = q = 2$, the discriminant $\Delta_{\mathbf{u}}(\mathbf{x})$ has simple roots if and only if $\Delta_{\mathbf{x}}(\mathbf{u})$ does. (However, it is not true that if $\Delta_{\mathbf{u}}(\mathbf{x})$ has a double root then $\Delta_{\mathbf{x}}(\mathbf{u})$ has a double root, although it does have a root of multiplicity at least two.)

We begin by outlining the known canonical forms in the case of planar elasticity, so $p = q = 2$. The discriminant $\Delta_{\mathbf{u}}(\mathbf{x})$ is a homogeneous quartic polynomial of the two variables $\mathbf{x} = (x, y)$, which has either two complex conjugate pairs of simple roots, or a complex conjugate pair of double roots. In the former case, we can find a real linear change of variables which moves the roots onto the imaginary axis, to $(1, \pm\tau\, i)$, $(1, \pm\tau^{-1} i)$, for some $\tau > 1$. (The constant τ is an invariant associated with the roots of the quartic.) In the latter case, we move the roots to $(1, \pm i)$. Performing the same change of variables on the other discriminant $\Delta_{\mathbf{x}}(\mathbf{u})$ (where, according to theory, the value of τ is necessarily the same), it can be proved, [1], that the symbol thereby reduces to one of "strongly orthotropic" form

$$x^2 u^2 + y^2 v^2 + \alpha\,(y^2 u^2 + x^2 v^2) + 2\beta\, x\, y\, u\, v, \tag{14}$$

where the canonical moduli α, β satisfy the inequalities

$$\alpha > 0, \qquad \beta \geq 0, \qquad |\alpha - 1| > \beta, \tag{15}$$

in the case when the discriminant has simple roots, or

$$0 < \alpha \leq 1, \qquad \beta = 1 - \alpha, \tag{16}$$

in the case of double roots. The corresponding stored energy function is given by the *orthotropic Lagrangian*

$$u_x^2 + \alpha\, u_y^2 + 2\beta\, u_x v_y + \alpha\, v_x^2 + v_y^2, \tag{17}$$

where the parameters α and β represent the two *canonical elastic moduli.* In fact, the Lagrangian (17) is, modulo a null Lagrangian, just a rescaled version of the standard stored energy of a linear, planar orthotropic elastic material

$$W = c_{1111} u_x^2 + c_{1212} (u_y + v_x)^2 + 2 c_{1122} u_x v_y + c_{2222} v_y^2 .$$

Indeed, after adding the null Lagrangian $c_{1212} (u_x v_y - u_y v_x)$, a simple rescaling will place this stored energy into the form (17), where

$$\alpha = \frac{c_{1212}}{\sqrt{c_{1111} c_{2222}}}, \qquad \beta = \frac{c_{1212} + c_{1122}}{\sqrt{c_{1111} c_{2222}}} .$$

Note especially that the discriminant has a complex conjugate pair of double roots if and only if the material is equivalent to an isotropic material, with $\alpha = \mu / (2\mu + \lambda)$, $\beta = (\mu + \lambda) / (2\mu + \lambda)$, where μ and λ are the classical Lamé moduli. Two isotropic Lagrangians determine the same orthotropic Lagrangian if and only if they have the same value for Poisson's ratio. Moreover, the isotropic stored energies are distinguished by the presence of a one-parameter symmetry group corresponding to the rotational invariance of (17) when $\alpha + \beta = 1$. The cases when the discriminant has simple roots, and the Lagrangian has at most discrete symmetries, correspond to "truly" anisotropic materials. Therefore, we have our first canonical form result in linear elasticity.

Theorem 1. Let $W(\nabla \mathbf{u})$ be a homogeneous first order planar quadratic Lagrangian which satisfies the Legendre-Hadamard strong ellipticity condition. Then W is equivalent to a orthotropic Lagrangian (17), where the canonical elastic moduli α and β satisfy the strong ellipticity inequalities $\alpha > 0$, $|\beta| < \alpha + 1$. The corresponding Euler-Lagrange equations are thus equivalent to the "orthotropic Navier equations"

$$u_{xx} + \alpha u_{yy} + \beta v_{xy} = 0, \qquad \beta u_{xy} + \alpha v_{xx} + v_{yy} = 0. \qquad (18)$$

See [1] for the explicit formulas for the change of variables taking a given stored energy function into its canonical orthotropic form. One can reduce a general strongly elliptic orthotropic stored energy (17) to a *unique* strongly orthotropic Lagrangian satisfying the more restrictive inequalities (15) or (16) using one or more of the three basic discrete equivalences taking the moduli (α, β) to either

$$(\alpha, -\beta), \quad \text{or} \quad \left(\frac{1}{\alpha}, \frac{1}{\beta}\right), \quad \text{or} \quad \left(\frac{1+\alpha-\beta}{1+\alpha+\beta}, \frac{2-2\alpha}{1+\alpha+\beta}\right).$$

Therefore, except in a few "exceptional" cases, each orthotropic Lagrangian is equivalent to seven different orthotropic Lagrangians. One further remark is that a complete set of canonical forms for general quadratic variational problems in the case $p = q = 2$ are known, [2]. To date, this is the only such classification which has appeared in the literature.

Turning to the case of planar deformations of a three-dimensional material, i.e. $p = 2$, $q = 3$, we are confronted with the problem of determining canonical forms for a positive definite "bi-ternary quadratic"

$$Q(x, y; u, v, w) > 0, \quad (x, y) \neq 0, \quad (u, v, w) \neq 0.$$

Such a symbol will be the planar restriction of a three-dimensional elastic stored energy function W provided it satisfies

$$Q(x, y; u, v, 0) = Q(u, v; x, y, 0). \tag{19}$$

The discriminant $\Delta_{\mathbf{u}}(\mathbf{x})$ is a homogeneous sextic polynomial in (x, y), which, according to the strong ellipticity assumption, has three complex conjugate pairs of roots (which may, in special instances, coincide). It is not hard to show that this polynomial coincides with Stroh's sextic, [3], [4], under the identification of (x, y) with $(1, \lambda)$, where λ is the eigenvalue parameter, so the roots of our discriminant have the same eigenvalue interpretation as in the Stroh formalism.

A stored energy function is called *separable* if there exist coordinates $\mathbf{x}, \mathbf{u}$ such that its symbol takes the form

$$Q(x, y; u, v, w) = R(x, y; u, v) + s(x, y)\, w^2.$$

Note that in this case, the Euler-Lagrange equations separate into a linear system for u, v, and a single separate second order elliptic equation for w, so that the problem essentially reduces to a problem for purely planar elasticity. In particular, we can introduce canonical coordinates whereby the planar part R is in canonical orthotropic form (14). If R is isotropic, then the rotational symmetry group can be used to diagonalize the quadratic polynomial $s(\mathbf{x})$, but, in general, we are left with the 4 parameter class of separable canonical forms

$$W = u_x^2 + \alpha u_y^2 + 2\beta u_x v_y + \alpha v_x^2 + v_y^2 + \gamma u_x^2 + 2\delta w_x w_y + \varepsilon w_y^2. \tag{20}$$

(One of the parameters $\gamma, \delta, \varepsilon$ can be eliminated by rescaling w.) Thus, the equilibrium equations reduce to the orthotropic Navier equations (18) together with a second order elliptic equation for w, which can be easily transformed into Laplace's equation, although not without changing the orthotropic form of the planar part.

A particular example of a separable material is that obtained from a three-dimensional elastic material which has the (x, y)-plane as a plane of symmetry, cf. [12]. In this case, all elastic moduli $c_{ijk\ell}$ which contain the index 3 either one or three times vanish, so that, restricting to planar deformations, we have (modulo null Lagrangians)

$$W = W_0(u_x, u_y, v_x, v_y) + c_{1313} w_x^2 + 2 c_{1323} w_x w_y + c_{2323} w_y^2,$$

where W_0 is an arbitrary planar stored energy function. This is clearly separable; in particular changing coordinates so that W_0 is in canonical orthotropic form we reduce W to the canonical form (20). If W itself is orthotropic then $c_{1323} = 0$, and, after rescaling, W_0 is in canonical form already, so the equilibrium equations reduce to the orthotropic Navier equations (18) plus a rescaled version of Laplace's equation for w.

As an example of an inseparable stored energy function, consider a material which has a reflectional symmetry with respect to a plane which is not the (x, y)-plane. Since the symmetry plane intersects the (x, y)-plane in a line, we can introduce a change of coordinates $(x, y, z) \longmapsto (a\,x + b\,y, c\,x + d\,y, e\,x + f\,y + g\,z)$ which changes the given plane into the (x, z)-plane. In the new coordinates, the stored energy function is

$$W = c_{1111}\, u_x^2 + c_{1212}\,(u_y + v_x)^2 + 2\,c_{1122}\, u_x v_y + c_{2222}\, v_y^2 + \\ + c_{1223}\,(u_y + v_x)\, w_y + 2\,c_{1322}\, w_x v_y + c_{1313}\, w_x^2 + c_{2323}\, w_y^2 .$$

The associated symbol has the form

$$p\,x^2 u^2 + q\,y^2 u^2 + 2\,r\,x\,y\,u\,v + s\,x^2 v^2 + t\,y^2 v^2 + \\ + a\,x^2 u\,w + b\,y^2 u\,w + 2\,c\,x\,y\,v\,w + d\,x^2 w^2 + e\,y^2 w^2 .$$

Consequently, the discriminant

$$\Delta_{\mathbf{u}}(\mathbf{x}) = \det \begin{vmatrix} p\,x^2 + q\,y^2 & r\,x\,y & a\,x^2 + b\,y^2 \\ r\,x\,y & s\,x^2 + t\,y^2 & c\,x\,y \\ a\,x^2 + b\,y^2 & c\,x\,y & d\,x^2 + e\,y^2 \end{vmatrix} = R(x^2, y^2),$$

where R is a homogeneous cubic polynomial in x^2, y^2. Such sextics occupy a distinguished role. According to Elliott, [13; p. 327, Ex. 21], a sextic polynomial $s(x, y)$ with no real roots can be written as a cubic in x^2, y^2 in some coordinate system if and only if it factors

$$s(x, y) = q_1(x, y)\, q_2(x, y)\, q_3(x, y)$$

into a product of three quadratic polynomials

$$q_j(x, y) = a_j\, x^2 + b_j\, x\,y + c_j\, y^2 ,$$

which form an *involution*, meaning

$$\langle\, q_1, q_2, q_3 \,\rangle \equiv \det \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 . \qquad (21)$$

Such sextics can be characterized in the following explicit invariant theoretic manner.

Theorem 2. Let $s(x, y)$ be a homogeneous sextic polynomial. Define the "skew invariant" R associated with s with the help of the basic covariants

$$i = s_{xxxx}\, s_{yyyy} - 4\, s_{xxxy}\, s_{xyyy} + 3\, s_{xxyy}^2 ,$$

$$\ell = i_{xxxx}\, s_{yyyy} - 4\, i_{xxxy}\, s_{xyyy} + 6\, i_{xxyy}\, s_{xxyy} - 4\, i_{xyyy}\, s_{xxxy} + i_{yyyy}\, s_{xxxx} ,$$

$$m = i_{xx}\, \ell_{yy} - 2\, i_{xy}\, \ell_{xy} + i_{yy}\, \ell_{xx} ,$$

$$n = i_{xx}\, m_{yy} - 2\, i_{xy}\, m_{xy} + i_{yy}\, m_{xx} ,$$

$$R = \langle\, \ell, m, n \,\rangle ,$$

cf. (21). (Note that i is a quartic polynomial, while ℓ, m, n are quadratics, and R is a constant.) Then $R = 0$ if and only if there exists a linear transformation $\tilde{x} = a x + b y$, $\tilde{y} = c x + d y$, such that either s is a cubic polynomial in $\tilde{x}^2, \tilde{y}^2$ or s is a product of $\tilde{x}^3$ times a cubic polynomial in $\tilde{x}, \tilde{y}$.

Note that the second possibility is excluded by our assumption that the symbol have only complex roots. As a consequence, we find the following explicit necessary condition for materials which admit a plane of symmetry.

Theorem 3. A necessary condition that an elastic material with $p = 2, q = 3$, be the planar restriction of a three-dimensional elastic material admitting a plane of symmetry is that its sextic discriminant $\Delta_{\mathbf{u}}(\mathbf{x})$ have vanishing skew invariant $R = 0$.

(Note that the separable stored energies have sextics with purely imaginary roots, which therefore also satisfy the involution criterion of Theorem 2.) I suspect that this condition is both necessary and sufficient, but have been unable to prove it. The reader should contrast the explicit (albeit complicated) nature of this criterion with the more implicit (since it requires the solution to a simultaneous eigenvalue problem) criterion of Cowin and Mehrabadi, [14], for fully three-dimensional materials to have planes of symmetry. It would be very interesting to determine a three-dimensional analogue of this theorem using the invariant theory of ternary sextics (which is however much less developed than that of binary sextics).

Turning to the problem of finding an explicit canonical form for the case of planar displacements of three-dimensional materials, according to our earlier dimension count, we are required to determine a suitable six-parameter family of stored energy functions. Moreover, the elasticity condition (19) imposes additional constraints on the physically relevant forms. Nevertheless, we will be able to determine an "elastic" canonical form for an arbitrary strongly elliptic quadratic variational problem with $p = 2$, $q = 3$. (This implies that only in the fully three-dimensional situation do the differences between elastic and general quadratic variational problems materialize.)

Theorem 4. Any strongly elliptic quadratic variational problem in $p = 2$ independent variables and $q = 3$ dependent variables can be written in either the separable canonical form

$$W = u_x^2 + \alpha u_y^2 + 2\beta u_x v_y + \alpha v_x^2 + v_y^2 + \gamma w_x^2 + 2\delta w_x w_y + \varepsilon w_y^2 . \tag{22}$$

or in the canonical form

$$W = u_x^2 + \alpha u_y^2 + 2\beta u_x v_y + \alpha v_x^2 + v_y^2 + + \gamma u_x w_x + \delta u_y w_y + \varepsilon v_x w_x + \theta v_y w_y + \rho w_x^2 + \sigma w_y^2 . \tag{23}$$

In particular, all non-elastic quadratic variational problem are equivalent to ones which satisfy the elastic criterion (19).

The canonical form (23) really only depends on 6 independent moduli, since the two-parameter family of rescalings $(x, y, u, v, w) \longmapsto (\lambda x, \lambda y, \lambda u, \lambda v, \mu w)$ will preserve the orthotropic moduli α, β of W, and so can be used to suitably normalize two of the remaining six moduli.

Proof.

To proceed, we begin by focussing attention on a single complex conjugate pair of roots of the discriminant $\Delta_{\mathbf{u}}(\mathbf{x})$ (which, by the Fundamental Theorem of Algebra, always exist). By a suitable real linear transformation, we can arrange that the roots are $(1, \pm i)$. We then write the symbol in the form

$$\begin{aligned} Q(\mathbf{x}, \mathbf{u}) &= (x + i\, y)^2\, p(\mathbf{u}) + (x - i\, y)^2\, \overline{p(\mathbf{u})} + (x + i\, y)(x - i\, y)\, s(\mathbf{u}) \\ &= (x^2 - y^2)\, q(\mathbf{u}) + 2\, x\, y\, r(\mathbf{u}) + (x^2 + y^2)\, s(\mathbf{u}), \end{aligned} \tag{24}$$

where $p(\mathbf{u}) = q(\mathbf{u}) + i\, r(\mathbf{u})$ is a complex-valued, and $q(\mathbf{u}), r(\mathbf{u}), s(\mathbf{u})$ are real-valued homogeneous quadratic polynomials of the real variables $\mathbf{u} = (u, v, w)$; moreover, according to our placement of the roots, $\operatorname{rank} p \le 2$. We proceed by placing the complex quadratic polynomial p into a suitable canonical form, and therefore need a complete list of canonical forms for complex quadratic polynomials under real linear transformations. To accomplish this, we begin by summarizing known results on the canonical forms for the associated pencil of quadratic polynomials $\lambda\, q(\mathbf{u}) + \mu\, r(\mathbf{u})$ under linear transformations of $\mathbf{u}$ and the parameters (λ, μ). These are a consequence of the general Kronecker-Weierstrass theory of complex matrix pencils, [15], along with results of Muth, [16], on the real case. See also Dickson, [17], for a statement of Kronecker's Theorem on singular pairs, and Gurevich, [11; pp. 258-259], for the real canonical forms of a pencil in two variables. (Unfortunately, the complete general theorem does not appear to have been written down in one place, but must be pieced together from the above references.) The case when one of the quadratic forms is positive definite is classical, and we know that the other can then be diagonalized. However, things get much more complicated if this is not the case.

Theorem 5. Any pencil $\lambda\, q(u, v, w) + \mu\, r(u, v, w)$ of real quadratic polynomials on $\mathbb{R}^3$ can, by a *real* linear transformation of the coordinates (u, v, w) and the parameters (λ, μ) be placed into one of the following thirteen canonical forms:

1. 0,
2. $\lambda\, u^2$,
3. $\lambda\, (u^2 \pm v^2)$,
4. $\lambda\, u^2 + 2\, \mu\, u\, v$,
5. $\lambda\, u^2 + \mu\, v^2$,
6. $\lambda\, (u^2 - v^2) + 2\, \mu\, u\, v$,
7. $\lambda\, (u^2 + v^2 \pm w^2)$,
8. $\lambda\, (u^2 \pm v^2) + \mu\, w^2$,
9. $\lambda\, u^2 + \mu\, v^2 \pm (\lambda + \mu)\, w^2$,
10. $\lambda\, (u^2 - v^2 + w^2) + 2\, \mu\, u\, v$,
11. $\lambda\, (u^2 + 2\, v\, w) + 2\, \mu\, v^2$,
12. $\lambda\, (u^2 + 2\, v\, w) + 2\, \mu\, u\, v$,
13. $\lambda\, u\, v + \mu\, u\, w$.

Corollary 6. Suppose $p(u, v, w)$ is a complex-valued quadratic function on $\mathbb{R}^3$, such that $\operatorname{rank} p \leq 2$. Then there is a real change of variables such that p is equal to a complex multiple of one of the following five families of canonical forms:

1. 0,
2. $u^2 + \sigma v^2$, $\quad \sigma \in \mathbb{C}$,
3. $u^2 + \sigma u v$, $\quad \sigma \in \mathbb{C} \setminus \mathbb{R}$,
4. $u^2 - v^2 + \sigma u v$, $\quad \sigma \in \mathbb{C} \setminus \mathbb{R}$,
5. $u^2 - v^2 + w^2 + 2 i u v$,
6. $u v + i u w$.

We now apply Corollary 6 to (24). Note that if we replace p by αp, where α is any complex number, this can be absorbed into the independent variables by replacing $x + i y$ by $\alpha^{-1} (x + i y)$, which has the effect of scaling and rotating $\mathbf{x}$. It is not hard to see that, by suitably redefining w, cases $1 - 4$ of Corollary 6 are found to give separable symbols, which are completely classified above. Thus, only the complex eigenvalue case 5 and the singular case 6 give genuinely three-dimensional problems, i.e. problems that do not decouple to a planar system plus a scalar equation. In the nondegenerate case 5, the symbol takes the preliminary canonical form

$$(x^2 - y^2)(u^2 - v^2 + w^2) + 4 x y u v + (x^2 + y^2) s(\mathbf{u}),$$

where

$$s(\mathbf{u}) = a u^2 + b u v + c v^2 + d u w + e v w + f w^2. \qquad (25)$$

Note that the given canonical form does indeed depend on six independent parameters; however, it does *not* satisfy the elasticity constraint (19) unless $b = 0$ and $a = c$. There is an easy way to fix this. If $d = e = 0$, then we are back to a separable symbol, which we know how to treat; otherwise, we can assume without loss of generality that $e \neq 0$. Then replacing w by $w - (b / e) u$ and appropriately rescaling u and v produces a symbol of the canonical form (23).

Turning to the singular case 6, the symbol now has the form

$$(x^2 - y^2) u w + 2 x y u v + (x^2 + y^2) s(\mathbf{u}),$$

where $s(\mathbf{u})$ is as above, cf. (25). If $e \neq 0$, then replacing w by $w - (b / e) u$ and appropriately rescaling u and v again produces a symbol of the canonical form (23). However, if $e = 0$, but $f \neq 0$, placing the symbol in the canonical form (23) is more tricky. Apparently it can't be done by looking just at the given pair of roots $(1, \pm i)$ of the discriminant. What can be proved, however, is first, that the discriminant necessarily has (at least) a second distinct complex conjugate pair of roots, (i.e. $(1, \pm i)$ is not a triple root), and second, at the other pair of roots, the corresponding complex polynomial $p(\mathbf{u})$ cannot be in the singular canonical form 6, and hence, by the previous method, the symbol is either separable, or can be reduced to our canonical form (23). Thus, interestingly, the singular canonical form can appear at at most one complex conjugate pair of roots of the discriminant. This completes the proof of Theorem 4.

Applications to Conservation Laws. In general, a conservation law for a system of differential equations is a divergence expression

$$\mathrm{Div}\ P = 0$$

which vanishes on all solutions. In elastostatics, conservation laws provide path- (or surface-) independent integrals, which can be used to great effect in the study of singularities such as cracks or dislocations. In the nonlinear theory, Noether's Theorem, [18], relating symmetry groups of the stored energy functional to conservation laws of the associated equilibrium equations, is used to great effect to derive the well known Eshelby energy-momentum tensor and other related integrals, [18; Example 4.32]. Conservation laws for linear isotropic elasticity, both two and three-dimensional, were completely classified in [5]. One of my motivations for developing the theory of canonical elastic moduli was my initial attempts to extend the results in the isotropic case to more general anisotropic materials, and being frustrated by the complications of the determining equations in the general coordinate system. Only after the introduction of the canonical orthotropic form was completed was I able to extend these results to planar anisotropic materials, [6]. In this section, I shall briefly summarize these results.

Any linear self-adjoint system of partial differential equations $\Delta[\mathbf{u}] = 0$ always possesses a *reciprocity relation*, which is a divergence identity of the general form

$$\mathbf{v} \cdot \Delta[\mathbf{u}] - \mathbf{u} \cdot \Delta[\mathbf{v}] = \mathrm{Div}\, P[\mathbf{u}, \mathbf{v}], \tag{26}$$

where P is some bilinear expression involving $\mathbf{u}$ and $\mathbf{v}$. (P is *not* uniquely determined since there are trivial reciprocity relations $\mathrm{Div}\, P_0 \equiv 0$; see [6].) If $\mathbf{v}$ is a solution to the system, then $P[\mathbf{u}, \mathbf{v}]$ forms a conservation law of the system. For a linearly elastic material (9), one explicit form of the Betti reciprocal theorem is

$$P[\mathbf{u}, \mathbf{v}] = \mathbf{v} \cdot \sigma[\mathbf{u}] - \mathbf{u} \cdot \sigma[\mathbf{v}], \tag{27}$$

where $\sigma[\mathbf{u}]$ is the stress tensor associated with the displacement $\mathbf{u}$.

Although isotropic and more general orthotropic materials have similar looking Lagrangians, (17), the structure of their associated conservation laws is different, but reminiscent of the differences between the Jordan canonical form of matrices with equal or distinct eigenvalues.

Theorem 7. Let $\mathcal{W}[u]$ be a strongly elliptic quadratic planar variational problem.

1. *The Isotropic Case.* If W is equivalent to an isotropic material, then there exists a complex linear combination z of the coordinates (x, y), a complex linear combination ω of the displacement components (u, v), and two complex linear combinations ξ, η of the components of the displacement gradient (u_x, u_y, v_x, v_y) having the properties:

a) The two Euler-Lagrange equations can be written as a single complex differential equation in form $D_z \eta = 0$.

b) Any conservation law is a real linear combination of

i) the Betti reciprocity relations,

ii) the two families of complex conservation laws

$$\operatorname{Re}[\, D_z F \,] = 0, \qquad \text{and} \qquad \operatorname{Re}\{\, D_z [\, (\xi + z)\, G_\eta + \bar{G} \,] \,\} = 0$$

where $F(z, \eta)$ and $G(z, \eta)$ are arbitrary complex analytic functions,

iii) the extra conservation law

$$\operatorname{Re}\{\, D_z [\, \omega\eta - i z \eta^2 \,] \,\} = 0 .$$

2. *The Anisotropic Case.* If W is equivalent to a strongly orthotropic material, then there exist two complex linear combinations z, w of the coordinates (x, y), and two corresponding complex linear combinations ξ, η of the components of the displacement gradient (u_x, u_y, v_x, v_y) with the properties:

a) The two Euler-Lagrange equations can be written as a single complex differential equation in either of the two forms $D_z \xi = 0$, or $D_w \eta = 0$.

b) Any conservation law is a real linear combination of

i) the Betti reciprocity relations, and

ii) the two families of complex conservation laws

$$\operatorname{Re}[\, D_z F \,] = 0, \qquad \text{and} \qquad \operatorname{Re}[\, D_w G \,] = 0,$$

where $F(z, \xi)$ and $G(w, \eta)$ are arbitrary complex analytic functions.

Thus one has the striking result that in *both* isotropic and anisotropic planar elasticity, there are three infinite families of conservation laws. One family is the well-known Betti reciprocity relations. The other two are determined by two arbitrary analytic functions of two complex variables. However, the detailed structure of these latter two families is markedly different depending upon whether one is in the isotropic or truly anisotropic (orthotropic) case. The two orthotropic families degenerate to a single isotropic family, but a second family makes its appearance in the isotropic case. In addition, the isotropic case is distinguished by the existence of one extra anomalous conservation law, the significance of which is not at all clear. Applications of these families of conservation laws to crack and dislocation problems remains uninvestigated.

The canonical form for the planar displacements of a three-dimensional linear elastic material given in Theorem 4 is new, and has, as yet, not been applied to the determination of conservation laws and path-independent integrals; this will be addressed in a future publication.

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