Math. Proc. Camb. Phil. Soc. (1985), **97**, 511
Printed in Great Britain

Conservation laws and null divergences
II. Non-negative divergences

By PETER J. OLVER
School of Mathematics, University of Minnesota, Minneapolis, MN 55455, U.S.A.

(*Received* 2 *July* 1984)

Abstract

It is shown that P, depending on x, u and derivatives of u, satisfies $\operatorname{Div} P \geqslant 0$ for all such x, u, if and only if $\operatorname{Div} P = \phi(x) \geqslant 0$ where ϕ is independent of u. Applications to theories of continuum thermomechanics are discussed.

1. *Introduction*

In the foundations of the theory of continuum thermodynamics, significant restrictions on the constitutive relations of materials can be deduced from the basic laws of thermodynamics through a procedure popularized by Coleman and Noll in their seminal paper [**3**]. In this procedure, one postulates certain balance laws, such as those of linear momentum or energy, together with certain thermodynamical inequalities such as the Clausius–Duhem inequality reflecting the imbalance of entropy. The local forms of these laws and inequalities usually involve divergences of certain process fields, such as internal energy, entropy, Cauchy stress and so on. The underlying constitutive hypotheses of the theory require these process fields to depend on certain 'fundamental' variables, usually including material coordinates x, time t, temperature θ, and deformation u, and their derivatives or gradients up to some prescribed order. Thus, for instance, entropy η might be postulated to be a function of temperature θ, deformation gradient $F = \nabla u$ and temperature gradient $\nabla\theta$,

$$\eta = \eta(\theta, F, \nabla\theta).$$

For a higher grade material, η could also depend on higher order gradients, e.g. $\nabla F = \nabla^2 u, \nabla^2\theta, \ldots$ as well as temporal derivatives $\dot{\theta}$, $\dot{F} = \nabla\dot{u}$, etc. At this stage, the basic axiom of Coleman and Noll comes into play. They postulate that the resulting balance equalities or inequalities must hold for all possible sufficiently smooth values of the fundamental dependent variables, i.e. all smooth functions $\theta(x,t)$, $u(x,t)$, etc. determining temperature, deformation, etc. This relative freedom in the specification of the 'processes' available has the net effect of significantly reducing the possible dependence of the process fields on the fundamental variables themselves. For example, in the classical theories of continuum thermomechanics, the Coleman–Noll procedure requires that the free energy be a function of deformation gradient and temperature alone, so no higher order gradients or temporal derivatives can occur, and be related to the entropy, energy and stress as a 'potential' for them. (See [**3**], [**4**].) More recently, Dunn and Serrin, [**4**], [**5**], have introduced additional process variables, (the 'interstitial work flux') in an attempt to circumvent this restriction, with a view towards the Korteweg theory of phase transitions.

In any analysis requiring some version of the Coleman–Noll procedure, one is required, eventually, to deal with the following types of complicated algebraic problems: Given independent variables $x = (x^1, \ldots, x^p)$ and dependent (field) variables $u = (u^1, \ldots, u^q)$, suppose $P = (P_1, \ldots, P_p)$ is a p-tuple of smooth functions depending on x, u and derivatives of u, $u^\alpha_J = \partial^n u^\alpha / \partial x^{j_1} \ldots \partial x^{j_n}$ up to some order n. The problem is then to characterize all those P's which satisfy either

$$\operatorname{Div} P \geqslant 0 \tag{1}$$

for all smooth $u = u(x)$, or

$$\operatorname{Div} P = 0 \tag{2}$$

for all smooth $u = u(x)$. The second question, characterizing all *null divergences*, was extensively studied in the companion paper [**6**] to the present one, and we assume that the reader is familiar with the results and notations therein. Effectively, [**6**] provides a complete characterization of all null divergences $P(x, u^{(n)})$ as 'generalized curls' of functions $Q(x, u^{(n)})$. Moreover, the dependence of P on the highest order derivatives occurring therein was completely determined, being given as linear combinations of certain special types of 'Jacobian null divergences' whose entries are given in terms of Jacobian determinants of the variables u.

The purpose of the present paper is to demonstrate that the more general problem (1) of characterizing non-negative divergences actually reduces to the previously analysed problem (2) of characterizing null divergences. Specifically, we will prove the following:

THEOREM 1. *Let $P(x, u^{(n)})$ be a p-tuple of smooth functions of x, u and derivatives of u up to order n. Then P is a non-negative divergence,*

$$\operatorname{Div} P(x, u^{(n)}) \geqslant 0,$$

if and only if

$$\operatorname{Div} P(x, u^{(n)}) = \phi(x) \geqslant 0, \tag{3}$$

where $\phi(x)$ is a smooth, non-negative function of x alone. In particular, if P vanishes when u and all its derivatives vanish,

$$P(x, 0) = 0, \tag{4}$$

or if $P = P(u^{(n)})$ does not depend on x, then $\phi(x) \equiv 0$; hence P is a non-negative divergence if and only if it is a null divergence.

In the more general case, by choosing any p-tuple $\psi(x) = (\psi_1(x), \ldots, \psi_p(x))$ such that $\operatorname{div} \psi = \phi$, the slightly modified p-tuple $\tilde{P} = P - \psi$ becomes a null divergence. Thus, in all cases, we reduce the study of divergence inequalities (1) to divergence equalities (2), the latter having been investigated in full detail in [**6**].

In contrast to the direct analysis of the Coleman–Noll procedure, which requires exceedingly complicated and intricate algebraic manipulations owing to the appearance of determinantal expressions, (see [4], [**5**]), Theorem 1, when coupled with the previous results on null divergences, will lead to a significant simplification of these problems. Applications of this method appear in the revised version of the work of Dunn and Serrin, where a far wider class of materials is open to analysis by this method.

The proof of Theorem 1 is surprisingly simple using the techniques presented in the earlier paper [**6**]. The key point is to concentrate on the divergence

$$L = \operatorname{Div} P \tag{5}$$

rather than P itself, since this will eliminate any extraneous null divergence components in P which would only complicate the analysis. Recall ([**6**]; theorem 2·4) that L is a divergence (5) for some P if and only if L is a *null Lagrangian*,

$$E(L) \equiv 0$$

for all x, u, where E is the Euler–Lagrange operator or variational derivative for the variational problem $\mathscr{L}[u] = \int L dx$. The key lemma is the following:

LEMMA 2. *Suppose $L(x, u^{(n)})$, depending on* nth *and lower order derivatives of u, is a null Lagrangian. Then L is an affine function of each* nth *order derivative*

$$u_J^\alpha,\ J = (j_1, \ldots, j_n),$$

so
$$L = A u_J^\alpha + B, \tag{6}$$

where A, B are independent of u_J^α (but may depend on other nth *order derivatives.)*

The proof of Lemma 2 is immediate from the more general characterization of the dependence of null Lagrangians on nth order derivatives, ([**1**], theorem 4·3) (see also section 4 of [**6**]). Alternatively, this can be proved by direct analysis as in ([**2**]; theorem 3·4).

It is important to remark that L is *not* affine in all the highest order derivatives simultaneously. For instance

$$u_x v_y - u_y v_x = D_x(u v_y) - D_y(u v_x)$$

is affine in u_x, u_y, v_x and v_y individually, but not an affine function of the 4-tuple (u_x, u_y, v_x, v_y)! The best that can be said is that L is a 'multi-affine' (as in 'multi-linear') function of the highest order derivatives.

Once this has been established, the proof of Theorem 1 becomes elementary. Namely, if $L = \operatorname{Div} P \geqslant 0$ depends on nth order derivatives of u, then L is affine in each nth order derivative. Since u is arbitrary, for any nth order derivative u_J^α, the only way an affine function (6) can be non-negative for all u is if the coefficient A of u_J^α vanishes, requiring L to be independent of u_J^α. Thus L is, contrary to our assumption, independent of *all* nth order derivatives u_J^α. We conclude that L must be independent of u and its derivatives entirely, so

$$L = \phi(x) = \operatorname{Div} P,$$

and we have proved (3).

The other statements in Theorem 1 are easily established. If $P(x, u^{(n)})$ vanishes whenever u and all its derivatives vanish, the same is clearly true of the total divergence $L = \operatorname{Div} P$. Thus $L = \phi(x) = 0$ for all x, and we have proved the first of these results. If $P = P(u^{(n)})$ does not depend on x explicitly, the same is true of $\operatorname{Div} P = \phi$. Thus $\phi(x) = a$ is a nonnegative constant, with

$$\operatorname{Div} P(u^{(n)}) = a \geqslant 0.$$

Now set u and all its derivatives to 0, so

$$\operatorname{Div} P(0) = a$$

also. But $P(0)$ is constant, so $\operatorname{Div} P(0) = 0$ and hence $a = 0$. This completes the proof of the second statement.

This paper arose in conversations with Ernest Dunn during a seminar on phase transitions held at the Mathematics Research Center of the University of Wisconsin. It is a pleasure to thank him for sparking my interest in the applications of these results to thermodynamics.

This research was supported in part by NSF Grant MCS 81-00786.

REFERENCES

[1] I. M. ANDERSON and T. DUCHAMP. On the existence of global variational principles. *Amer. J. Math.* **102** (1980), 781–868.

[2] J. M. BALL, J. C. CURRIE and P. J. OLVER. Null Lagrangians, weak continuity and variational problems of arbitrary order. *J. Funct. Anal.* **41** (1981), 135–174.

[3] B. D. COLEMAN and W. NOLL. The thermodynamics of elastic materials with heat conduction and viscosity. *Arch. Rational Mech. Anal.* **13** (1963), 167–178.

[4] J. E. DUNN. Interstitial working and a nonclassical continuum thermodynamics. In *New Perspectives in Thermodynamics* (Springer-Verlag, in the Press).

[5] J. E. DUNN and J. SERRIN. On the thermomechanics of interstitial working. *Arch. Rational Mech. Anal.* (in the Press).

[6] P. J. OLVER. Conservation laws and null divergences. *Math. Proc. Cambridge Philos. Soc.* **94** (1983), 529–540.