Conservation Laws in Continuum Mechanics

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Preface.

This article reviews some recent work on the conservation laws of the equations of continuum mechanics, with especial emphasis on planar elasticity. The basic material on conservation laws and symmetry groups of systems of partial differential equations is given an extensive treatment in the author's book, [6], so this paper will only give a brief overview of the basic theory. Some of the applications appear in the published papers cited in the references, while others are more recent.

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1. Conservation Laws of Partial Differential Equations.

The equations of non-dissipative equilibrium continuum mechanics come from minimizing the energy functional

$$\mathcal{W}[u] = \int_\Omega W(x,u^{(n)})\, dx. \qquad (1)$$

Here the independent variables $x=(x^1,\ldots,x^p) \in \Omega$ represent the material coordinates in the body, and the dependent variables $u=(u^1,\ldots,u^p)$ the deformation, where $p=2$ for planar theories, while $p=3$ for fully three-dimensional bodies. In the absence of body forces, the stored energy W will usually depend just on x and the deformation gradient ∇u, but may, in a theory of higher grade material, depend on derivatives of u up to order n, denoted $u^{(n)}$. Smooth minimizers will satisfy the Euler-Lagrange equations

$$E_\nu(W) = 0, \qquad \nu = 1, \ldots, p, \tag{2}$$

which, in the case of continuum mechanics, form a strongly elliptic system of partial differential equations of order 2n. Strong ellipticity implies that this system is totally nondegenerate (in the sense of [6; Definition 2.83]).

Given the system of partial differential equations (2), a *conservation law* is a divergence expression

$$\mathrm{Div}\, P = \sum_{i=1}^{p} D_i P_i = 0 \tag{3}$$

which vanishes on all solutions to (1), where the p-tuple $P(x,u^{(m)})$ can depend on x, u and the derivatives of u. For static problems, conservation laws provide path-independent integrals, which are of use in determining the behavior at singularities such as cracks or dislocations. For dynamic problems, conservation laws provide constants of the motion, such as conservation of mass or energy.

Two conservation laws are *equivalent* if they differ by a sum of trivial conservation laws, of which there are two types. In the first type of triviality, the p-tuple P itself vanishes on all solutions to (2), while the second type are the *null divergences*, where the identity (3) holds for all functions u=f(x) (not just solutions to the system). As trivial laws provide no new information about the solutions, we are only interested in equivalence classes of nontrivial conservation laws.

An elementary integration by parts shows that any conservation law for the nondegenerate system (2) is always equivalent to a conservation law in *characteristic form*

$$\mathrm{Div}\, P = Q.E(W) = \sum_{\nu=1}^{p} Q_\nu E_\nu(W) \tag{4}$$

where the p-tuple $Q=(Q_1,\ldots,Q_p)$ is the *characteristic* of the conservation law. A characteristic is called *trivial* if it vanishes on solutions to (2), and two characteristics are equivalent if they differ by a trivial characteristic. For nondegenerate systems of partial differential equations, each conservation law is uniquely determined by its characteristic, up to equivalence.

Theorem. If the system (2) is nondegenerate, then there is a one-to-one correspondence between (equivalence classes of) nontrivial conservation laws and (equivalence classes of) nontrivial characteristics.

2. Symmetries and Noether's Theorem.

A *generalized vector field* is a first order differential operator

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u^{(m)}) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{p} \varphi_\alpha(x,u^{(m)}) \frac{\partial}{\partial u^\alpha}.$$

If the coefficients ξ^i and φ_α depend only on x and u, then **v** generates a one-parameter group of *geometrical* transformations, which solve the system of ordinary differential equations

$$\frac{dx^i}{d\varepsilon} = \xi^i(x,u), \qquad \frac{du^\alpha}{d\varepsilon} = \varphi_\alpha(x,u).$$

For general **v**, the group transformations are nonlocal, and determined as solutions of a corresponding system of evolution equations.

The vector field **v** is a *symmetry* of the system (2) if and only if the infinitesimal invariance condition

$$\mathrm{pr}\,\mathbf{v}\,[E_\nu(W)] = 0, \qquad \nu = 1,\ldots,p,$$

holds on all solutions to (2). Here pr **v** denotes the *prolongation* of **v**, which determines how **v** acts on the derivatives of u. An elementary lemma says that we can always replace **v** by the simpler *evolutionary vector field*

$$\mathbf{v}_Q = \sum_{\alpha=1}^{p} Q_\alpha(x,u^{(m)}) \frac{\partial}{\partial u^\alpha}$$

where the *characteristic* $Q = (Q_1, \dots, Q_p)$ of $\mathbf{v}$ is defined by

$$Q_\alpha = \varphi^\alpha - \sum_{i=1}^{p} \xi^i \cdot \frac{\partial u^\alpha}{\partial x^i}$$

(See [6; Chapter 5] for the explicit formulas.) The infinitesimal invariance condition

$$\mathrm{pr}\, \mathbf{v}_Q\, [E_\nu(W)] = 0, \text{ whenever } E(W) = 0, \qquad \nu = 1, \dots, p, \tag{5}$$

constitutes a large system of elementary partial differential equations for the components of the characteristic Q. Fixing the order of Q, the defining equations (5) can be systematically solved so as to determine the most general symmetry of the given order of the system.

An evolutionary vector field $\mathbf{v}_Q$ is a *trivial symmetry* of (2) if the characteristic $Q(x,u^{(m)})$ vanishes on all solutions to (2). Two symmetries are *equivalent* if they differ by a trivial symmetry. Clearly we are only interested in determining classes of inequivalent symmetries of a given system of partial differential equations.

More restrictively, the evolutionary vector field $\mathbf{v}_Q$ is called a *variational symmetry* of the variational problem (1) if the infinitesimal invariance condition

$$\mathrm{pr}\, \mathbf{v}_Q\, (W) = \mathrm{Div}\, B \tag{6}$$

holds for some p-tuple $B(x,u^{(k)})$. Every variational symmetry of a variational integral (1) is a symmetry of the associated Euler-Lagrange equations (2), but the converse is *not* always true. (The most common counterexamples are scaling symmetry groups.) It is easy to check which of the symmetries of the Euler-Lagrange equations satisfy the additional variational criterion (6); see also [6; Proposition 5.39].

Noether's Theorem provides the connection between variational symmetries of a variational integral and conservation laws of the associated Euler-Lagrange equations $E(W) = 0$.

Theorem. Suppose we have a variational integral (1) with non-degenerate Euler-Lagrange equations (2). Then a p-tuple $Q(x,u^{(m)})$ is the characteristic of a conservation law for the Euler-Lagrange equations (2) if and only if it is the characteristic of a variational symmetry of (1). Moreover, equivalent conservation laws correspond to equivalent variational symmetries and vice versa.

Thus there is a one-to-one correspondence between equivalence classes of nontrivial variational symmetries and equivalence classes of nontrivial conservation laws. The proof rests on the elementary integration by parts formula

$$\mathrm{pr}\,\mathbf{v}_Q(W) = Q \,.\, E(W) + \mathrm{Div}\, A, \tag{7}$$

for some p-tuple $A = (A_1, \ldots, A_p)$ depending on Q and W. (There is an explicit formula for A, but it is a bit complicated; see [6; Proposition 5.74].) Comparing (7) and the symmetry condition (6), we see that

$$\mathrm{Div}(B - A) = Q \,.\, E(W),$$

and hence $P = B - A$ constitutes a conservation law of (2) with characteristic Q. The nontriviality follows from the theorem of section 1.

3. Finite Elasticity.

As an application of the general theory, we consider the case of an elastic material, so the stored energy function $W(x,\nabla u)$ depends only on the deformation gradient. We show how simple symmetries lead to well-known conservation laws. Material frame indifference implies that W is invariant under the Euclidean group

$$u \to Ru + a$$

of rotations R and translations a. The translational invariance is already implied by the fact that W does not depend explicitly on u, while rotational invariance requires that $W(x,R.\nabla u) = W(x,\nabla u)$ for all rotations R. The conservation laws coming from translational invariance are just the Euler-Lagrange equations themselves

$$\sum_{i=1}^{p} D_i\left\{ \frac{\partial W}{\partial u_i^\alpha} \right\} = 0,$$

written in divergence form. The rotational invariance provides $p(p-1)/2$ further conservation laws

$$\sum_{i=1}^{p} D_i\left\{ u^\alpha \frac{\partial W}{\partial u_i^\beta} - u^\beta \frac{\partial W}{\partial u_i^\alpha} \right\} = 0.$$

If the material is homogeneous, then W does not depend on x, and we have the additional symmetry group of translations

$$x \to x + b$$

in the material coordinates. There are thus p additional conservation laws

$$\sum_{i=1}^{p} D_i\left\{ \sum_{\alpha=1}^{p} u_j^\alpha \frac{\partial W}{\partial u_i^\alpha} - \delta_j^i W \right\} = 0,$$

whose entries form the components of Eshelby's celebrated energy-momentum tensor. If the material is isotropic, then W is invariant under the group of rotations in the material coordinates, so $W(\nabla u.R) = W(\nabla u)$ for all rotations R. There are an additional $p(p-1)/2$ conservation laws

$$\sum_{i=1}^{p} D_i\left\{ \sum_{\alpha=1}^{p} [x^j u_k^\alpha - x^k u_j^\alpha] \frac{\partial W}{\partial u_i^\alpha} - [\delta_j^i x^k - \delta_k^i x^j] W \right\} = 0.$$

Scaling symmetries can produce conservation laws under the assumption that W is a homogeneous function of the deformation gradient

$$W(\lambda .\nabla u) = \lambda^n . W(\nabla u), \quad \lambda > 0.$$

The scaling group $(x,u) \to (\lambda x, \lambda^{(n-p)/n}u)$ is a variational symmetry group, leading to the conservation law

$$\sum_{i=1}^{p} D_i\{ \sum_{\alpha=1}^{p} [\frac{n-p}{n}u^\alpha - \sum_{j=1}^{p} x^j u_j^\alpha] \frac{\partial W}{\partial u_i^\alpha} + x^i W \} = 0.$$

Of course, stored energy functions which are invariant under the scaling symmetry group are rather special. If one writes out the above divergence in the more general case, then we obtain the divergence identity

$$\sum_{i=1}^{p} D_i\{ \sum_{\alpha=1}^{p} [u^\alpha - \sum_{j=1}^{p} x^j u_j^\alpha] \frac{\partial W}{\partial u_i^\alpha} + x^i W \} = pW.$$

This was used by Knops and Stuart, [3], to prove the uniqueness of the solutions corresponding to homogeneous deformations. This latter identity is closely related to the general dentities determined by Pucci and Serrin, [10]. Indeed the general formula used by Pucci and Serrin to determine their identities is a special case of the integration by parts formula (7) in the case that the characteristic Q comes from a geometrical vector field. Particular choices of the coefficient functions ξ^i and φ_α lead to the particular identities that are used to study eigenvalue problems and uniqueness of solutions, generalizing earlier ideas of Rellich and Pohozaev.

4. Linear Planar Elasticity.

Although the general structure of symmetries and conservation laws for many of the variational problems of continuum mechanics remains an open problem, the case of linear planar elasticity, both isotropic and anisotropic, is now well understood. In this case, the stored energy function $W(\nabla u)$ is a quadratic function of the deformation gradient, which is usually written in terms of the strain tensor $e=(\nabla u+\nabla u^T)/2$. We have

$$W(\nabla u) = \sum c_{ijkl}\, e_{ij}\, e_{kl}, \qquad (8)$$

where the constants c_{ijkl} are the elastic moduli which describe the physical properties of the elastic material of which the body is composed. The elastic moduli must satisfy certain inequalities stemming from the Legendre-Hadamard strong ellipticity condtion. This states that the quadratic stored energy function $W(\nabla u)$ must be positive definite whenever the deformation gradient is a rank one tensor, i.e. $\nabla u = a \otimes b$ for vectors a, b. Following [7], we define the symbol of the quadratic variational problem with stored energy (8) to be the biquadratic polynomial $Q(x,u) = W(x \otimes u)$ obtained by replacing ∇u by the rank one tensor $x \otimes u$. In this case, the Legendre-Hadamard strong ellipticity condtion requires that

$$Q(x,u) > 0 \quad \text{whenever} \quad x \neq 0, \text{ and } u \neq 0. \tag{9}$$

The symmetry of the stress tensor and the variational structure of the equations impose the symmetry conditions

$$c_{ijkl} = c_{jikl} = c_{ijlk}, \qquad c_{ijkl} = c_{klij}.$$

on the elastic moduli, which are equivalent to the symmetry condition

$$Q(x,u) = Q(u,x)$$

on the symbol.

For each fixed u, $Q(x,u)$ is a homogeneous quadratic polynomial in x, and so we can form its discriminant $\Delta_x(u)$ (i.e. b^2-4ac), which will be a homogeneous quartic polynomial in u. The nature of the roots of $\Delta_x(u)$ provides the key to the structure of the problem. First, the Legendre-Hadamard condition (9) requires that $\Delta_x(u)$ has all complex roots. There are then only two distinct cases.

Theorem. Let $W(\nabla u)$ be a strongly elliptic quadratic planar variational problem, and let $\Delta_x(u)$ be the discriminant of its symbol. Then exactly one of the following possibilities holds.

1. The Isotropic Case. If $\Delta_x(u)$ has a complex conjugate pair of double roots, then there exists a linear change of variables

$$x \to Ax, \quad u \to Bu, \qquad A, B \text{ invertible } 2x2 \text{ matrices}$$

which changes W into an isotropic stored energy function.

2. The Orthotropic Case. If $\Delta_x(u)$ has two complex conjugate pairs of simple roots, then there exists a linear change of variables

$$x \to Ax, \quad u \to Bu, \qquad A, B \text{ invertible } 2x2 \text{ matrices}$$

which changes W into an orthotropic (but not isotropic) stored energy function.

(Recall, [2], that an orthotropic elastic material is one which has three orthogonal planes of symmetry. Thus, this theorem states that any planar elastic material is equivalent to an orthotropic (possibly isotropic) material, and so has three (not necessarily orthogonal) planes of symmetry. The analogous result is not true in three dimensions, cf. [1].)

This theorem is a special case of a general classification of quadratic variational problems in the plane, [7], and results in the construction of "canonical elastic moduli" for two-dimensional elastic media, [8]. One consequence is that in planar linear elasticity, there are, in reality, only two independent elastic moduli, since one can rescale any orthotropic stored energy to one whose elastic moduli have the "canonical form"

$$c_{1111} = c_{2222} = 1, \quad c_{1122} = c_{2211} = \alpha, \quad c_{1212} = \beta, \quad c_{1112} = c_{1222} = 0.$$

Thus the constants α and β play the role of canonical elastic moduli, with the special case $2\alpha + \beta = 1$ corresponding to an isotropic material. This confirms a conjecture made in [5]. Extensions to three-dimensional materials are currently under investigation.

Although isotropic and more general orthotropic materials have similar looking Lagrangians, the structure of their associated conservation laws is quite dissimilar. (For simplicity of notation, we write (x,y) for the independent variables and (u,v) for the dependent variables from now on.)

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Theorem. Let $\mathcal{W}[u]$ be a strongly elliptic quadratic planar variational problem, with corresponding Euler-Lagrange equations $E(W) = 0$.

1. <u>The Isotropic Case</u>. If W is equivalent to an isotropic material, then there exists a complex linear combination z of the variables (x, y), a complex linear combination w of the variables (u, v), and two complex linear combinations ξ, η of the components of the deformation gradient (u_x, u_y, v_x, v_y) with the properties:

a) The two Euler-Lagrange equations can be written as a single complex differential equation in form

$$D_z \eta = 0.$$

(Recall that if $z = x + iy$, then the complex derivative D_z is defined as $(D_x - iD_y)$.)

b) Any conservation law is a real linear combination of the Betti reciprocity relations, the complex conservation laws

$$\mathrm{Re}[\, D_z F \,] = 0,$$

and

$$\mathrm{Re}[D_z\{(\xi+z)G_\eta + \overline{G}\}] = 0,$$

where $F(z,\eta)$ and $G(z,\eta)$ are arbitrary complex analytic functions of their two arguments, and the extra conservation law

$$\mathrm{Re}[\, D_z\{w\eta - iz\eta^2\} \,] = 0.$$

2. <u>The Orthotropic Case</u>. If W is equivalent to an orthotropic, non-isotropic material, then there exist two complex linear combinations z, ζ of the variables (x, y), and two corresponding complex linear combinations ξ, η of the components of the deformation gradient (u_x, u_y, v_x, v_y) with the properties:

a) The two Euler-Lagrange equations can be written as a single complex differential equation in either of the two forms

$$D_z \xi = 0,$$

or

$$D_\zeta \eta = 0.$$

b) Any conservation law is a real linear combination of the Betti reciprocity relations, and the complex conservation laws

$$\mathrm{Re}[\, D_z F \,] = 0, \qquad \text{and} \qquad \mathrm{Re}[\, D_\zeta G \,] = 0,$$

where $F(z,\xi)$ and $G(\zeta,\eta)$ are arbitrary complex analytic functions of their two arguments.

Thus one has the striking result that in *both* isotropic and anisotropic planar elasticity, there are three infinite families of conservation laws. One family is the well-known Betti reciprocity relations. The other two are determined by two arbitrary analytic functions of two complex variables. However, the detailed structure of these latter two families is markedly different depending upon whether one is in the isotropic or truly anisotropic (orthotropic) case. The two orthotropic families degenerate to a single isotropic family, but a second family makes its appearance in the isotropic case. In addition, the isotropic case is distinguished by the existence of one extra anomalous conservation law, the significance of which is not fully understood. The details of the proof of this theorem in the isotropic case have appeared in [4; Theorem 4.2] (although there is a misprint, corrected in an Errata to [4] appearing recently in the same journal); the anisotropic extension will appear in [9].

I suspect that a similar result even holds in the case of nonlinear planar elasticity, but have not managed to handle the associated "vector conformal equations", cf. [5]. Extensions to three-dimensional elasticity have only been done in the isotropic case; see [4] for a complete classification of the conservation laws there. In this case, beyond Betti reciprocity, there are just a finite number of conservation laws, some of which were new.

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