

# Hamiltonian structures for systems of hyperbolic conservation laws

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The bi-Hamiltonian structure for a large class of one-dimensional hyperbolic systems of conservation laws in two field variables, including the equations of gas dynamics, shallow water waves, one-dimensional elastic media, and the Born–Infeld equation from nonlinear electrodynamics, is exhibited. For polytropic gas dynamics, these results lead to a quadri-Hamiltonian structure. New higher-order entropy-flux pairs (conservation laws) and higher-order symmetries are exhibited.

## I. HAMILTONIAN SYSTEMS OF CONSERVATION LAWS

With Gardner’s discovery of the Hamiltonian structure of the Korteweg–de Vries equation,<sup>1</sup> and Arnol’d’s determination of the Lie–Poisson structure underlying the Euler equations of fluid flow,<sup>2</sup> the range of applications of the Hamiltonian formalism embraced truly infinite-dimensional systems. Subsequent progress in the field has been rapid, especially after Magri proved his remarkable theorem on the complete integrability of bi-Hamiltonian systems.<sup>3</sup> This has led to a large number of papers on the applications of Hamiltonian methods to the integrable soliton systems arising as models for nonlinear wave phenomena. More recently, a number of researchers have exploited Arnol’d’s original idea to vastly broaden the range of Hamiltonian systems in fluids, plasmas, and relativity.<sup>4</sup> Yet a third branch of infinite-dimensional Hamiltonian mechanics was opened with the investigations of Manin,<sup>5</sup> Cavalcante and McKean,<sup>6</sup> Dubrovin and Novikov,<sup>7</sup> and Nutku,<sup>8</sup> into the Hamiltonian structures arising in first-order quasilinear systems of partial differential equations, known as “systems of hydrodynamic type.” These are beginning to have applications to the analysis of hyperbolic conservation laws, and the study of discontinuous (shock wave) solutions.<sup>9–11</sup>

This paper represents a further investigation of the last class of equations, with two-component hyperbolic systems of conservation laws in one spatial variable being the type of system under consideration, and with particular attention being paid to gas dynamics and some model equations arising in elasticity. Our results build on earlier work of Sheftel,<sup>12,13</sup> who was primarily concerned with the symmetry structure of these systems. We connect Sheftel’s results with the Hamiltonian framework via Magri’s theorem, and deduce the bi-Hamiltonian structure of a broad class of systems of conservation laws, as well as the quadri-Hamiltonian structure of the equations of polytropic gas dynamics themselves. New conservation laws (entropy-flux pairs), substantially extending earlier results of Verosky,<sup>14</sup> are also found. Interesting examples of incompatible bi-Hamiltonian systems are found. Indeed, the fundamental message of this paper is that, in the case of two-dimensional Hamiltonian systems, and particularly for polytropic gas dynamics, nature appears to be overly generous in the amount of structure

she provides. Why these systems should be this way remains mysterious. We hope to return to this topic, and to applications of these results to the analysis of smooth solutions and shock waves in a subsequent paper. As a prerequisite for studying this paper, we assume that the reader is familiar with the fundamentals of symmetry groups, Hamiltonian systems of evolution equations, conservation laws, and Magri’s theorem, as presented, for instance, in Olver.<sup>15</sup>

The general form of a two-component hyperbolic system of conservation laws of Hamiltonian type is as follows. The unknowns  $\mathbf{u}(x,t) = \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}$  depend on the real-valued temporal and spatial variables  $t$  and  $x$ . The Hamiltonian functional or energy is

$$\mathcal{H}[\mathbf{u}] = \int H(u,v)dx,$$

where the integrand or *Hamiltonian density*  $H(u,v)$  is a smooth, but otherwise arbitrary function of  $u$  and  $v$ . The system takes the explicit form

$$\begin{aligned} u_t &= D_x(H_v) = H_{uv} \cdot u_x + H_{vv} \cdot v_x, \\ v_t &= D_x(H_u) = H_{uu} \cdot u_x + H_{uv} \cdot v_x, \end{aligned} \quad (1.1)$$

with the subscripts denoting partial derivatives. We can also write it in the convenient vector format

$$\mathbf{u}_t = \mathbf{H} \cdot \mathbf{u}_x, \quad (1.2)$$

where

$$\mathbf{H} = \sigma_1 \cdot D^2 H = \begin{pmatrix} H_{uv} & H_{vv} \\ H_{uu} & H_{uv} \end{pmatrix}, \quad (1.3)$$

and  $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . The system (1.2) is in the elementary Hamiltonian form

$$\mathbf{u}_t = \mathcal{D}_0 E[H]. \quad (1.4)$$

Here  $E = E_{\mathbf{u}}$  denotes the Euler operator or variational derivative, and the Hamiltonian operator is the matrix differential operator

$$\mathcal{D}_0 = \sigma_1 \cdot D_x = \begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix}. \quad (1.5)$$

The corresponding Poisson bracket is

$$\begin{aligned} \{F, H\} &= \int E(F) \cdot \mathcal{D}_0 E(H) dx \\ &= \int \{E_u(F) \cdot D_x E_v(H) + E_v(F) D_x E_u(H)\} dx, \end{aligned}$$

which arises in a number of physically important systems.

There are three important examples. The equations of gas dynamics<sup>16</sup>

$$u_t + uu_x + f(v)v_x = 0, \quad v_t + (uv)_x = 0, \quad (1.6)$$

are of the form (1.1), with

$$H(u, v) = -\left(\frac{1}{2}u^2v + F(v)\right), \quad \text{where } f(v) = F'(v). \quad (1.7)$$

In physical applications,  $u(x, t)$  represents the velocity, and  $v(x, t)$  the density (more customarily denoted by  $\rho$ ) of the fluid; the function  $f(v)$  is related to the physical pressure  $P(v)$  according to the equation  $f(v) = P'(v)/v$ . The equations of *polytropic gas dynamics* correspond to the choice  $F(v) = v^\gamma/\gamma(\gamma - 1)$  for some exponent  $\gamma \neq 0, 1$ . In particular, the special case  $F(v) = \frac{1}{2}v^2$  also arises in shallow water theory.<sup>6</sup> A second important example is provided by the Hamiltonian

$$H(u, v) = u/v + v/u, \quad (1.8)$$

in which case (1.1) is equivalent to the Born-Infeld equation from nonlinear electrodynamics.<sup>17,18</sup> Finally, we mention the simple model

$$u_t = [\sigma(v)]_x, \quad v_t = u_x, \quad (1.9)$$

for a one-dimensional nonlinear elastic medium.<sup>9,10,19</sup> In this case, the Hamiltonian density is

$$H(u, v) = \frac{1}{2}u^2 + F(v), \quad (1.10)$$

where  $F'(v) = \sigma(v)$ . The case when  $\sigma$  is a monotone function of  $v$  corresponds to an ideal fluid or elastic solid; models of phase transitions arise with more general functions  $\sigma$ , the most interesting being that of a van der Waals fluid, where  $\sigma(v) = c/(v - b) - a/v^2$ . Furthermore, the case  $\sigma(v) = (1 + v)^{-(1+\gamma)}$  corresponds to the Euler equation arising in nonlinear acoustics.<sup>20</sup>

## II. A PROTOTYPE: THE RIEMANN EQUATION

A remarkable feature of the two-component Hamiltonian system (1.1) is that, suitably interpreted, all the structure already manifests itself in the simple scalar nonlinear wave equation

$$u_t = uu_x. \quad (2.1)$$

This equation has many names (Riemann, inviscid Burgers', Hopf, etc.), and serves as a prototype for so many of the phenomena associated with hyperbolic systems, that its role in the Hamiltonian structure and conservation laws will not be surprising. See Whitham<sup>16</sup> for a good review of the classical theory of this equation.

The purpose of this introductory section is to list some basic results for the Riemann equation (2.1), all of which have direct counterparts in the equations for polytropic gas dynamics (1.6) and the model elasticity equations (1.9), and many of which have counterparts for more general two-

dimensional hyperbolic systems. The rather easy proofs are left to the reader, as they serve as good warm-up exercises for the much harder calculations in two dimensions.

(1) There is an infinite sequence of zeroth-order conserved densities

$$H_n(u) = u^n, \quad n = 1, 2, 3, \dots \quad (2.2)$$

In fact, any function  $F(u)$  is a conserved density for (2.1). [By the *order* of a function  $F[u] = F(u, u_x, \dots)$ , we mean the highest order derivative of  $u$  on which it depends.]

(2) The equation admits three first-order Hamiltonian operators

$$\mathcal{D}_0 = D_x, \quad \mathcal{D}_1 = 2u \cdot D_x + u_x, \quad \mathcal{D}_2 = u^2 \cdot D_x + uu_x, \quad (2.3)$$

and so can be written in Hamiltonian form in three distinct ways:

$$u_t = \mathcal{D}_0 \cdot E(\frac{1}{6}H_3) = \mathcal{D}_1 \cdot E(\frac{1}{6}H_2) = \mathcal{D}_2 \cdot E(H_1).$$

Moreover, since

$$\mathcal{D}_2 \neq \mathcal{D}_1 \cdot \mathcal{D}_0^{-1} \cdot \mathcal{D}_1,$$

these Hamiltonian operators are not trivially related.<sup>21</sup>

(3) The three Hamiltonian operators  $\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2$ , are compatible, meaning that any two of them form a Hamiltonian pair in the sense of Magri's theorem.<sup>3,22</sup> Therefore, Eq. (2.1) is a "tri-Hamiltonian system." The resulting recursion operators

$$\mathcal{R}_1 = \mathcal{D}_1 \cdot \mathcal{D}_0^{-1}, \quad \mathcal{R}_2 = \mathcal{D}_2 \cdot \mathcal{D}_0^{-1}, \quad \mathcal{R}_3 = \mathcal{D}_2 \cdot \mathcal{D}_1^{-1}, \quad (2.4)$$

which map symmetries to symmetries, are trivially related by

$$\mathcal{R}_2 = \mathcal{R}_3 \cdot \mathcal{R}_1,$$

but are otherwise independent "integro-differential operators."

(4) There is an infinite sequence of commuting Hamiltonian flows corresponding to the evolutionary vector fields  $v_n = \mathcal{Q}_n[u] \cdot \partial_u$ , with characteristics  $\mathcal{Q}_n[u] = u^n \cdot u_x$ . The  $n$ th flow can be written in Hamiltonian form in three distinct ways:

$$\begin{aligned} u_t = \mathcal{Q}_n[u] &= u^n \cdot u_x \\ &= \mathcal{D}_0 \cdot E([(n+2)(n+1)]^{-1} H_{n+2}) \\ &= \mathcal{D}_1 \cdot E([(2n+1)(n+1)]^{-1} H_{n+1}) \\ &= \mathcal{D}_2 \cdot E(n^{-2} H_n), \end{aligned} \quad (2.5)$$

whose Hamiltonians  $H_n$  are given by (2.2). These Hamiltonians are in involution with respect to any of the three Poisson brackets determined by the Hamiltonian operators:

$$\{H_n, H_m\}_j = \int E(H_n) \cdot \mathcal{D}_j E(H_m) dx = 0, \quad j = 0, 1, 2.$$

The recursion operators (2.4) reproduce the hierarchy of Hamiltonian flows,

$$\mathcal{R}_1(\mathcal{Q}_n) = [(2n+3)/(n+1)] \mathcal{Q}_{n+1},$$

$$\mathcal{R}_2(\mathcal{Q}_n) = [(n+2)/(n+1)] \mathcal{Q}_{n+2}.$$

Thus, starting with the initial flow (2.1),  $\mathcal{R}_1$  produces every flow in the hierarchy (2.5), whereas  $\mathcal{R}_2$  produces every oth-

er one. Interestingly, even though there are two independent recursion operators, the two hierarchies of zeroth-order conserved densities and corresponding first-order symmetries of (2.1) guaranteed by Magri's theorem happen to coincide. However, this remark is special to the polynomial hierarchy (2.2); on other hierarchies of Hamiltonian densities  $H_n(u)$  defined by the recursion relation, starting with a nonpolynomial function  $H_0(u)$ , these recursion operators will act differently.

(5) The equation admits an additional third-order Hamiltonian operator

$$\mathcal{E} = D_x \cdot (1/u_x) \cdot D_x \cdot (1/u_x) \cdot D_x. \quad (2.6)$$

Each of the Hamiltonian systems (2.5) can be written in this new Hamiltonian form using the Hamiltonian operator  $\mathcal{E}$ ,

$$u_t = u^n u_x$$

$$= \mathcal{E} \cdot E[(n+1)(n+2)(n+3)(n+4)]^{-1} H_{n+4}.$$

(6) The Hamiltonian operators  $\mathcal{D}_0$  and  $\mathcal{E}$  are compatible; however,  $\mathcal{E}$  is not compatible with either  $\mathcal{D}_1$  or  $\mathcal{D}_2$ . Thus, the Riemann equation, and its higher degree generalizations (2.5) are in the rather anomalous position of being "quadri-Hamiltonian systems," meaning there are four distinct local Hamiltonian structures, but of the six different possible pairs of Hamiltonian operators, four are compatible, whereas two are not!

(7) The recursion operator

$$\hat{\mathcal{R}} = \mathcal{E} \cdot \mathcal{D}_0^{-1} = D_x \cdot (1/u_x) \cdot D_x \cdot (1/u_x),$$

is the square of a simpler first-order recursion operator

$$\mathcal{R} = D_x \cdot (1/u_x). \quad (2.7)$$

This latter operator acts on the first-order flows according to

$$\mathcal{R}(Q_n) = nQ_{n-1},$$

and so, up to multiple, "inverts" the first-order recursion operator  $\mathcal{R}_1$ . [Again, as in part (4), this is special to the polynomial hierarchy (2.5).]

(8) There is a rational second-order generalized symmetry  $\hat{v}_2 = \hat{Q}_2 \cdot \partial_u = u_x^{-2} \cdot u_{xx} \cdot \partial_u$  with corresponding flow governed by the evolution equation

$$u_t = \hat{Q}_2 = u_{xx}/u_x^2.$$

Thus the recursion operator  $\mathcal{R}$  generates an additional hierarchy of higher order symmetries  $\hat{v}_n = \hat{Q}_n \cdot \partial_u$ , where

$$\hat{Q}_{n+1} = \mathcal{R}(\hat{Q}_n), \quad n = 2, 3, \dots \quad (2.8)$$

(9) The third-order generalized symmetry  $\hat{v}_3$  in the hierarchy (2.8) is Hamiltonian relative to the first-order Hamiltonian operator  $\mathcal{D}_0$ . Thus there is a rational first-order conserved density

$$\hat{H}_1 = 1/u_x$$

such that the third-order evolution equation

$$u_t = \hat{Q}_3 = (u_x \cdot u_{xxx} - 3u_{xx}^2)/u_x^4 \quad (2.9)$$

corresponding to  $\hat{v}_3$  is in fact bi-Hamiltonian

$$u_t = \mathcal{D}_0 \cdot E(\hat{H}_1) = \mathcal{E} \cdot E(\hat{H}_0).$$

Here

$$\hat{H}_0 = xu + \frac{1}{2}tu^2$$

is a (time-dependent) zeroth-order conserved density for (2.1). Consequently, every other member of the hierarchy (2.8), i.e., the vector fields  $\hat{v}_{2n+1}$ , is bi-Hamiltonian, corresponding to a hierarchy of higher-order rational conserved densities  $\hat{H}_n$ ,  $n = 0, 1, 2, \dots$ .

(10) For any solution to the general first-order flow (2.5) corresponding to the vector field  $v_n$ , each of the higher-order quantities  $\int \hat{H}_m [u] dx$  is a linear function of  $t$  (provided that the integral converges). In fact, if  $H(u)$  is any zeroth-order conserved density, with flow  $u_t = D_x E(H)$ , then

$$\hat{H}_m + t \frac{\partial^{2m+1} H}{\partial u^{2m+1}}$$

is a conserved density for such a flow. In particular,  $\hat{H}_m$  is a conserved density for  $v_n$  whenever  $2m \geq n - 1$ .

There are some additional facts concerning higher order symmetries and conservation laws for (2.1) which do not appear to have counterparts for the equations of gas dynamics. [This demonstrates that one must be careful when deciding which of the properties of (2.1) one wishes to generalize to two-dimensional systems in general, and gas dynamics in particular.] The proof of the following complete characterization of symmetries and conservation laws is not difficult.

**Theorem 2.1:** Define the rational differential functions

$$K_0 = u, \quad K_1 = x - tu, \quad K_2 = u/u_x - x, \quad K_3 = u_{xx}/u_x^2, \\ K_{n+1} = (1/u_x) D_x K_n, \quad n \geq 3.$$

(i)  $T[u]$  is conserved density of (2.1) if and only if

$$T = u_x \cdot F(K_0, K_1, K_2, \dots, K_n),$$

where  $F$  is an arbitrary smooth function of the differential functions  $K_0, \dots, K_n$ .

(ii)  $Q[u]$  is the characteristic of a generalized symmetry  $v_Q = Q \cdot \partial_u$  of (2.1) if and only if

$$Q = u_x \cdot G(K_0, K_1, K_2, \dots, K_n),$$

where  $G$  is an arbitrary smooth function of the functions  $K_0, \dots, K_n$ .

### III. ZERO-ORDER CONSERVED DENSITIES

We now return to the general two-component Hamiltonian system (1.1), and investigate its symmetries, conserved densities, and Hamiltonian structures. We will follow the basic outline of the properties for the elementary one-dimensional wave equation (2.1) discussed in Sec. II. We therefore begin by discussing item (1), the existence of zeroth-order conserved densities. Much of this material is well-known.<sup>5,13,23</sup>

Associated with each hyperbolic system (1.1) is a second-order linear partial differential equation

$$A(u,v)F_{uu} = B(u,v)F_{vv}, \quad (3.1)$$

with  $A = H_{vv}$ ,  $B = H_{uu}$ . The solutions of (3.1) are the zeroth-order conserved densities (entropies) for the hyperbolic system (1.1).<sup>10,13</sup>

**Proposition 3.1:** A functional  $\mathcal{F}[u] = \int F(u,v) dx$  is conserved for the Hamiltonian system (1.1) if and only if  $F$  is a solution to Eq. (3.1).

Note that  $H$  itself is always a solution to (3.1). Also, if  $F$

is a conserved density for the Hamiltonian system governed by  $H$ , then, reciprocally,  $H$  is a conserved density for the Hamiltonian system governed by  $F$ . In addition, we always have four elementary solutions given by the functions

$$1, u, v, u \cdot v. \quad (3.2)$$

These are conserved densities for *any* Hamiltonian system of the form (1.1).

The most important class of Hamiltonian systems (1.1) are those for which the corresponding partial differential equation (3.1) admits a separation of variables in the rectangular  $(u, v)$  coordinates.

**Definition 3.2:** A Hamiltonian density  $H(u, v)$  is called *separable* if (a) the second-order partial derivatives  $H_{uu}$  and  $H_{vv}$  do not vanish identically, and (b) there exist functions  $\lambda(u)$  and  $\mu(v)$  such that

$$H_{uu}/H_{vv} = \lambda(u)/\mu(v). \quad (3.3)$$

If the Hamiltonian density is separable, then Eq. (3.1) determining the zeroth-order conserved densities takes the form

$$F_{uu}/\lambda(u) = F_{vv}/\mu(v). \quad (3.4)$$

In this paper, we will deal exclusively with separable Hamiltonian systems (1.1); an interesting open question is how many of these results can be generalized to the nonseparable cases. Gas dynamics, (1.6), is an example of a separable system, where

$$\lambda(u) \equiv 1 \text{ and } \mu(v) = \frac{F''(v)}{v} = \frac{f(v)}{v} = \frac{P'(v)}{v^2}.$$

In fact, the special case when  $\lambda \equiv 1$  has added importance.

**Definition 3.3:** Let  $H(u, v)$  be a Hamiltonian density. If the ratio  $H_{uu}/H_{vv} = \mu(v)$  is a function of  $v$  alone, then  $H$  is said to be a *generalized gas dynamics Hamiltonian density*.

The elastic models (1.9) are of gas dynamics type, with  $\mu(v) = \sigma'(v)$ . The Born-Infeld Hamiltonian (1.8) is separable, with  $\lambda(u) = u^{-4}$  and  $\mu(v) = v^{-4}$ , but not of gas dynamics type. [However, Verosky<sup>24</sup> noticed that the transformation

$$\tilde{u} = -(u^{-2} + v^{-2}), \quad \tilde{v} = \frac{1}{2}uv, \quad (3.5)$$

will change the Born-Infeld system into a polytropic gas dynamics system with adiabatic index  $\gamma = -1$ , i.e.,  $F(v) = 1/2v$ . This remarkable transformation will be discussed in more detail in a forthcoming publication.<sup>17]</sup>

In the separable case, there are four fundamental hierarchies of solutions to the wave equation (3.4). Each of them takes the form

$$H_n(u, v) = \sum_{i=0}^n F_i(u) \cdot G_{n-i}(v),$$

where the functions  $F_i$  and  $G_i$  are generated by the recursion relations

$$\frac{\partial^2 F_i}{\partial u^2} = \lambda(u) F_{i-1}, \quad \frac{\partial^2 G_i}{\partial v^2} = \mu(v) G_{i-1}, \quad (3.6)$$

and normalized so that

$$F_i(0) = F'_i(0) = 0, \quad G_i(0) = G'_i(0) = 0.$$

The hierarchies depend on the initial selection of  $H_0 = F_0 \cdot G_0$ , and there are four obvious possibilities, given

by the four elementary conserved densities (3.2):

$$\begin{aligned} H_0 &= 1, & F_0 &= G_0 = 1, \\ H_0 &= u, & F_0 &= u, & G_0 &= 1, \\ H_0 &= v, & F_0 &= 1, & G_0 &= v, \\ H_0 &= uv, & F_0 &= u, & G_0 &= v. \end{aligned}$$

In practice, it is expedient to combine the first and second, and the third and fourth hierarchies, leading to two fundamental hierarchies of conserved densities, which we denote by  $H_n$  and  $\tilde{H}_n$ , respectively, so that  $H_{2n}$  is the  $n$ th member of the first hierarchy,  $H_{2n+1}$  is the  $n$ th member of the second hierarchy,  $\tilde{H}_{2n}$  is the  $n$ th member of the third hierarchy, and  $\tilde{H}_{2n+1}$  is the  $n$ th member of the fourth hierarchy.

For reference, we list the first few members of each sequence in the gas dynamics case  $\lambda \equiv 1$ :

$$\begin{aligned} H_0 &= 1, & \tilde{H}_0 &= v, \\ H_1 &= u, & \tilde{H}_1 &= uv, \\ H_2 &= \frac{1}{2}u^2 + G_1(v), & \tilde{H}_2 &= \frac{1}{2}u^2v + \tilde{G}_1(v), \\ H_3 &= \frac{1}{6}u^3 + uG_1(v), & \tilde{H}_3 &= \frac{1}{6}u^3v + u\tilde{G}_1(v), \\ H_4 &= \frac{1}{24}u^4 + \frac{1}{2}u^2G_1(v) + G_2(v), \\ \tilde{H}_4 &= \frac{1}{24}u^4v + u^2\tilde{G}_1(v) + \tilde{G}_2(v), \end{aligned} \quad (3.7)$$

etc., where, according to (3.6),

$$\begin{aligned} G_1(v) &= \int_0^v (v-w) \cdot \mu(w) dw, \\ G_2(v) &= \int_0^v (v-w) \cdot \mu(w) \cdot G_1(w) dw, \\ \tilde{G}_1(v) &= \int_0^v (v-w) \cdot w \cdot \mu(w) dw, \\ \tilde{G}_2(v) &= \int_0^v (v-w) \cdot \mu(w) \cdot \tilde{G}_1(w) dw. \end{aligned}$$

Note that the elastic Hamiltonian (1.10) appears in the first hierarchy as  $H_2$ , whereas the gas dynamics Hamiltonian (1.7) appears in the alternative hierarchy as  $-\tilde{H}_2$ .

**Lemma 3.4:** Let  $H(u, v)$  be a generalized gas dynamics Hamiltonian. If  $F(u, v)$  is a conserved density, then so is  $\partial F / \partial u$ .

In fact, the hierarchies (3.7) satisfy

$$\frac{\partial H_n}{\partial u} = H_{n-1}, \quad (3.8)$$

and similarly for  $\tilde{H}_n$ .

For a general separable system, each of the Hamiltonian functions generates a Hamiltonian flow, governed by the corresponding evolutionary system (1.1). We let

$$\mathbf{Q}_n = \mathcal{D}_0 E [H_n] = \mathbf{H}_n \cdot \mathbf{u}_x \quad (3.9)$$

[cf. (1.3) and (1.5)] denote the right-hand side of this equation, which is also the characteristic for the symmetry vector field  $\mathbf{v}_n = \mathbf{Q}_n \cdot \partial_{\mathbf{u}}$ . We define  $\tilde{\mathbf{H}}_n$ ,  $\tilde{\mathbf{Q}}_n$ , and  $\tilde{\mathbf{v}}_n$  from the alternative hierarchy  $\tilde{H}_n$  similarly. In particular,  $\mathbf{Q}_0 = \mathbf{Q}_1 = \tilde{\mathbf{Q}}_0 = 0$ , meaning that the densities  $H_0, H_1, \tilde{H}_0$  determine distinguished functionals (Casimirs) for the Hamiltonian structure (1.5), while  $\tilde{\mathbf{Q}}_1 = \mathbf{u}_x$  is the common translational symmetry of the systems. All the Hamiltonians  $H_n$  and  $\tilde{H}_n$

are in involution with respect to the Poisson bracket determined by the Hamiltonian operator (1.5), i.e.,

$$\{H_n, H_m\} = \{H_n, \tilde{H}_m\} = \{\tilde{H}_n, \tilde{H}_m\} = 0,$$

for all  $n, m$ . Consequently, the vector fields  $v_n$  and  $\tilde{v}_m$  all mutually commute.

#### IV. FIRST-ORDER HAMILTONIAN STRUCTURES: POLYTROPIC GAS DYNAMICS

We have already seen the first Hamiltonian structure of the gas dynamics system (1.6). We now turn to a discussion of the other first-order Hamiltonian structures in the case of polytropic gas dynamics. These results correspond to items

$$\mathcal{D}_1 = \begin{pmatrix} v^{\gamma-2} \cdot D_x + D_x \cdot v^{\gamma-2} & (\gamma-1)u \cdot D_x + u_x \\ (\gamma-1)u \cdot D_x + (\gamma-2)u_x & v \cdot D_x + D_x \cdot v \end{pmatrix}.$$

They have yet another Hamiltonian form,

$$u_t = \mathcal{D}_2 \cdot E(\tilde{H}_0), \tag{4.2}$$

where

$$\mathcal{D}_2 = \begin{pmatrix} uv^{\gamma-2} \cdot D_x + D_x \cdot uv^{\gamma-2} & \left\{ \frac{1}{2}(\gamma-1)u^2 + 2[v^{\gamma-1}/(\gamma-1)] \right\} \cdot D_x \\ \left\{ \frac{1}{2}(\gamma-1)u^2 + 2[v^{\gamma-1}/(\gamma-1)] \right\} \cdot D_x & uv \cdot D_x + D_x \cdot uv \\ + uu_x + v^{\gamma-2}v_x & \\ + (\gamma-2)uu_x + v^{\gamma-2}v_x & \end{pmatrix}.$$

The Hamiltonian operators  $\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2$  are mutually compatible, leading to three distinct Hamiltonian pairs. The corresponding recursion operators are just as in (2.4), i.e.,

$$\mathcal{R}_1 = \mathcal{D}_1 \cdot \mathcal{D}_0^{-1}, \quad \mathcal{R}_2 = \mathcal{D}_2 \cdot \mathcal{D}_0^{-1}, \quad \mathcal{R}_3 = \mathcal{D}_2 \cdot \mathcal{D}_1^{-1}, \tag{4.3}$$

and, as before, are trivially related by the identity  $\mathcal{R}_2 = \mathcal{R}_3 \cdot \mathcal{R}_1$ , but are otherwise distinct. Nevertheless, they both give rise to the same series of gas dynamics Hamiltonians, since

$$\mathcal{R}_1(\mathbf{Q}_n) = \mathbf{Q}_{n+1}, \quad \mathcal{R}_2(\mathbf{Q}_n) = \mathbf{Q}_{n+2},$$

and similarly for the alternative hierarchy  $\tilde{v}_n$ . In other words, just as with the one-dimensional prototype (2.1),  $\mathcal{R}_2$  leads to every other member of the hierarchies (3.9), and so for some reason  $\mathcal{R}_3$  acts the same as  $\mathcal{R}_1$  on these hierarchies. (As with the Riemann equation recursion operators, these recursion operators will act differently on other hierarchies.) Strangely, there does not appear to be a counterpart of these two recursion operators in the general nonpolytropic case, i.e., when the pressure is not proportional to a power of the density. We do not fully understand why this should be the case.

A similar situation exists for the ‘‘polytropic’’ versions of the elasticity models (1.9), i.e., when  $F(v) = v^{\gamma-1}/(\gamma-2) \cdot (\gamma-1)$ ,  $\gamma \neq 0, 1, 2$ . Here we can write it in a second Hamiltonian form

$$u_t = \mathcal{D}_1 \cdot E((\gamma-2)^{-1}H_1). \tag{4.4}$$

However, even though this system cannot be written in Hamiltonian form using the operator  $\mathcal{D}_2$ , nevertheless it still admits all three recursion operators (4.3).

The only other case of (1.9) which appears to admit a

(2)–(4) for the one-dimensional equation (2.1). Strangely, these results do not seem to generalize to the general gas dynamics system, much less the more general hyperbolic system (1.1).

Nutku<sup>8</sup> has shown that, in the case of polytropic gas dynamics, there are two additional first-order Hamiltonian structures. Using the Hamiltonian hierarchies (3.7), we find that we can write the polytropic gas dynamics equations (1.6), with  $f(v) = v^{\gamma-2}$ , in the alternative Hamiltonian form

$$u_t = \mathcal{D}_1 \cdot E(\gamma^{-1}\tilde{H}_1), \tag{4.1}$$

where  $\mathcal{D}_1$  is the Hamiltonian operator

second Hamiltonian structure is the curious choice  $\sigma(v) = e^v$ . Here we have the Hamiltonian operator

$$\mathcal{D}_1 = \begin{pmatrix} u \cdot e^v \cdot D_x + D_x \cdot u \cdot e^v & (e^v + \frac{1}{2}u^2) \cdot D_x + D_x \cdot e^v \\ e^v \cdot D_x + D_x \cdot (e^v + \frac{1}{2}u^2) & u \cdot D_x + D_x \cdot u \end{pmatrix},$$

and the equation can be written as

$$u_t = \mathcal{D}_1 \cdot E(\tilde{H}_0).$$

Interestingly, the resulting recursion operator  $\mathcal{R}_1 = \mathcal{D}_1 \cdot \mathcal{D}_0^{-1}$  alternates between the two hierarchies of conserved densities (3.7), i.e.,

$$\mathcal{R}_1(\mathbf{Q}_n) = \tilde{\mathbf{Q}}_{n+2}, \quad \mathcal{R}_1(\tilde{\mathbf{Q}}_n) = \mathbf{Q}_{n+2}.$$

Note that this also implies that the gas dynamics equations (1.6), with  $f(v) = e^v$ , can also be written in the Hamiltonian form  $u_t = \mathcal{D}_1 \cdot E(H_1)$ . Apart from obvious rescalings and translations of  $v$ , this appears to be the only nontrivial case where this happens.

#### V. HIGHER-ORDER HAMILTONIAN STRUCTURE

In this section, we present the second Hamiltonian structure of a general separable Hamiltonian system (1.2). We begin by exhibiting the analog of the third-order Hamiltonian operator (2.6), thereby finding analogies to items (5) and (6) of Sec. II. We will then show how to write the system itself in bi-Hamiltonian form.

Following Sheftel<sup>13</sup> define the functions

$$L(u) = \int_0^u \lambda(s) ds, \quad M(v) = \int_0^v \mu(s) ds,$$

and the matrix variables

$$U(u,v) = \begin{pmatrix} u & M(v) \\ v & L(u) \end{pmatrix}, \quad V(u,v) = \begin{pmatrix} L(u) & M(v) \\ v & u \end{pmatrix}. \quad (5.1)$$

Note that  $\sigma_1 \cdot U = V^T \cdot \sigma_1$ , where  $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . Let

$$U_x = \begin{pmatrix} u_x & \mu(v)v_x \\ v_x & \lambda(u)u_x \end{pmatrix} \quad \text{and} \quad U_t = \begin{pmatrix} u_t & \mu(v)v_t \\ v_t & \lambda(u)u_t \end{pmatrix}$$

denote the total  $x$  and  $t$  derivatives of the matrix  $U$ ; define  $V_x$  and  $V_t$  analogously. Also let

$$U_x^{-1} = \frac{1}{\delta} \begin{pmatrix} \lambda(u)u_x & -\mu(v)v_x \\ -v_x & u_x \end{pmatrix},$$

$$V_x^{-1} = \frac{1}{\delta} \begin{pmatrix} u_x & -\mu(v)v_x \\ -v_x & \lambda(u)u_x \end{pmatrix},$$

$$\delta = \lambda(u) \cdot u_x^2 - \mu(v) \cdot v_x^2,$$

denote the matrix inverses of  $U_x$  and  $V_x$ .

The basic third-order Hamiltonian operator for a general separable Hamiltonian system can be written compactly as follows.

**Theorem 5.1:** The operator

$$\begin{aligned} \mathcal{E} &= D_x \cdot V_x^{-1} \cdot D_x \cdot U_x^{-1} \cdot \sigma_1 \cdot D_x \\ &= D_x \cdot V_x^{-1} \cdot D_x \cdot \sigma_1 \cdot V_x^{-T} \cdot D_x \end{aligned} \quad (5.2)$$

is Hamiltonian.

*Proof:* The fact that  $\mathcal{E}$  is a skew-adjoint differential operator follows immediately from the second formula in (5.2). Rather than prove the Poisson form<sup>22</sup> of the Jacobi identity for  $\mathcal{E}$ , it is simpler to prove that the symplectic two-form

$$\Omega = \int \{du^T \wedge \mathcal{E}^{-1} du\} dx,$$

is closed<sup>25</sup>:  $d\Omega = 0$ . Let  $\varphi, \psi$  denote potential functions for  $u, v$ , respectively, so

$$\varphi_x = u, \quad \psi_x = v.$$

Thus, formally,

$$D_x^{-1}(d\varphi) = du, \quad D_x^{-1}(d\psi) = dv,$$

and hence

$$\Omega = \int \{(v_x d\varphi + u_x d\psi)$$

$$\wedge D_x^{-1}(\lambda(u)u_x d\varphi + \mu(v)v_x d\psi)\} dx$$

$$= \int \{(\psi_{xx} d\varphi + \varphi_{xx} d\psi)$$

$$\wedge D_x^{-1}(\lambda(\varphi_x)\varphi_{xx} d\varphi + \mu(\psi_x)\psi_{xx} d\psi)\} dx.$$

Let  $L, M$  be as in (5.1), and let

$$\widehat{L}(u) = \int_0^u s \cdot \lambda(s) ds,$$

$$\widetilde{L}(u) = \int_0^u (u-s)\lambda(s) ds = uL(u) - \widehat{L}(u),$$

$$\widehat{M}(u) = \int_0^u s \cdot \mu(s) ds,$$

$$\widetilde{M}(u) = \int_0^u (u-s)\mu(s) ds = uM(u) - \widehat{M}(u).$$

Performing an inspired series of integration by parts, we find

$$\begin{aligned} \Omega &= - \int \{(\psi_x d\varphi + \varphi_x d\psi) \wedge (\lambda(\varphi_x)\varphi_{xx} d\varphi + \mu(\psi_x)\psi_{xx} d\psi) \\ &\quad + d(\varphi_x \psi_x) \wedge D_x^{-1}(\lambda(\varphi_x)\varphi_{xx} d\varphi + \mu(\psi_x)\psi_{xx} d\psi)\} dx \\ &= \int \{(\varphi_x \lambda(\varphi_x)\varphi_{xx} - \psi_x \mu(\psi_x)\psi_{xx}) d\varphi \wedge d\psi \\ &\quad - d(\varphi_x \psi_x) \wedge (L(\varphi_x) d\varphi + M(\psi_x) d\psi) + d(\varphi_x \psi_x) \wedge D_x^{-1} d(\widetilde{L}(\varphi_x) + \widetilde{M}(\psi_x))\} dx, \\ &= \int \{(-\widehat{L}(\varphi_x) + \widehat{M}(\psi_x))(d\varphi_x \wedge d\psi + d\varphi \wedge d\psi_x) \\ &\quad - d(\varphi_x \psi_x) \wedge (L(\varphi_x) d\varphi + M(\psi_x) d\psi) + d(\varphi_x \psi_x) \wedge D_x^{-1} d(\widetilde{L}(\varphi_x) + \widetilde{M}(\psi_x))\} dx. \end{aligned}$$

Therefore,

$$\begin{aligned} d\Omega &= \int \{-\varphi_x \lambda(\varphi_x) d\varphi_x \wedge d\varphi \wedge d\psi_x \\ &\quad + \psi_x \mu(\psi_x) d\psi_x \wedge d\varphi_x \wedge d\psi \\ &\quad + d(\varphi_x \psi_x) \wedge (\lambda(\varphi_x) d\varphi_x \wedge d\varphi \\ &\quad + \mu(\psi_x) d\psi_x \wedge d\psi)\} dx, \end{aligned}$$

which clearly vanishes. This completes the proof that  $\Omega$  is closed, and hence  $\mathcal{E}$  is a Hamiltonian operator.

**Proposition 5.2:** The operators  $\mathcal{E}$  and  $\mathcal{D}_0$  form a Hamiltonian pair.

*Proof:* Here it is more convenient to use Poisson meth-

ods.<sup>22</sup> Let  $\theta = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$  be the basis univectors associated with the variables  $u = \begin{pmatrix} u \\ v \end{pmatrix}$ . Since we already know that  $\mathcal{E}$  and  $\mathcal{D}_0$  are Hamiltonian operators, we need only verify the additional compatibility condition<sup>26</sup>

$$\text{pr } \mathbf{v}_{\mathcal{D}_0, \theta}(\Theta_{\mathcal{E}}) + \text{pr } \mathbf{v}_{\mathcal{E}, \theta}(\Theta_{\mathcal{D}_0}) = 0, \quad (5.3)$$

where

$$\Theta_{\mathcal{D}_0} = \frac{1}{2} \int \theta^T \wedge \mathcal{D}_0(\theta) dx = \int \xi \wedge \eta_x dx$$

is the functional bivector associated with the Hamiltonian operator  $\mathcal{D}_0$ , and

$$\begin{aligned}\Theta_{\mathcal{E}} &= \frac{1}{2} \int \theta^T \wedge \mathcal{E}(\theta) dx \\ &= \frac{1}{2} \int [V_x^{-T} \theta]^T \wedge \sigma_1 \cdot D_x [V_x^{-T} \theta] dx\end{aligned}$$

is the bivector associated with the Hamiltonian operator  $\mathcal{E}$ . Now  $\Theta_{\mathcal{D}_0}$  is a constant coefficient bivector, and hence

$$\begin{aligned}\text{pr } \mathbf{v}_{\mathcal{E}\theta}(\Theta_{\mathcal{D}_0}) &= 0, \\ \text{automatically. Thus, to verify (5.3), we need only compute} \\ \text{pr } \mathbf{v}_{\mathcal{D}_0\theta}(\Theta_{\mathcal{E}}) &= \int [\text{pr } \mathbf{v}_{\mathcal{D}_0\theta}(V_x^{-T}) \cdot \theta]^T \wedge \sigma_1 \cdot D_x [V_x^{-T} \theta] dx, \\ &= - \int [V_x^{-T} \cdot \text{pr } \mathbf{v}_{\mathcal{D}_0\theta}(V_x^T) \cdot V_x^{-T} \cdot \theta]^T \\ &\quad \wedge \sigma_1 \cdot D_x [V_x^{-T} \theta] dx, \tag{5.4}\end{aligned}$$

where

$$\begin{aligned}\text{pr } \mathbf{v}_{\mathcal{D}_0\theta}(V_x^T) &= \text{pr} \left\{ \eta_x \frac{\partial}{\partial u} + \xi_x \frac{\partial}{\partial v} \right\} \begin{pmatrix} \lambda(u) \cdot u_x & v_x \\ \mu(v) \cdot v_x & u_x \end{pmatrix} \\ &= \begin{pmatrix} \lambda(u) \cdot \eta_{xx} + \lambda'(u) \cdot u_x \cdot \eta_x & \eta_{xx} \\ \mu(v) \cdot \xi_{xx} + \mu'(v) \cdot v_x \cdot \xi_x & \xi_{xx} \end{pmatrix}.\end{aligned}$$

It now appears that the only recourse is a rather long calculation to check that the functional trivector (5.4) vanishes. We will not reproduce this calculation here, but just remark that, after several integrations by parts, the final result only depends on the four basis trivectors

$$\begin{aligned}\xi_x \wedge \eta_x \wedge \xi_{xx}, \quad \xi_x \wedge \eta_x \wedge \eta_{xx}, \\ \xi_x \wedge \xi_{xx} \wedge \eta_{xx}, \quad \eta_x \wedge \xi_{xx} \wedge \eta_{xx}.\end{aligned} \tag{5.5}$$

Thus one need only check that the coefficient of each of these trivectors is zero, an exercise we leave to the reader.

We now show how to combine the two Hamiltonian operators  $\mathcal{E}$  and  $\mathcal{D}_0$  to make a separable Hamiltonian system bi-Hamiltonian.

**Theorem 5.3:** Let  $H(u,v)$  be a separable Hamiltonian density. Then there exists a second zeroth-order conserved density  $H^*(u,v)$  such that the corresponding Hamiltonian system (1.1) can be written in bi-Hamiltonian form

$$\mathbf{u}_t = \mathcal{D}_0 E[H] = \mathcal{E} E[H^*]. \tag{5.6}$$

*Proof:* Here, all the calculations are local, i.e., over a suitably small domain in  $(u,v)$  space. Let  $H^*(u,v)$  be a separable density satisfying the same equation (3.3) as  $H$ . A straightforward calculation using (3.3) shows that

$$\mathcal{E} E[H^*] = D_x \begin{pmatrix} H_{uv}^* / \lambda \\ H_{uv}^* / \mu \end{pmatrix}.$$

This will coincide with (1.1) provided  $H^*$  satisfies

$$H_v = H_{uv}^* / \lambda, \quad H_u = H_{uv}^* / \mu. \tag{5.7}$$

Note first that, given  $H^*$ , we can always determine a corresponding Hamiltonian density  $H$ , satisfying (3.3), since the compatibility conditions for (5.7), i.e.,

$$\begin{aligned}\frac{\partial^2}{\partial u \partial v} \left( \frac{H_{uv}^*}{\lambda} \right) &= \frac{\partial}{\partial u} \left( \frac{H_{uv}^*}{\lambda} \right) \\ &= \frac{\partial}{\partial v} \left( \frac{H_{uv}^*}{\mu} \right) = \frac{\partial^2}{\partial u \partial v} \left( \frac{H_{uv}^*}{\mu} \right), \tag{5.8}\end{aligned}$$

are clearly satisfied since  $H^*$  solves (3.3). Conversely, given  $H(u,v)$ , which satisfies (3.3), we need to show that there exists a function  $H^*(u,v)$ , which also satisfies (3.3), and satisfies (5.7). To accomplish this, we first determine a function  $G(u,v)$  which satisfies

$$G_u = \lambda H_v, \quad G_v = \mu H_u.$$

This is possible (locally) by virtue of (3.3). Further, let  $F(u,v)$  be any function such that

$$F_{uv} = G.$$

Now according to (5.7), we will have  $H_{uv}^* = G$  also, hence

$$H^*(u,v) = F(u,v) + \rho(u) + \sigma(v) \tag{5.9}$$

for certain functions  $\rho(u)$  and  $\sigma(v)$  of one variable. We only need to determine  $\rho$  and  $\sigma$  so that  $H^*$  satisfies (3.3). Now, according to (5.8)

$$\frac{\partial^2}{\partial u \partial v} \left( \frac{F_{uu}}{\lambda} - \frac{F_{vv}}{\mu} \right) = 0,$$

hence

$$F_{uu} / \lambda - F_{vv} / \mu = \varphi(u) + \psi(v). \tag{5.10}$$

If we choose  $\rho$  and  $\sigma$  to satisfy

$$\rho'' = -\lambda \cdot \varphi, \quad \sigma'' = -\mu \cdot \psi,$$

then it is not hard to see that  $H^*$ , as determined by (5.9), satisfies (3.3), since

$$\frac{H_{uu}^*}{\lambda} - \frac{H_{vv}^*}{\mu} = \frac{F_{uu}}{\lambda} + \frac{\rho''}{\lambda} - \frac{F_{vv}}{\mu} - \frac{\sigma''}{\mu} = 0$$

by virtue of (5.10). This completes the construction of the appropriate density  $H^*$ , and hence proves the theorem.

If the Hamiltonian density  $H$  in Theorem 5.3 is one of the densities  $H_n$  in the hierarchy (3.7), then it is not hard to see that the corresponding density  $H^*(u,v)$  can be taken to be the density  $H_{n+2}$ ; similarly, if  $H = \tilde{H}_n$ , then  $H^* = \tilde{H}_{n+2}$ . (See also Lemma 6.2 below.)

*Example 5.4:* Consider Eqs. (1.6) of polytropic gas dynamics, so that  $f(v) = v^{\gamma-2}$ . Here  $\lambda(u) = 1$ , and  $\mu(v) = v^{\gamma-3}$ . Therefore the matrix variables (5.1) coincide

$$U = V = \begin{pmatrix} u & v^{\gamma-2}/(\gamma-2) \\ v & u \end{pmatrix}.$$

According to the proof of Theorem 5.3, we can write the gas dynamics equations in the new Hamiltonian form

$$\mathbf{u}_t = \mathcal{E} E[H^*].$$

The Hamiltonian operator is

$$\mathcal{E} = D_x \cdot U_x^{-1} \cdot D_x \cdot U_x^{-1} \cdot \sigma_1 \cdot D_x,$$

and the new Hamiltonian is

$$H^*(u,v) = \tilde{H}_4(u,v) = \frac{1}{2\lambda} u^4 v + u^2 \cdot v^{\gamma}/\gamma(\gamma-1)$$

$$+ v^{2\gamma-1}/2\gamma(\gamma-1)^2(2\gamma-1),$$

provided  $\gamma \neq \frac{1}{2}$  (otherwise the last summand is logarithmic).

Similar formulas hold for nonpolytropic gas dynamics.

As discussed in Sec. IV, polytropic gas dynamics has two additional Hamiltonian structures. A similar, but even lengthier, calculation along the lines of that in Proposition 5.2 shows that for polytropic gas dynamics, the Hamiltonian operators  $\mathcal{E}$  and  $\mathcal{D}_1$  are *not* compatible, nor are the Hamiltonian operators  $\mathcal{E}$  and  $\mathcal{D}_2$ . [In this case, there are other basis trivectors besides the ones listed in (5.5) which show up in the compatibility conditions (5.3), e.g.,  $\xi \wedge \eta_x \wedge \xi_{xx}$ , and one checks that the coefficient of at least one of these trivectors does not vanish.] Thus, as with the one-dimensional Riemann equation (2.1), the polytropic gas dynamic equations form a quadri-Hamiltonian system with four compatible and two incompatible Hamiltonian pairs.

## VI. RECURSION OPERATORS AND HIGHER-ORDER SYMMETRIES

We now turn to items (7) and (8) of Sec. II, and discuss the higher-order symmetries for hyperbolic systems. We begin by invoking Magri's theorem<sup>3</sup> to construct recursion operators from the Hamiltonian pairs of differential operators. The first consequence of the developments in the preceding section is the following result of Sheftel'.<sup>13</sup>

*Proposition 6.1:* The operator

$$\hat{\mathcal{R}} = \mathcal{E} \cdot \mathcal{D}_0^{-1} = D_x \cdot V_x^{-1} \cdot D_x \cdot U_x^{-1} \quad (6.1)$$

is a recursion operator for the separable Hamiltonian system (1.1).

Thus applying the recursion operator  $\hat{\mathcal{R}}$  to any symmetry of the gas dynamics system leads to another symmetry. A straightforward calculation shows that for the zeroth-order conserved densities, this recursion operator does not lead to anything new.

*Lemma 6.2:* Let  $\mathbf{v}_n$  and  $\tilde{\mathbf{v}}_n$  be the  $n$ th-order Hamiltonian symmetries determined by the two hierarchies (3.7). Then the recursion operator acts on their characteristics according to

$$\hat{\mathcal{R}}(\mathbf{Q}_n) = \mathbf{Q}_{n-2}, \quad \hat{\mathcal{R}}(\tilde{\mathbf{Q}}_n) = \tilde{\mathbf{Q}}_{n-2}. \quad (6.2)$$

In the case of a generalized gas dynamics Hamiltonian, so  $\lambda(u) \equiv 1$ , the matrix variables (5.1) coincide:

$$U = V = \begin{pmatrix} u & M(v) \\ v & u \end{pmatrix}.$$

Thus, just as in the one-dimensional case, the recursion operator (6.1) is the square of a simpler recursion operator<sup>13</sup>

$$\mathcal{R} = D_x \cdot U_x^{-1}, \quad (6.3)$$

On the zeroth-order symmetries,

$$\mathcal{R}(\mathbf{Q}_n) = \mathbf{Q}_{n-1}, \quad \mathcal{R}(\tilde{\mathbf{Q}}_n) = \tilde{\mathbf{Q}}_{n-1}. \quad (6.4)$$

In the polytropic case,  $\mathcal{R}$  is the "inverse" to the recursion operator  $\mathcal{R}_1$  on the hierarchies (3.7), although as always, this is special to these particular hierarchies.

Since we cannot obtain any higher-order symmetries by applying Sheftel's recursion operator to the first-order symmetries, we need to look elsewhere for the analogies to the higher order rational symmetries of the Riemann equation. We begin by stating the basic condition for a generalized vector field to be a symmetry of (1.1).

*Proposition 6.3:* The function  $\mathbf{Q}[\mathbf{u}] = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$  is the characteristic of a symmetry vector field  $\mathbf{v}_Q = \mathbf{Q} \cdot \partial_{\mathbf{u}}$  of (1.2) if and only if  $\mathbf{Q}[\mathbf{u}]$  satisfies the identity

$$D_t \mathbf{Q} = D_x (\mathbf{H} \cdot \mathbf{Q}) \quad (6.5)$$

on solutions to the system (1.2).

This result is standard<sup>27</sup> using the fact that the operator  $D_t - D_x \cdot \mathbf{H}$  is the Fréchet derivative operator for the system of differential equations (1.2). As a consequence of Proposition 6.3 and the usual condition for a recursion operator,<sup>28</sup> we also deduce the following.

*Corollary 6.4:* The Fréchet derivative and recursion operators commute:

$$(D_t - D_x \cdot \mathbf{H}) \cdot \hat{\mathcal{R}} = \hat{\mathcal{R}} \cdot (D_t - D_x \cdot \mathbf{H}). \quad (6.6)$$

This can also be proved directly.<sup>13</sup> The commutation identity (6.6) also clearly holds for the first-order recursion operator (6.3) in the generalized gas dynamics case.

**Theorem 6.5:** Let  $H = H_n$  be a Hamiltonian in the first hierarchy constructed in Sec. III, and let  $\mathbf{v}_n$  be the corresponding Hamiltonian vector field. Let  $\hat{\mathbf{v}}_{2m}$  denote the generalized vector field of order  $2m$  with characteristic

$$\hat{\mathbf{Q}}_{2m} = \hat{\mathcal{R}}^m(x\mathbf{u}_x). \quad (6.7)$$

Then  $\hat{\mathbf{v}}_{2m}$  is a symmetry for the flow generated by  $\mathbf{v}_n$  provided  $2m \geq n - 1$ . Similarly,  $\hat{\mathbf{v}}_{2m}$  is a symmetry for the flow generated by  $\tilde{\mathbf{v}}_n$  corresponding to the Hamiltonian  $\tilde{H}_n$  provided  $2m \geq n$ .

*Proof:* We need to verify the symmetry criterion (6.5) of Proposition 6.3. Using the recursion condition (6.6), (6.2), and (6.7), we see that, on solutions to (1.2) with  $\mathbf{H} = \mathbf{H}_n$ ,

$$\begin{aligned} (D_t - D_x \cdot \mathbf{H}_n) \hat{\mathbf{Q}}_{2m} &= (D_t - D_x \cdot \mathbf{H}_n) \cdot \hat{\mathcal{R}}^m(x\mathbf{u}_x) \\ &= \hat{\mathcal{R}}^m \cdot (D_t - D_x \cdot \mathbf{H}_n)(x\mathbf{u}_x) \\ &= \hat{\mathcal{R}}^m \cdot \{x \cdot D_x(\mathbf{u}_t - \mathbf{H}_n \cdot \mathbf{u}_x) - \mathbf{H}_n \cdot \mathbf{u}_x\} \\ &= -\hat{\mathcal{R}}^m(\mathbf{Q}_n) = -\mathbf{Q}_{n-2m}, \end{aligned}$$

which vanishes if  $n - 2m \leq 1$ . The proof for  $\tilde{H}_n$  is the same, but now  $\tilde{\mathbf{Q}}_{n-2m}$  vanishes if  $n - 2m \leq 0$ .

In the case of generalized gas dynamics, there is a more extensive hierarchy of symmetries because the recursion operator is a first-order differential operator. The same calculation yields the following theorem.

**Theorem 6.6:** Suppose  $H = H_n$  be one of the  $n$ th-order generalized gas dynamics Hamiltonians, as in (3.7), and let  $\mathbf{v}_n$  be the corresponding first-order Hamiltonian flow. Let  $\hat{\mathbf{v}}_m$  denote the generalized vector field of order  $m$  with characteristic

$$\hat{\mathbf{Q}}_m = \mathcal{R}^m(x\mathbf{u}_x) = \mathcal{R}^{m-1} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (6.8)$$

Then  $\hat{\mathbf{v}}_m$  is a symmetry for the flow generated by  $\mathbf{v}_n$  provided  $m \geq n - 1$ . Similarly,  $\hat{\mathbf{v}}_m$  is a symmetry for the flow generated by  $\tilde{\mathbf{v}}_n$  corresponding to the Hamiltonian  $H = \tilde{H}_n$  provided  $m \geq n$ .

Finally, we note that in polytropic gas dynamics, we can still form additional recursion operators by combining the Hamiltonian operator  $\mathcal{E}$  with the operators  $\mathcal{D}_1$ ,  $\mathcal{D}_2$ , even though they are not compatible.<sup>29</sup> However, the resulting higher-order symmetries appear to always be nonlocal since we cannot explicitly invert  $\mathcal{D}_1$  or  $\mathcal{D}_2$ .



## VII. HIGHER-ORDER CONSERVATION LAWS FOR GENERALIZED GAS DYNAMICS

Proposition 3.1 gives a complete description of the zeroth-order conservation laws for the general Hamiltonian system (1.1). For the classical gas dynamics Hamiltonian (1.6), Verosky<sup>14</sup> found an additional first-order conserved density,

$$\hat{H}_1[u,v] = \frac{v_x}{\delta} = \frac{v_x}{u_x^2 - \mu(v) \cdot v_x^2}. \quad (7.1)$$

We now show how Verosky's conservation law fits into our general framework, and derive analogous laws for generalized gas dynamics Hamiltonians. This will complete our extension of the results of Sec. II to two-dimensional Hamiltonian systems. We first note that a generalized gas dynamics Hamiltonian system can be cast into a suggestive matrix form, which is very reminiscent of the scalar Riemann equation (2.1).

*Lemma 7.1:* Suppose  $H(u,v)$  is a generalized gas dynamics Hamiltonian. Then the corresponding Hamiltonian system (1.2) is equivalent to the matrix equation

$$U_t = \mathbf{H} \cdot U_x. \quad (7.2)$$

This follows from an elementary direct computation. The key to Verosky's conservation law and its higher order generalizations is the following matrix divergence identity.

*Lemma 7.2:* Let  $H(u,v)$  be a generalized gas dynamics Hamiltonian. Then

$$D_t \cdot (U_x^{-1}) - D_x (\mathbf{H} \cdot U_x^{-1}) = -(\mathbf{H}_x \cdot U_x^{-1} + U_x^{-1} \cdot \mathbf{H}_x) \quad (7.3)$$

holds on solutions to the system (1.2).

*Proof:* We first note that the matrices  $\mathbf{H}$  and  $U_x$  commute:

$$\mathbf{H} \cdot U_x = U_x \cdot \mathbf{H}.$$

Also, differentiating (7.2) we find

$$U_{xt} = \mathbf{H} \cdot U_{xx} + \mathbf{H}_x \cdot U_x.$$

Therefore we immediately deduce (7.3):

$$\begin{aligned} D_t \cdot (U_x^{-1}) - D_x (\mathbf{H} \cdot U_x^{-1}) &= -U_x^{-1} \cdot U_{xt} \cdot U_x^{-1} - \mathbf{H}_x \cdot U_x^{-1} \\ &\quad + \mathbf{H} \cdot U_x^{-1} \cdot U_{xx} \cdot U_x^{-1} \\ &= -\mathbf{H}_x \cdot U_x^{-1} - U_x^{-1} \cdot \mathbf{H}_x. \end{aligned}$$

In particular, the (2,1) entry of the matrix identity (7.3) reads

$$D_t \left( \frac{v_x}{\delta} \right) + D_x \left( \frac{H_{uu} \cdot v_x - H_{uv} \cdot u_x}{\delta} \right) = -2H_{uuu}. \quad (7.4)$$

For the classical gas dynamics Hamiltonian (1.7),  $H_{uuu} \equiv 0$ , and we recover Verosky's conservation law, with density (7.1). For more general gas dynamics Hamiltonians,  $H_{uuu}$  will no longer be 0, and  $v_x/\delta$  will no longer be a conserved density; however, we can simply modify it to get a time-dependent conservation law.

**Theorem 7.4:** Let  $H(u,v)$  be a gas dynamics Hamiltonian. Then the function

$$\hat{H}_1^* = v_x/\delta + 2tH_{uuu} \quad (7.5)$$

is a conserved density for the corresponding flow (1.1).

*Proof:* Let  $T(u,v) = H_{uuu}$ . Note first that according to Lemma 3.4,  $T(u,v)$  is a conserved density for the flow. According to (7.4), to prove that (7.5) is also a conserved density, it suffices to show that there exists a function  $X(u,v)$  such that the divergence identity

$$D_t(t \cdot T(u,v)) - D_x(t \cdot X(u,v)) = T(u,v) \quad (7.6)$$

holds. Evaluating the left-hand side of (7.6), and using Eq. (1.1), we find that  $X$  must satisfy the pair of first-order partial differential equations

$$X_u = H_{uv} T_u + H_{uu} T_v, \quad X_v = H_{vv} T_u + H_{uv} T_v.$$

The integrability condition  $X_{uv} = X_{vu}$  simplifies to the condition

$$H_{vv} T_{uu} = H_{uu} T_{vv}. \quad (7.7)$$

But this follows immediately from Lemma 3.1, using the fact that  $T$  is a conserved density. This completes the proof.

*Corollary 7.5:* The integral

$$\hat{\mathcal{H}}_1 = \int \frac{v_x}{\delta} dx,$$

when it converges, is a linear function of  $t$  for any gas dynamics Hamiltonian.

The first-order conserved density  $\hat{H}_1 = v_x/\delta$  leads to a Hamiltonian flow using the basic Hamiltonian operator  $\mathcal{D}_0$ . We now connect this flow with the symmetries generated in Theorem 6.6. This will allow us to apply the theorem of Magri to the Hamiltonian pair  $\mathcal{E}$  and  $\mathcal{D}_0$ , and thereby generate a new hierarchy of higher order conservation laws in gas dynamics. The starting point is the following straightforward result.

*Proposition 7.6:* The symmetry  $\hat{v}_3$  is Hamiltonian with respect to the Hamiltonian structure determined by  $\mathcal{D}_0$ , and the corresponding conserved density is  $-2$  times Verosky's density (7.1).

*Proof:* According to (6.8), the characteristic for the symmetry  $\hat{v}_3$  is given by

$$\hat{\mathbf{Q}}_3 = \mathcal{R}^2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -D_x \cdot U_x^{-2} \cdot U_{xx} \cdot U_x^{-1} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

On the other hand, the Hamiltonian flow corresponding to Verosky's density has characteristic

$$D_x \cdot \begin{pmatrix} E_v(\hat{H}_1) \\ E_u(\hat{H}_1) \end{pmatrix}.$$

Therefore, it suffices to verify that

$$\begin{pmatrix} E_v(\hat{H}_1) \\ E_u(\hat{H}_1) \end{pmatrix} = 2U_x^{-2} \cdot U_{xx} \cdot U_x^{-1} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We have thus reduced the proposition to a straightforward computation, which we leave to the reader.

**Theorem 7.7:** For a generalized gas dynamics Hamiltonian, there is a hierarchy of higher order Hamiltonian densities  $\hat{H}_m$ ,  $m = 1, 2, \dots$ , with  $m$  indicating the order of derivatives on which they depend, and corresponding commuting bi-Hamiltonian systems

$$\mathbf{u}_t = \hat{\mathbf{Q}}_{2m+1} = \mathcal{D}_0 E[\hat{H}_m] = \mathcal{E} E[\hat{H}_{m-1}], \quad m \geq 1. \quad (7.8)$$

The Hamiltonians are in involution with respect to both the  $\mathcal{D}_0$  and  $\mathcal{E}$  Poisson brackets.

This follows directly from Magri's theorem.<sup>3,22</sup> There is, however, a technical point to be resolved: whether we can always invert the differential operator  $\mathcal{D}_0$  to continue the recursive construction of the densities from (7.8). However this follows from the theorem in Ref. 30 that shows that this always is the case when one of the members of a Hamiltonian pair is a constant coefficient differential operator.

Note that if one of the higher-order densities  $\hat{H}_m$  is conserved, then so is every subsequent density  $\hat{H}_k$ ,  $k > m$ . Thus we need only know which is the first conserved density for our Hamiltonian system (1.1).

**Theorem 7.8:** If  $H = H_n$  is a Hamiltonian density in the first generalized gas dynamics hierarchy, then the higher-order density  $\hat{H}_m$  is conserved for the Hamiltonian system (1.2) provided  $n \leq 2m + 1$ . If  $H = \tilde{H}_n$  is in the second generalized gas dynamics hierarchy, then  $\hat{H}_m$  is conserved provided  $n \leq 2m + 2$ .

*Proof:* We apply the recursion operator  $\hat{\mathcal{R}} = \mathcal{R}^2$  to the (time-dependent) flow corresponding to the Hamiltonian density  $\hat{H}_1^*$  [cf. (7.5)]. Using (3.8) and (6.4), we conclude that, as in the one-dimensional case, the function

$$\hat{H}_m + t \frac{\partial^{2m+1} H}{\partial u^{2m+1}}$$

is a conserved density for (1.1) with  $H = H_n$  or  $\tilde{H}_n$ . In particular, if  $2m + 1 > n$ , the second summand vanishes; more particularly, if  $H = H_n$ , and  $n = 2m + 1$ , then the second summand is just a constant multiple of  $t$ , and can be ignored. This completes the proof.

For more general separable Hamiltonian systems not of gas dynamics type, higher order conservation laws do not appear to exist in general. The principal reason for this is that the recursion operator (6.1) is second order, and the hierarchy of higher-order symmetries (6.7) omits the obvious Hamiltonian candidates. However, we do not have a complete proof that these systems do not have higher-order conservation laws, and, indeed, Verosky's transformation (3.5) shows that at least one special example—the Born–Infeld equation—they do exist. This is an area that requires further investigation.

In a later publication, we hope to return to the application of our results to physically interesting initial value problems and discontinuous solutions/shock waves. Another interesting direction for further research is to try to extend these results to three-dimensional systems of conservation laws, especially those of nonisentropic gas dynamics for which Verosky has discovered additional higher order conservation laws.<sup>31</sup>

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