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Group Theoretic Classification of Conservation Laws in Elasticity

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1. Introduction

Although Noether's celebrated theorem relating symmetry groups and conservation laws for systems arising from variational principles has been available for over sixty years, and despite the well-acknowledged importance of group theory in the study of elasticity, for historical reasons I do not fully understand there has never been a systematic classification of general symmetry groups and their associated conservation laws for even the simplest elastic materials. In fact, not until the work of Günther, [7], and Knowles and Sternberg, [10], was even a limited variant of Noether's general theorem applied in this subject. This is especially surprising in that the relevant techniques are completely constructive, amenable to straight-forward computational methods. In this note, the basic theory underlying Noether's theorem will be outlined, and illustrated by some recent applications to elasticity, work which is very much still in progress. It is a pleasure to thank John Ball for sparking my interest in applying these general techniques to elasticity, and to all the organizers of the NATO/LMS institute for a most enjoyable and productive conference.

2. Hyperelasticity, [9]

Although most techniques apply much more generally, we specialize to the case of homogeneous elastostatics in the absence of body forces. The variational problem is

$$I = \int_\Omega W(\nabla u)\,dx \ , \tag{2.1}$$

where W is the stored energy function, $x \in \mathbb{R}^p$, $u \in \mathbb{R}^q$ (usually

$p = q = 2$ or 3) and $\nabla u = (u^i_\alpha) \equiv (\partial u^i / \partial x^\alpha)$ the deformation gradient. The Euler-Lagrange equations are

$$E^i = \Sigma W^{ij}_{\alpha\beta} u^j_{\alpha\beta} = 0 , \quad W^{ij}_{\alpha\beta} \equiv \partial^2 W / \partial u^i_\alpha \partial u^j_\beta . \tag{2.2}$$

Occasionally, the Legendre-Hadamard condition that the $q \times q$ matrix $Q(\xi)$ with entries

$$Q_{ij}(\xi) = W^{ij}_{\alpha\beta} \xi^\alpha \xi^\beta , \qquad \xi \in \mathbb{R}^p ,$$

be positive definite for $\xi \neq 0$, will be imposed.

3. Conservation Laws, [1], [10], [13], [15].

A divergence expression

$$\text{Div } A = \Sigma D_\alpha A^\alpha = 0 \tag{3.1}$$

is a conservation law if it vanishes for all solutions of (2.2). Equivalently, $\int_\Gamma A \cdot dS$ is path - (or surface) independent. Such integrals, of which Eshelby's energy momentum tensor, [5], was the first important example, are of importance in the analysis of strain concentrations at crack tips, [2], [18].

Trivial examples arise when a) $A = 0$ for all solutions and b) (3.1) holds identically; equivalently $A^\alpha = \Sigma D_\beta B^{\alpha\beta}$ for some skew-tensor B. Many of the results described here rely on a new characterization, [14], of the second variety of trivial laws. For simplicity, we only consider laws in which $A = A(x,u,\nabla u)$, i.e. no higher derivatives of u enter.

4. Symmetry Groups, [1], [13], [17].

A symmetry group of a system of differential equations, e.g. (2.2), is characterized by its property of transforming solutions of the system to other solutions. Geometrical or Lie symmetries are (local) diffeomorphisms $g:(x,u) \to (\tilde{x},\tilde{u})$ and act on solutions by point-wise transforming their graphs (think of a rotation in the x,u - plane). Leaving aside discrete symmetries (e.g. reflections), each one-parameter subgroup is characterized by its infinitesimal generator- the vector field

$$\vec{v} = \Sigma\, \eta^{\alpha}(x,u) \frac{\partial}{\partial x^{\alpha}} + \Sigma\, \varphi_i(x,u) \frac{\partial}{\partial u^i} , \qquad (4.1)$$

from which the group transformations are recovered as solutions of the system of o.d.e.'s

$$dx^{\alpha}/d\varepsilon = \eta^{\alpha} , \qquad du^{i}/d\varepsilon = \varphi_i ,$$

ε being the group parameter. The group transformations, and hence infinitesimal generator, "prolong" to the derivatives of u, so

$$\text{pr}\ \vec{v} = \vec{v} + \Sigma\, \varphi_i^{\alpha} \frac{\partial}{\partial u^i_{\alpha}} + \dots ,$$

where

$$\varphi_i^{\alpha} = D_{\alpha}\psi_i + \Sigma\, u^i_{\alpha\beta}\, \eta^{\beta} , \qquad (4.2)$$

$$\psi_i = \varphi_i - \Sigma\, u^i_{\beta}\, \eta^{\beta} . \qquad (4.3)$$

Theorem For the system $E = 0$ nondegenerate (cf. [14]) (4.1) generates a symmetry group if and only if

$$\mathrm{pr}\ \vec{v}(E) = 0 \quad \text{whenever} \quad E = 0 \ . \tag{4.4}$$

Condition (4.4) forms a large system of elementary differential equations for the coefficients η^α, φ_i of $\vec{v}$, whose solution (a straight-forward, albeit tedious, task) gives the most general symmetry of the system.

A generalized symmetry has the same form for its infinitesimal generator, but now η^α, φ_i can depend on derivatives of u. The symmetry criterion (4.4) is unchanged, and again can be solved. The standard form of such a symmetry is

$$\vec{v}_\psi = \Sigma\, \psi_i\, \partial/\partial u^i \ , \tag{4.5}$$

cf. (4.3), and it easily follows that $\vec{v}$ is a symmetry if and only if $\vec{v}_\psi$ is. The corresponding group transformations are realized by solving the evolutionary system.

$$\partial u^i/\partial\epsilon = \psi_i(x,u,\nabla u,\ldots), \qquad u\big|_{\epsilon=0} = u_0 \ ,$$

so that if $u_0(x)$ is a solution, so is $u(x,\epsilon)$ for any fixed ϵ.

5. Noether's theorem, [1], [3], [11], [13]

For systems arising from variational principals a (geometrical) symmetry is a variational symmetry if the integral (2.1) is unchanged by the relevant transformation for arbitrary subdomains $\Omega \subset \mathbb{R}^p$. The corresponding infinitesimal criterion is

$$\mathrm{pr}\ \vec{v}(W) + W\ \mathrm{Div}\ \eta = 0 \ . \tag{5.1}$$

Bessel-Hagen, [3], replaced the right hand side of (5.1) by Div B for some unspecified B, calling the corresponding symmetries divergence transformations (here the precise geometrical significance is less clear). Equivalently,

$$\mathrm{pr}\ \vec{v}_{\psi}(W) = \mathrm{Div}\ \tilde{B} \tag{5.2}$$

for some $\tilde{B}$, cf. (4.5). Any generalized vector field satisfying (5.2) will be termed a variational symmetry. A nontrivial computation, [13], proves every variational symmetry is a symmetry of the corresponding Euler-Lagrange equations, but not conversely, scale symmetries being the most common counterexamples. Often, the easiest way to compute variational symmetries is to first find all symmetries of the Euler-Lagrange equation and then check which of these satisfy (5.2).

Noether's theorem comes from the trivial identity

$$\Sigma\ \psi_i E^i = \mathrm{Div}\ A\ , \tag{5.3}$$

holding for any variational symmetry, where A , which yields a conservation law, can be explicitly given in terms of ψ, W and $\tilde{B}$. Thus each variational symmetry gives a conservation law and, under mild nondegeneracy hypotheses, each conservation law is equivalent (up to addition of a trivial law) to one satisfying (5.3) and hence yields a corresponding symmetry.

6. The Symmetry Equations, [15]

A simplifying feature in the search for conservation laws of (2.2) is that if $A(x,u,\nabla u)$ satisfies (3.1), so does $A(x_o,u_o,\nabla u)$ for fixed x_o, u_o . Thus, the x,u - independent laws, say $A^{(1)}(\nabla u),\ldots,A^{(m)}(\nabla u)$ are calculated first, from which the most general law takes the form

$$A = \Sigma\, \gamma^{(k)}(x,u)A^{(k)} + \Sigma\, \delta^{(k)}(x,u)T^{(k)}$$

for suitable $\gamma^{(k)}, \delta^{(k)}$, and where $T^{(1)},\dots,T^{(n)}$ form a complete set of trivial x,u - independent laws, [14].

For (2.2), the conditions (5.2) reduce to the system.

$$\frac{\partial A^{\alpha}}{\partial u^{i}_{\beta}} + \frac{\partial A^{\beta}}{\partial u^{i}_{\alpha}} = \Sigma\, \psi_j G^{ij}_{\alpha\beta}\ , \quad G^{ij}_{\alpha\beta} = W^{ij}_{\alpha\beta} + W^{ji}_{\alpha\beta}\ . \tag{6.1}$$

For $q = 1$, these are the equations for an infinitesimal conformal symmetry for the metric $g_{\alpha\beta} = G^{11}_{\alpha\beta}$; this immediately leads to bounds on the number of x,u - independent conservation laws ($\leq \frac{1}{2}(p+1)(p+2)$ if $p \geq 3$). Presumably, the "vector conformal equations" (6.1) lead to similar results, but have not be treated before.

<u>Proposition</u> The system (2.2) admits nonvariational symmetries (respectively generalized symmetries) if and only if there is a $q \times q$ matrix $N(\xi) \neq 0$ (resp. $\neq n(\xi)I$) of linear functions of ξ such that $Q(\xi)N(\xi)$ is skew symmetric (respectively symmetric) for all ξ.

For $p = q = 3$, with the Legendre-Hadamard condition, QN skew symmetric implies that $Q(\xi) = C^T Q_o(\xi) C$ (C independent of ξ) with Q_o either diagonal or of the form $Q_o = p(\xi)I + \ell(\xi) \otimes \ell(\xi)$. For linear elasticity this implies the material is isotropic; the meaning for nonlinear materials is less apparent. The condition QN symmetric imposes restrictions on the form of Q, but I have been unable to discern their meaning.

7. Linear, Isotropic Elasticity, [8], [16]

To illustrate the theory, we classify all conservation laws with $A = A(x,u,\nabla u)$ for the case of linear, isotropic elasticity. Here

$$W = \mu\|\nabla u + \nabla u^T\|^2 + \frac{1}{2}\lambda(\nabla\cdot u)^2 , \tag{7.1}$$

where μ,λ are the Lamé moduli, so (2.2) reads

$$E = \mu\Delta u + (\mu+\lambda)\nabla(\nabla\cdot u) = 0 . \tag{7.2}$$

Strong ellipticity is implied by $\mu > 0$, $2\mu+\lambda > 0$, but we only require $\mu(\mu+\lambda)(2\mu+\lambda) \neq 0$. Conserved densities are given in tensorial form, so $A = (A_i^\alpha)$ yields the laws $\Sigma D_\alpha A_i^\alpha = 0$ for each i.

Theorem Assume $p = q = 3$. For $7\mu + 3\lambda \neq 0$, the following densities arise from the indicated symmetry groups:

A. Geometrical symmetries ($E(3) \ltimes \mathbb{R}$ - Euclidean group and scaling).

$S = 2\mu\nabla u + (\mu+\lambda)(\nabla\cdot u)\underset{\sim}{1}$,

$P = \mu\nabla u^T\nabla u + (\mu+\lambda)\nabla u^T(\nabla\cdot u) - \frac{1}{2}[\mu\|\nabla u\|^2 + (\mu+\lambda)(\nabla\cdot u)^2]\underset{\sim}{1}$,

$R = x\wedge P + u\wedge S$,

$Y = x^T P + \frac{1}{2}u^T S$,

B. Generalized symmetries ($E(3)$)

$Q = \mu(2\mu+\lambda)\nabla u(\nabla\cdot u) + \mu^2\nabla u(\nabla u - \nabla u^T) + \frac{1}{2}(\mu+\lambda)(2\mu+\lambda)(\nabla\cdot u)^2\underset{\sim}{1}$,

$T = (\mu+\lambda)x\wedge Q + \mu(3\mu+\lambda)u\wedge S + \frac{1}{2}\mu^2(\mu+\lambda)[(u\wedge\nabla u)^T - \operatorname{tr}(u\wedge\nabla u)\underset{\sim}{1}]$,

C. Addition of solutions $\epsilon(x)$ of (7.2)

$K_\epsilon = \epsilon^T S - u^T(\mu\nabla\epsilon + (\mu+\lambda)(\nabla\cdot\epsilon)\underset{\sim}{1})$.

If $7\mu + 3\lambda = 0$, the following additional densities, corresponding to the full conformal group $O(3,1)$ of geometrical symmetries and a conformal group $O(3,1)$ of generalized symmetries, hold:

$$I = 2(x\otimes x)P - |x|^2 P + (x\otimes u - 2u\otimes x)S + 2(x.u)S - 2\mu\ u\otimes u - \tfrac{1}{2}\mu|u|^2 \underset{\sim}{1} ,$$

$$Z = x^T Q + \mu\ u^T S + \mu^2[(\nabla.u)u - u^T\nabla u] ,$$

$$J = 2(x\otimes x)Q - |x|^2 Q + \mu(2x\otimes u - u\otimes x)S + \mu(x.u)S + \mu^2[2x\otimes(u\nabla.u - (\nabla u)u) + u^T(\nabla u)\, x\underset{\sim}{1} - (\nabla u^T u)\otimes x + x\wedge(\nabla u^T\wedge u) - \mathrm{tr}(u\wedge\nabla u)x\wedge\underset{\sim}{1}] .$$

Knowles and Sternberg, [10], derived the densities in A ; they restricted attention just to geometrical symmetries satisfying (5.1), a point raised by Edelen, [4], in his critique of their claims of completeness of conservation laws. In particular, P is Eshelby's energy-momentum tensor, [5], [18]. The K_ε correspond to Betti reciprocity; the remaining laws are new.

The conformal case $7\mu + 3\lambda = 0$ is curious; I am unaware of any physical interpretation. Even if $7\mu + 3\lambda \neq 0$, interesting divergence identities of the form

$$\mathrm{Div}\ A = \|\nabla u\|^2 \quad \text{or} \quad (\nabla.u)^2 \quad \text{or} \quad x^\alpha\|\nabla u\|^2 \quad \text{or} \quad x^\alpha(\nabla.u)^2 \qquad (7.3)$$

can be constructed by taking suitable combinations of I,Z,J , so the conformal group still plays an important role. For two dimensional elasticity, the results are even more striking.

<u>Theorem</u> Assume $p = q = 2$. Using complex notation $z = x^1 + i\, x^2$, $w = u^1 + i\, u^2$, $\xi = 2\partial w/\partial\bar{z}$, $\eta = \mu(u^2_1 - u^1_2) + i(2\mu+\lambda)(u^1_1 + u^2_2)$. If $(3\mu+\lambda) \neq 0$, the following complex densities are conserved:

$2\mu(2\mu+\lambda)\xi\ \ \partial B/\partial\eta + (\mu+\lambda)i\bar{B}$, $\qquad B(\bar{z},\eta)$ analytic

$C(\bar{z},\eta)$ $\qquad C$ analytic

$w\eta + iz\eta^2$

$i(\tilde{w}\eta - \tilde{w}\eta)$ $\qquad \tilde{w}$ an arbitrary solution with corresponding $\tilde{\eta}$.

(If $A = A^1 + iA^2$, then $D_1A^1 + D_2A^2 = 0$ is the conservation law.)

If $3\mu + \lambda = 0$, (7.2) reduces to $\partial^2 w/\partial z^2 = 0$, and there are many more conservation laws. This theorem reflects the fact that any complex-analytic function gives rise to a conformal transformation in the plane.

8. Further Directions of Investigation

1) Classify all conservation laws and symmetries for elastic materials with relatively simple constitutive relations, e.g. elastic fluid, Mooney-Rivlin materials, anisotropic linear elasticity, etc, [9].

2) Extend the results to elastodynamics. Fletcher, [6], has shown how each of Knowles and Sternberg's laws has a dynamic counterpart. Presumably, Hamiltonian techniques in [12], will be of use here.

3) Discuss applications of the laws found to crack problems, cf. [2].

4) The appearance of conformal symmetries seems especially significant. For nonlinear wave equations, Morawetz and Strauss, cf. [19], have shown the importance of conservation laws arising from conformal invariance in scattering problems. The identities (7.3) could be equally useful for scattering in elastic bodies with nonlinear body forces. The two dimensional case looks especially promising.

5) For linear problems each symmetry gives rise to an infinite family of generalized symmetries, and, often, conservation laws depending on higher order derivatives of u , [13]. Are these of use? (This question has not even been investigated for the wave equation!) For nonlinear equations, the existence of infinite families of symmetries is closely tied with complete integrability of the system, the prototypical example being the Korteweg-de Vries equation, [1]. Do any elastic materials have such a property?

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