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THE CONNECTION BETWEEN PARTIAL DIFFERENTIAL EQUATIONS SOLUBLE BY INVERSE SCATTERING AND ORDINARY DIFFERENTIAL EQUATIONS OF PAINLEVÉ TYPE*

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Abstract. A completely integrable partial differential equation is one which has a Lax representation, or, more precisely, can be solved via a linear integral equation of Gel'fand–Levitan type, the classic example being the Korteweg–de Vries equation. An ordinary differential equation is of Painlevé type if the only singularities of its solutions in the complex plane are poles. It is shown that, under certain restrictions, if G is an analytic, regular symmetry group of a completely integrable partial differential equation, then the reduced ordinary differential equation for the G-invariant solutions is necessarily of Painlevé type. This gives a useful necessary condition for complete integrability, which is applied to investigate the integrability of certain generalizations of the Korteweg–de Vries equation, Klein–Gordon equations, some model nonlinear wave equations of Whitham and Benjamin, and the BBM equation.

Key words. Completely integrable partial differential equations, inverse scattering method, Gel'fand–Levitan equation, KdV equation, Klein–Gordon equations, nonlinear Schrödinger equation, similarity solutions, Painleve transcendents

1. Introduction.

The recent discovery of nonlinear partial differential equations which can be exactly solved by the linear integral equations of inverse scattering theory has provoked considerable interest in the range of applicability of these methods for the integration of nonlinear equations in mathematical physics. The original investigations of Gardner, Kruskal and Miura [26] and Lax [22] for the Korteweg–de Vries (KdV) equation have now been extended to solve a surprising number of differential equations of physical interest, including the sine-Gordon, nonlinear Schrödinger, three-wave interaction and other equations (cf. [1], [19], [37], [38], [39]). In all of these examples, the given equation is recast into a "Lax representation,"

$$\frac{dL}{dt}=[B,L]=BL-LB, \tag{1.1}$$

where L, B are linear differential operators depending on the solution $u(x,t)$ of the equation, with B skew-adjoint. This representation implies that the spectrum of L has an elementary time evolution, and hence the original equation can be integrated once the inverse scattering problem of reconstructing the potential $u(x,t)$ from the spectral data of the corresponding operator L has been solved. In all known examples, this inverse scattering problem is effected through the solution of a linear integral equation of the form

$$K(x,y;t)+F(x,y;t)+\int_x^\infty K(x,z;t)H(z,y;t)\,dz=0, \tag{1.2}$$

known as the *Gel'fand–Levitan equation*. Here F and H are constructed from the spectral data of L; the potential $u(x,t)$ is recovered from the values of K on the diagonal $x=y$.

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Hereafter, any partial differential equation which can be solved by such a linear integral equation will be termed *completely integrable*, this terminology stemming from the interpretation of the KdV equation as a completely integrable Hamiltonian system [11], [23]. Of course, only certain types of solutions can be obtained in this fashion, so this definition is subject to further refinement (cf. Definition 2.1). Completely integrable equations all seem to have many other remarkable properties in common including cleanly interacting soliton solutions, existence of infinitely many conservation laws, Bäcklund transformations, etc. (cf. [21]). However, the precise interrelationship among these properties remains to be rigorously formulated; thus reasons of practicality necessitate the adoption of the Gel'fand–Levitan type of linear integral equation as the distinguishing characteristic of complete integrability.

The most notable drawback in the applicability of inverse scattering techniques is that there is as yet no systematic method for determining whether a given differential equation is completely integrable, i.e., can be solved by such a linear integral equation. In this paper we find a useful necessary condition for integrability based on the nature of the complex singularities of group-invariant solutions to the equation. Whereas we are thus no closer to finding a scattering problem if it exists, this condition is useful for determining when no such solution is possible. In the applications to be considered, a number of nonlinear partial differential equations (p.d.e.'s) of interest will be shown not to be integrable by inverse scattering methods.

This condition was inspired by an observation of Ablowitz, Ramani and Segur [2], [4] that the ordinary differential equations for group-invariant (self-similar) solutions of known examples of completely integrable equations inevitably are equations of the type studied by Painlevé and his students; these are characterized by the property that all their solutions are meromorphic in the complex plane (cf. [17], [18]). Such an equation will be referred to as an equation of Painlevé type. (Painlevé also allowed fixed singularities of arbitrary type, but we will not.) This leads immediately to the conjecture proposed by Ablowitz, Ramani and Segur [2] and Hastings and McLeod [16]:

CONJECTURE. *If a system of partial differential equations is completely integrable, and G is a symmetry group of this system, then the reduced system of ordinary differential equations for the G-invariant solutions is of Painlevé type.*

This conjecture, if true, would provide a powerful necessary condition to test for complete integrability. Here we will prove a somewhat weakened version of the conjecture, which nevertheless proves useful in several applications. There are two restrictions. First, if, in the Lax operator L, some combination of the solution u and its spatial derivatives occurs, say $Q(u)$, then it is this combination (or combinations) that must have only poles as singularities. For instance, if $L=D^2+u_x$, then only u_x is required to have poles, and thus we may allow logarithmic branch points as singularities of the solutions of the reduced ordinary differential equations. Usually we will assume that Q is a linear combination of u and its spatial derivatives, calling this case *linearly completely integrable*. Secondly, the same combination Q must satisfy certain preconditions for the inverse scattering formalism to go through; this means that, when restricted to the real axis, Q either is periodic or satisfies decay conditions at $x=\pm\infty$, which implies corresponding restrictions on the solutions u that can be considered. It is only for such solutions such that $Q(u)$ must be meromorphic. If a system of ordinary differential equations has the property that, for such solutions u, the combination $Q(u)$ is meromorphic, we say that the system is of *restricted Painlevé type* relative to Q. Our basic result, in rough form, replaces "Painlevé type" by "restricted Painlevé type" in the above conjecture.

The main tool in our proof is a theorem of Steinberg [32] which states that if $T(z)$ is an analytic family of compact operators in a Banach space, then $(I-T(z))^{-1}$, provided this inverse exists for at least one value of z, is a meromorphic family of operators. Under appropriate assumptions on the initial data of our completely integrable system (to ensure that the functions F and H in the Gel'fand–Levitan equation satisfy certain analyticity criteria) we can conclude from Steinberg's result that Q must be a meromorphic function of (x,t). Now suppose that G is a one-parameter, analytic, regular local group of transformations acting on the space of independent and dependent variables which leaves the set of solutions of the system of partial differential equations invariant. Then the G-invariant (self-similar) solutions can all be found by integrating a system of ordinary differential equations on the quotient manifold whose points correspond to the orbits of G. The analyticity of G implies that for any G-invariant solution whose initial data satisfies the inverse scattering assumptions, the function Q on the quotient manifold can have only poles for singularities. In other words, the reduced system of ordinary differential equations must be of restricted Painlevé type relative to Q.

Ablowitz, Ramani and Segur [2], [3] have also given proofs of a version of the above conjecture. They restrict their attention to Gel'fand–Levitan equations of Fredholm type, and their groups are only groups of scaling transformations. Thus our result is somewhat more general. Both proofs are necessarily restricted to certain types of solutions, in particular, solutions decaying sufficiently rapidly as $|x|\to\infty$ are allowed. Extensions to the case of spatially periodic solutions can be inferred from the work of McKean and Trubowitz on the Korteweg–de Vries equation [23], [34], although the analogue of the Gel'fand–Levitan equation is not explicitly written down. We strongly suspect, however, that solutions are in general meromorphic in the periodic case also, and therefore include solutions of this type in our test for complete integrability. It would be of great interest to remove all restrictions on the types of solutions for which such a result can be proved and thereby prove the complete version of the conjecture.

In § 3 we discuss some applications of this result. First we show that the generalized KdV equation

$$u_t+u^p u_x+u_{xxx}=0 \tag{1.3}$$

can be linearly completely integrable only if $p=0, 1$, or 2. These exceptional cases correspond to the Airy equation in moving coordinates, the KdV, and the modified KdV equations, which are well known to be completely integrable. Secondly we consider a nonlinear Klein–Gordon equation in characteristic coordinates:

$$u_{xt}=f'(u). \tag{1.4}$$

It is shown that if $f(u)$ is a rational function, real for real u and with two consecutive zeros, simple or double, on the real axis, and if (1.4) is linearly completely integrable, then f is a polynomial of degree at most 4. Also, if $f(u)$ is a linear combination of exponentials $e^{\alpha_j u}$ with the α_j all rational multiples of some complex number α, again real for real u and with two consecutive simple or double zeros, and if (1.4) is linearly completely integrable, then

$$f(u)=c_2e^{2\beta u}+c_1e^{\beta u}+c_0+c_{-1}e^{-\beta u}+c_{-2}e^{-2\beta u}$$

for some β. This includes the sine- and sinh-Gordon, and an equation due to Mikhailov [24], [25], which are known to be integrable, and the double sine-Gordon equation, whose status is a matter of dispute. The next application shows that certain nonlinear model wave equations considered by Benjamin, Bona and Mahony [5] and Whitham [36] cannot be linearly completely integrable. The last example deals with the BBM equation [5],

$$u_t+uu_x-u_{xxt}=0. \tag{1.5}$$

Although this cannot be treated rigorously by the methods of the present paper, we show that if the full conjecture were true, then (1.5) could not be linearly completely integrable.

Finally we discuss the general Lax representations of Gel'fand and Dikii for scalar differential operators L of order n (see [12], [13]). For n a composite number, there exist steady state solutions of the corresponding evolutionary systems with arbitrary complex singularities. This suggests that the inverse scattering problem for such an L is not amenable to solution by a Gel'fand–Levitan type equation, at least in the form discussed here. Indeed, only for second and third order L (see [20]) has the inverse problem been solved, so the theory for fourth order operators becomes of great interest. From those results, it can be seen that our criterion for complete integrability is a powerful preliminary test to determine whether a given system can be integrated by inverse scattering.

2. Analyticity properties of completely integrable differential equations. Consider a system of partial differential equations

$$\Delta(t,x,u)=0, \tag{2.1}$$

where $x,t\in\mathbf{R}$ and $u=(u^1,\cdots,u^m)\in\mathbf{R}^m$ is a vector-valued function. We assume that the initial value problem of (2.1) with

$$u(x,0)=f(x) \tag{2.2}$$

is well posed for f in some Banach space $\mathfrak{B}$ of functions, so that for t sufficiently small, there is a unique solution $u(x,t)$ of (2.1)–(2.2). In practice $\mathfrak{B}$ is either a space of functions decreasing sufficiently rapidly at $\pm\infty$ or a space of periodic functions. Usually the presence of appropriate conservation laws will ensure that the solutions are actually global in t, but this will not be assumed a priori. The first task is to make precise what is meant by (2.1) being completely integrable.

DEFINITION 2.1. A system of partial differential equations is *completely integrable relative to* $Q(u)$ *in the Banach space* $\mathfrak{B}$ if there is a linear matrix integral equation of the form

$$K(x,y;t)+F(x,y;t)+\int_x^\infty K(x,z;t)H(z,y;t)\,dz=0, \tag{2.3}$$

called the *Gel'fand–Levitan equation*, satisfying the following properties:

i) F,H,K are $N\times N$ matrices of functions;

ii) F and H are uniquely determined by the initial data (2.2);

iii) for initial data in $\mathfrak{B}$, and for all real x,y, all complex ε, and t in some domain Ω in $\mathbf{C}$, the functions $F(x-\varepsilon t,y-\varepsilon t;t)$ and $H(x-\varepsilon t,y-\varepsilon t;t)$ are analytic in ε,t, and there is a Banach space $\mathfrak{B}^*$ (not necessarily the same as $\mathfrak{B}$) for which $F(x-\varepsilon t,y-\varepsilon t;t)\in\mathfrak{B}^*$ as a function of y and the operator

$$T(x,t)f(y)=\int_x^\infty f(z)H(z-\varepsilon t,y-\varepsilon t;t)\,dz$$

is a compact operator in $\mathfrak{B}^*$;

iv) the Gel'fand–Levitan equation has a unique solution (in $\mathfrak{B}^*$) for all x and at least one t in Ω;

v) the solution u of the system (2.1), (2.2) can be recovered from the solution K of the Gel'fand–Levitan equation via a relation of the form

$$Q[u(x,t)] = P[K(x,x,t)], \tag{2.4}$$

where Q is some function of u and its spatial derivatives, and P is a polynomial in K and its spatial derivatives.

Thus to recover the solution u of a completely integrable system of partial differential equations, we must solve the Gel'fand–Levitan equation for K and then solve the differential equation (2.4) for u. In practical examples, Q is a linear combination of the spatial derivatives of u, and in this case the system will be called *linearly completely integrable*. It should also be remarked that the requirement that iii) hold for all complex ε can certainly be relaxed, although there seems little practical point in doing so, and that the domain Ω will customarily include the origin or at least have the origin on its boundary (it might, as in the example of the KdV equation below, be a sector of a circle center the origin).

Example 2.2. *The Korteweg–de Vries equation.* This is the original example of the use of inverse scattering techniques [21], [22]. The equation is

$$u_t + 6uu_x + u_{xxx} = 0, \tag{2.5}$$

and has a Lax representation with operators

$$L = -D^2 - u, \qquad B = -\{4D^3 + 3(Du + uD)\}, \tag{2.6}$$

where $D = d/dx$. The Gel'fand–Levitan equation takes the form

$$K(x,y;t) + F(x+y;t) + \int_x^\infty K(x,z;t)F(z+y;t)\,dz = 0, \tag{2.7}$$

and we recover the solution of the KdV equation via

$$u(x,t) = 2\frac{d}{dx}K(x,x;t). \tag{2.8}$$

The kernel F is given by

$$F(x,t) = \sum_{j=1}^{n} c_j \exp\left(8k_j^3 t - k_j x\right) + \frac{1}{2\pi}\int_{-\infty}^{\infty} R(k)\exp(2kx + 8ik^3 t)\,dk, \tag{2.9}$$

where $\lambda_j = -k_j^2$ are the eigenvalues, c_j the corresponding norming constants and $R(k)$ the reflection coefficient associated with the potential $u(x,0) = f(x)$. This solution is valid provided

$$\int_{-\infty}^{\infty} (1+x^2)|f(x)|\,dx < \infty \tag{2.10}$$

(cf. [7], [21]).

The uniqueness of the solution of (2.7) in the KdV case is a standard result, and the only item remaining to be checked is Definition 2.1 in condition iii). So far as analyticity is concerned, the only part of F that could fail to be analytic is that corresponding to the continuous spectrum of L:

$$F_c(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(k)\exp[8ik^3 t + ikx]\,dk. \tag{2.11}$$

If we take any reasonable space of initial data for $\mathfrak{B}$, for example that given by (2.10), then $R(k)$ can be extended analytically into the upper half of the k-plane, and $|R(k)/k^2|$ is bounded as $|k| \to \infty$. (The function R is closely related to the spectral density function m of Titchmarsh, and the analyticity and estimates can be obtained by suitably translating the results in [33, Chap. V].) If therefore we write

$$F_c(x,t) = \frac{1}{2\pi}\left\{\int_{-\infty}^{0} + \int_{0}^{\infty}\right\} R(k)\exp[8ik^3t + ikx]\,dk = F_2 + F_1,$$

say, and consider F_1, then, if t is real and positive, we can deform the integral from $(0,\infty)$ to $(0,\alpha\infty)$, for any α with $0<\arg\alpha<\frac{1}{3}\pi$. We can now increase $\arg t$, but the range for α becomes $0<\arg\alpha<\frac{1}{3}(\pi-\arg t)$. Nonetheless this does allow us to define $F_1(x,t)$ as an analytic function of t for $0<\arg t<\pi$. (It is also an analytic function of x since for large k the term k^3t dominates kx.) If we decrease $\arg t$, the range for α becomes $-\frac{1}{3}\arg t<\arg\alpha<\frac{1}{3}\pi$, which allows us to define $F_1(x,t)$ as an analytic function of t for $-\pi<\arg t<0$, and so in fact in the whole complex plane cut along the negative axis. Similar remarks apply to F_2.

Further, by using the deformed contours and integrating by parts (integrating e^{ikx} and differentiating the remainder), we see that $F\in\mathfrak{B}$, the Banach space defined by (2.10), and that the operator T is compact in $\mathfrak{B}$, although $\mathfrak{B}$ is certainly not the only possible choice for $\mathfrak{B}^*$.

Example 2.3. A case in which the combination $Q(u)$ appearing in the definition 2.1 of complete integrability is nontrivial is provided by the sine-Gordon equation, which is

$$u_{xt} = \sin u. \tag{2.12}$$

The scattering problem which can be used to solve the sine-Gordon equation was first described by Zakharov and Shabat [39] and was developed in full detail by Ablowitz, Kaup, Newell and Segur [1]; it takes the form

$$\begin{aligned} v_x &= -i\zeta v - \tfrac{1}{2}u_x(x,t)w,\\ w_x &= i\zeta w + \tfrac{1}{2}u_x(x,t)v, \end{aligned} \tag{2.13}$$

in which the x-derivative u_x of the solution of the sine-Gordon equation appears as a potential.

The analogue of the Gel'fand–Levitan equation for (2.13) again takes the form (2.7), but in this case K and F are now 2×2 matrices of functions. The matrix F is constructed from the appropriate scattering data for (2.13); the precise details of this construction can be found in [1]. Since u_x appears as the potential in (2.13), the analogue of (2.8), used to recover the solution of the sine-Gordon equation, takes the form

$$u_x(x,t) = -2K_{12}(x,x;t),$$

where K_{12} denotes the upper right-hand entry of the matrix K. Thus for the sine-Gordon equation, $Q(u)=u_x$, a fact that will be of significance when we analyze the travelling wave solutions in §3.

We now investigate the properties of the solutions of a general integral equation of Gel'fand–Levitan type. Our main tool is the following theorem of Steinberg [32], generalizing a theorem of Dolph, McLeod and Thoe [9], for the case of Hilbert–Schmidt operators.

THEOREM 2.4. *Let $\mathfrak{B}$ be a Banach space, and let $T(z)$ be an analytic family of compact operators defined for $z \in \Omega \subset \mathbf{C}$. Then either $I - T(z)$ is nowhere invertible for $z \in \Omega$ or $(I - T(z))^{-1}$ is meromorphic for $z \in \Omega$.*

Let us write the Gel'fand–Levitan equation (2.3) in the symbolic form

$$(I + T(x,t))K(x,y;t) + F(x,y;t) = 0, \tag{2.14}$$

where $T(x,t)$ denotes the family of integral operators

$$T(x,t)f(y) = \int_x^\infty f(z)H(z,y;t)\,dz. \tag{2.15}$$

It will always be assumed that $T(x,t)$ is a compact operator for each fixed (x,t). For instance, this is guaranteed if

$$\int_x^\infty \int_x^\infty |H(y,z;t)|^2 dy\,dz < \infty;$$

indeed, in this case T is Hilbert–Schmidt.

To apply Steinberg's theorem, we treat the time t as the complex parameter. (Note that it would not do any good to look at x as this parameter since the domain of integration for $T(x,t)$ depends on x, and so the operators could not possibly be analytic for a large enough class of functions.) Now, for all x,y, if the kernel $H(x,y;t)$ depends analytically on t for $t \in \Omega$, then the operators $T(x,t)$ depend analytically on t. If furthermore $F(x,y;t)$ is analytic in t, then Steinberg's theorem implies that

$$K(x,y;t) = -(I + T(x,t))^{-1}F(x,y;t)$$

is, for each fixed (x,y), a meromorphic function of t. (It is one of the assumptions of complete integrability that the inverse exists for at least one t.) Therefore

$$Q[u(x,t)] = P[K(x,x;t)]$$

is also a meromorphic function of t for each fixed x.

THEOREM 2.5. *If a system of partial differential equations is Q-completely integrable in the Banach space $\mathfrak{B}$, and if the initial data $u(x,0) \in \mathfrak{B}$, then the function $Q[u(x,t)]$ is meromorphic in t for $t \in \Omega$ and each fixed x.*

A slight generalization of this theorem will prove to be of use in the sequel. Suppose that the time axis is "skewed", by making the change of variables

$$(\tilde{x}, \tilde{t}) = (x + \varepsilon t, t)$$

for some real ε. If $u = f(x,t)$ is the solution to the "unskewed" equation, then $\tilde{u} = \tilde{f}(\tilde{x}, \tilde{t}) = f(\tilde{x} - \varepsilon\tilde{t}, \tilde{t})$ is the solution in terms of the new coordinates. If we let

$$\tilde{K}(\tilde{x}, \tilde{y}; \tilde{t}) = K(\tilde{x} - \varepsilon\tilde{t}, \tilde{y} - \varepsilon\tilde{t}; \tilde{t}),$$

then $\tilde{K}$ is a solution of a Gel'fand–Levitan equation of the form

$$K(\tilde{x}, \tilde{y}; \tilde{t}) + F(\tilde{x} - \varepsilon\tilde{t}, \tilde{y} - \varepsilon\tilde{t}; \tilde{t}) + \int_{\tilde{x}}^\infty \tilde{K}(\tilde{x}, \tilde{z}; \tilde{t})H(\tilde{z} - \varepsilon\tilde{t}, \tilde{y} - \varepsilon\tilde{t}; \tilde{t})\,d\tilde{z} = 0.$$

Therefore the "skewed" equation is also completely integrable, which gives the following theorem.

THEOREM 2.6. *If a system of partial differential equations is Q-completely integrable in the Banach space $\mathfrak{B}$, and if the initial data are in $\mathfrak{B}$, then the function $Q[u(x,t)]$ is meromorphic in (x,t) for $x \in \mathbf{C}$, $t \in \Omega$.*

Consider now the particular solutions of a given completely integrable system which are invariant under the action of a one-parameter symmetry group of the system. In many examples, the group is either a group of translations, leading to travelling wave solutions, or a group of scale transformations, leading to the self-similar solutions of dimensional analysis. The theory for more general symmetry groups is no more difficult than for these particular well-known examples, but in order to preserve the continuity of the exposition, we relegate a brief overview of the theory of group invariant solutions of partial differential equations to an appendix. More comprehensive treatments may be found in [6], [30] and [27].

The main result required here, which is standard for the two main examples, is that, roughly speaking, all solutions invariant under a p-parameter group G of symmetries of a given system $\Delta=0$ of partial differential equations can be found by integrating a system $\Delta/G=0$ of differential equations involving p fewer independent variables. For example, if $\Delta=0$ is a single equation for the function $u(x,t)$, $x,t\in\mathbf{R}$, the solutions invariant under the translation group $G_c:(x,t,u)\rightarrow(x+c\varepsilon,t+\varepsilon,u)$, $\varepsilon\in\mathbf{R}$, where c, the wave speed, is fixed, are just the travelling wave solutions

$$u=w(\xi),\qquad \xi=x-ct,$$

obtained as solutions of an ordinary differential equation found by substituting the above expression into the given equation. Similarly, a scaling group $G_s:(x,t,u)\rightarrow(\lambda^\alpha x,\lambda^\beta t,\lambda^\gamma u)$, $0<\lambda\in\mathbf{R}$ has self-similar solutions of the form

$$u=t^{\gamma/\beta}w(\xi),\qquad \xi=x/t^{\alpha/\beta},$$

again obtained as solutions of an ordinary differential equation.

We now state the precise hypotheses required to prove our version of the general conjecture on completely integrable systems and Painlevé type equations. For a definition of terms the reader should consult the Appendix.

We restrict our attention to a Q-completely integrable system, $\Delta=0$, of partial differential equations in two independent variables (x,t). Let G be a one-parameter local projectable symmetry group of the given system, such that the transformations in G, when extended to complex values of the variables (x,t,u), are analytic. Let G_0 denote the projected group action on (x,t)-space. Assume further that the action of G_0 on some subdomain $D_0\subset\mathbf{C}\times\Omega$, Ω as in Definition 2.1, is regular in the sense of Palais [31], so that all the G-invariant solutions of $\Delta=0$ defined over D_0 are found by integrating a system of ordinary differential equations, $\Delta/G=0$, defined over the image M_0 of D_0 in the quotient manifold M.

THEOREM 2.7. *Suppose $\Delta=0$ is a Q-completely integrable system of partial differential equations in the Banach space $\mathfrak{B}$ with an analytic, regular, projectable, one-parameter symmetry group G. If $u=f(x,t)$ is a G-invariant solution of $\Delta=0$ with initial data lying in $\mathfrak{B}$, then the combination corresponding to Q of the solution of the reduced system of ordinary differential equations is meromorphic in M_0, the image of D_0 in M.*

Proof. Since G_0 is analytic, the orbits of G_0 in the (x,t)-plane must be analytic curves. If the solution of the reduced equation had a singularity other than a pole on M_0, the corresponding G-invariant solution would have a similar singularity along the orbit corresponding to the singular point. This, however, would contradict Theorem 2.6. □

Thus Theorem 2.7, in a certain restricted sense, states that the reduced equation for the G-invariant solutions must be of Painlevé type. However, since the initial data for the G-invariant solutions must lie in $\mathfrak{B}$, it is not for every solution of the reduced equation that Q is required to have only poles for singularities. In effect we can consider only those solutions which either decay sufficiently rapidly at $\pm\infty$ along the real axis, or are periodic along the real axis. This restriction seems inescapable given the particular method of proof. It would be extremely interesting to remove these restrictions and prove the conjecture of the introduction in full generality.

3. Applications.

3.1. The generalized KdV equations. Consider the equation

$$u_t + u^p u_x + u_{xxx} = 0, \tag{3.1}$$

where p is a nonnegative integer. This equation has scale-invariant solutions, but as the resulting third order ordinary differential equation is rather complex to analyze in full, we therefore apply our results to a simpler class of self-similar solutions, namely the travelling wave solutions. Here the symmetry group is

$$G_c\colon (x,t,u) \to (x+c\varepsilon, t+\varepsilon, u), \qquad \varepsilon \in \mathbf{R},$$

where c denotes the velocity of the wave. The invariants of G_c are $\xi = x - ct, u$, and the reduced equation for G_c-invariant solutions takes the form

$$u''' + u^p u' - cu' = 0,$$

primes denoting derivatives with respect to ξ. This can be integrated once:

$$u'' = \frac{-1}{p+1} u^{p+1} + cu + \frac{1}{2} d.$$

Multiplying by u', a further integration yields

$$(u')^2 = \frac{-2}{(p+1)(p+2)} u^{p+2} + cu^2 + du + e, \tag{3.2}$$

for some constants d, e. Thus the general travelling wave solution will be expressed in terms of the hyperelliptic function corresponding to the square root of the $(p+2)$nd order polynomial on the right of (3.2). The following two results characterize the singularities of the solutions of (3.2).

THEOREM 3.1 (*Painlevé's theorem*). *Consider the ordinary differential equation*

$$G(u', u, \xi) = 0,$$

where G is a polynomial in u' and u, and analytic in ξ. Then the movable singularities of the solutions are poles and/or algebraic branch-points.

THEOREM 3.2. *Consider the equation*

$$(u')^2 = R(u), \tag{3.3}$$

where R is a rational function of u. Then the solutions of (3.3) *are all meromorphic in* $\mathbf{C}$ *if and only if R is a polynomial of degree not exceeding* 4.

The proofs may be found in Ince [18] and Hille [17, p. 683]. Note that if u has an algebraic branch point, so also does any linear combination of u and its derivatives. Therefore, for (3.1) to be linearly completely integrable, (3.2) must satisfy Theorem 3.2. Thus $p = 0, 1$, or 2, and in these cases the solutions are given by elliptic or trigonometric functions. Note that $p = 0$ corresponds to the linear case, $p = 1$ to the KdV equation, and $p = 2$ to the modified KdV equation, all of which are known to be integrable by inverse scattering.

To complete the demonstration that the generalized KdV equations are not linearly completely integrable for $p \neq 0, 1, 2$, we must place the complete integrability in a suitable Banach space $\mathfrak{B}$, and to do so we check the asymptotic behavior of the travelling wave solutions at $\pm\infty$. If we require that $u, u_x \to 0$ as $|x| \to \infty$, then $d=e=0$ in (3.2). Moreover the polynomial on the right of (3.2) now has a double zero at $u=0$ and a simple zero at $u_0=[\frac{1}{2}(p+1)(p+2)c]^{1/p}$. Standard techniques (cf. [36]) allow us to conclude the existence of travelling wave solutions with positive velocities decaying exponentially for $|x| \to \infty$, and reaching an extreme value of u_0. Thus for p odd, the travelling waves are humps with u_0 the peak value, while for p even, both humps and troughs occur. The important point, however, is the exponential decay of these waves for $|x| \to \infty$, and the fact that for $p \neq 0, 1, 2$, they have complex nonpolar singularities. If therefore we take for the Banach space $\mathfrak{B}$ a space of functions vanishing exponentially, we have shown that the generalized KdV equations are not linearly completely integrable in $\mathfrak{B}$ for $p \neq 0, 1, 2$, and this completes the demonstration that these equations can be solved by inverse scattering only when $p=0, 1$ or 2. This result is in accordance with numerical evidence [10] that only in these special cases do the equations have soliton solutions.

3.2. Nonlinear Klein–Gordon equations. Consider the nonlinear Klein–Gordon equation in characteristic coordinates

$$u_{xt}=f'(u), \tag{3.4}$$

where f is an analytic function of u, real for real u, and prime denotes derivative. The cases we will be most interested in are when f is a polynomial or a finite sum of exponential functions. We will determine necessary conditions on f for (3.4) to be linearly completely integrable by analysis of the singularities of the travelling wave solutions. If c is the velocity, $\xi=x-ct$, then the reduced equation for the G_c-invariant solutions of (3.4) is

$$-cu''=f'(u). \tag{3.5}$$

Multiplying (3.5) by u' and integrating yields

$$-\frac{c}{2}(u')^2=f(u)+k \tag{3.6}$$

for some constant k. For simplicity we shall assume that k can be chosen so that u_1 (real) is a simple or double zero of $f(u)+k$ and there is a second consecutive simple or double zero for some real u_2. This assumption ensures that the initial data $u(x,0)$ can be chosen to lie in a suitable Banach space $\mathfrak{B}$:

i) if u_1 and u_2 are simple zeros, so that a solution of (3.6) oscillates between u_1 and u_2, we take $\mathfrak{B}$ to be a space of periodic functions;

ii) if u_1 is a double and u_2 a simple zero, so that a solution of (3.6) decays exponentially to u_1 as $|\xi| \to \infty$, we take $\mathfrak{B}$ to be a space of functions exponentially converging:

iii) if u_1 and u_2 are double zeros, so that a solution of (3.6) tends exponentially to u_1 as $\xi \to \infty$ and to u_2 as $\xi \to -\infty$ (or vice versa), we can again take $\mathfrak{B}$ to be a space of functions exponentially converging, but to different limits.

The following theorem (stated in the context of (3.4) although it applies generally) is an immediate consequence of considering a linear combination of u and its derivatives. It tells us what singularities are possible for solutions of linearly completely integrable equations.

THEOREM 3.3. *Suppose for some constant k that the analytic function $f(u)+k$ has two consecutive simple and/or double zeros on the real axis. Then, if the nonlinear Klein–Gordon equation* (3.4) *is linearly completely integrable in the relevant Banach space* *indicated above, it must be the case that any solution of* (3.6) (*with c having the opposite sign to* $f(u)+k$ *between the zeros*) *has as singularities only poles or logarithmic branch-points.*

A logarithmic branch-point is by definition a singularity such that some linear combination of derivatives has a pole. It arises in practice if the scattering operator L depends only on $u_x, u_{xx}, \cdots$, so that $Q[u]$ in turn depends only on derivatives; to demonstrate that this situation can indeed arise, consider the sine-Gordon equation

$$u_{xt}=\sin u.$$

It was indicated in Example 2.3 that this is completely integrable, and to examine it in the context of Theorem 3.3 we take

$$f(u)=-\cos u, \qquad k=0.$$

The solution of (3.6) is then

$$\sqrt{2}\sin\left(\tfrac{1}{2}u\right)=\operatorname{sn}\{c^{-1/2}(\xi+\delta)\},$$

where sn is the Jacobi elliptic function with modulus $k=1/\sqrt{2}$. This is well defined for $c>0$. Now sn has simple poles on a certain rectangular lattice in $\mathbf{C}$, and so u has logarithmic singularities at these lattice points. The reason for the appearance of these nonpolar singularities is the fact that u_x rather than u appears in the scattering operator L. We note that u_x on the other hand does have only poles for singularities.

THEOREM 3.4. *Suppose that* $f(u)$ *is a rational function, real for real u and such that, for some k,* $f(u)+k$ *has two consecutive simple and/or double zeros on the real axis. If the Klein–Gordon equation* $u_{xt}=f'(u)$ *is linearly completely integrable, then f is a polynomial of degree not exceeding* 4.

The proof is immediate from Theorems 3.1 and 3.2.

To discuss the case where f is a polynomial of degree ≤ 4, one can try other similarity solutions of (3.4), or else quite different tests. For example, it can be shown [8] that when f is of degree >2, so that f' is nonlinear, (3.4) has only finitely many polynomial conservation laws, while a theorem of Gel'fand and Dikii [12], [13] states that if a system of partial differential equations has a Lax representation, then there are an infinite number of polynomial conservation laws. Next we consider the case where f is a finite sum of exponential functions

$$f(u)=\sum_{j=0}^{m} c_j e^{\alpha_j u}, \qquad c_j, \alpha_j \in \mathbf{C}.$$

For simplicity, we restrict our attention to the case where $\alpha_j=n_j\alpha$ for some $\alpha\in\mathbf{C}$ and some rational numbers n_j. By dividing α by the common denominator of the n_j, we may assume the n_j are integers. Now let $v=\exp(\alpha u)$, so that $v'=\alpha v u'$. Thus v satisfies

$$-\frac{c}{2\alpha^2}(v')^2=\sum c_j v^{n_j+2}. \tag{3.7}$$

Note that Theorem 3.2 cannot be applied here since v may have singularities not shared by u. However, since $u'=v'/\alpha v$, it is necessary to find conditions on (3.7) such that the function v'/v, for solutions v, has no movable algebraic branch-points. This requires a more detailed investigation of the proof of Theorem 3.2. It suffices for our purposes to note the following:

LEMMA 3.5. *Consider the ordinary differential equation*

$$(v')^2=v^{-n}P(v),$$

where P is a polynomial with $P(0)\neq 0$ and n is a positive integer. Then for any $\xi_0\in\mathbf{C}$ there is a solution v with algebraic branch-point at ξ_0. This solution has a Puiseux expansion

$$v(\xi)=\sum_{j=1}^{\infty} a_j(\xi-\xi_0)^{jr}$$

with $a_1\neq 0$, and the rational number r is given by

i) $r=(m+1)^{-1}$ *if* $n=2m$,

ii) $r=2(2m+3)^{-1}$ *if* $n=2m+1$.

The proof of this result can be inferred from Hille, [17, pp. 681–682].

LEMMA 3.6. *Suppose v has an algebraic branch-point at ξ_0. Then v'/v has no branch-point at ξ_0 if and only if $v(\xi)=(\xi-\xi_0)^r f(\xi)$ for r rational and f meromorphic at ξ_0.*

Proof. Assume without loss of generality that $\xi_0=0$. Let v have the Puiseux expansion

$$v(\xi)=\xi^{mr}\sum_{j=0}^{\infty} a_j\xi^{jr},$$

where m is an integer and $a_0\neq 0$. Let a_k be the first nonzero coefficient for which kr is not an integer, if such exists. Now

$$\frac{1}{v}=\xi^{-mr}\sum_{j=0}^{\infty} b_j\xi^{jr},$$

where $b_0=a_0^{-1}$ and the first nonzero coefficient b_j with jr not an integer is $b_k=-a_k a_0^{-2}$. Furthermore

$$v'=\xi^{mr-1}\sum_{j=0}^{\infty}(m+j)ra_j\xi^{jr}.$$

Therefore

$$\frac{v'}{v}=\xi^{-1}\sum_{j=0}^{\infty} c_j\xi^{jr}$$

and the coefficient of ξ^{kr} is

$$c_k=-mra_k a_0^{-1}+(m+k)ra_k a_0^{-1},$$

which vanishes only when $a_k=0$. This proves the lemma. □

PROPOSITION 3.7. *Consider the ordinary differential equation*

$$(v')^2=\sum_{j=-n}^{N} b_j v^j. \tag{3.8}$$

Given $\xi_0\in\mathbf{C}$, there exists a solution v of (3.8) *such that v'/v has an algebraic branch-point at ξ_0, unless* (3.8) *is of the special form*

$$(v')^2=\sum_{j=-2}^{2} c_j v^{jk+2} \tag{3.9}$$

for some integer k.

Proof. Let $\xi_0=0$ and assume $b_N\neq 0$, $b_{-n}\neq 0$. By Lemma 3.6 all solutions must be of the form $v(\xi)=\xi^r f(\xi)$ with r rational and f meromorphic at 0 if we are to avoid an algebraic branch-point for v'/v. Thus

$$(v')^2=\xi^{2r}(r\xi^{-1}f+f')^2,$$

and

$$v^j=\xi^{jr}f^j,$$

so that, equating the fractional powers of ξ, we see that $b_j=0$ unless $jr=2r+\iota$ for some integer ι. If $n>0$, it follows from Lemma 3.5 that $b_j=0$ unless

i) $j\equiv 2 \bmod (m+1)$ for $n=2m$, or

ii) $2j\equiv 4 \bmod (2m+3)$ for $n=2m+1$.

In particular, the only negative values of j which satisfy these congruences are $1-\frac{1}{2}n$ and $-n$, the first value occurring only when n is even.

Next set $w=1/v$. Then (3.8) becomes

$$(w')^2=\sum_{j=-n}^{N} b_j w^{4-j}.$$

Since $w'/w=-v'/v$, w must satisfy the same conditions as v. Therefore, if $N>4$, $b_j=0$ unless

i) $j\equiv 2 \bmod (M-1)$ if $N=2M$, or

ii) $2j\equiv 4 \bmod (2M-2)$ if $N=2M+1$.

The only positive values of j satisfying these are N, $\frac{1}{2}N+1$ and 2, the second only if N is even. Comparison of the two sets of congruences then shows that (3.8) must be of the required form. □

THEOREM 3.8. *Suppose $f(u)$ is a linear combination of exponential functions $e^{\alpha_j u}$ with $\alpha_j=n_j\alpha$, n_j rational, α complex. Suppose further that $f(u)$ is real for u real, and that, for some real k, $f(u)+k$ has two consecutive simple and/or double zeros on the real axis. If the Klein–Gordon equation $u_{xt}=f'(u)$ is linearly completely integrable, then f must be of the special form*

$$f(u)=\sum_{j=-2}^{2} c_j e^{j\beta u}, \tag{3.10}$$

where β is a rational multiple of α.

It is interesting that the form (3.10) for f includes the double sine-Gordon equation

$$u_{xt}=a\sin\alpha u+b\sin\left(\frac{1}{2}\alpha u\right),$$

for which numerical studies of Dodd and Bullough [8] indicate the existence of soliton solutions. Mikhailov [24], [25] and Fordy and Gibbons [15], have shown that a special case of (3.10) when $f(u)=e^{2u}+e^{-u}$ does have a Lax representation, but it is not known whether the result extends to a general function $f(u)$ of the form (3.10).

3.3. Model wave equations of Whitham and Benjamin. The integro-differential equation

$$u_t+uu_x+\mathcal{H}[u_x]=0, \tag{3.11}$$

where $\mathcal{H}$ is the integral operator

$$\mathcal{H}[f](x)=\int_{-\infty}^{\infty}H(x-y)f(y)\,dy,$$

was proposed by Whitham [35], [36] as an alternative to the KdV equation for long waves in shallow water which could also model breaking and peaking. Here $\mathcal{H}$ is taken to be the Fourier transform of the desired phase velocity $c(k)$, where k is the wave number. Of particular interest is the case

$$c(k)=\frac{1}{\nu^2+k^2}, \qquad \nu>0,$$

so that

$$H(x)=\frac{1}{2\nu}e^{-\nu|x|}.$$

Note that $\mathcal{H}$ is the Green's function of the operator $D^2-\nu^2=\mathcal{D}$ so that (3.11) is equivalent to the differential equation

$$\mathcal{D}[u_t+uu_x]+u_x=0. \tag{3.12}$$

It can be shown [10] that (3.12) possesses travelling wave solutions u, with $|u|\to 0$ as $|x|\to\infty$, and amplitudes between 0 and some maximum height. Computer studies indicate that these waves may be solitons, i.e., they may interact cleanly. One possibly undesirable feature of (3.11) is the extremely fast propagation of short-wave components, and for this reason Benjamin, Bona and Mahony [5] proposed the alternative model

$$u_t+uu_x-\mathcal{H}[u_t]=0. \tag{3.13}$$

Again, in the special case, (3.13) can be rewritten as

$$\mathcal{D}[u_t+uu_x]-u_t=0. \tag{3.14}$$

In general, we will let $\mathcal{D}$ be any constant coefficient linear differential operator

$$\mathcal{D}=\sum_{i=0}^{n} c_iD^i, \qquad c_n\neq 0.$$

We show here that the model equations (3.12), (3.14) cannot be integrable by inverse scattering methods. As usual, consider the travelling wave solutions of these equations. If c denotes the velocity, then the reduced equation, after integration, is

$$\mathcal{D}\left[\frac{1}{2}(u-c)^2\right]+a(u+d)=0. \tag{3.15}$$

Here d is a constant of integration, $a=1$ in the Whitham model, $a=c$ in the Benjamin model, and D now denotes $d/d\xi$, $\xi=x-ct$. Since nth order equations of Painlevé type have not been classified, we resort to Painlevé's original "α-method" to analyze the singularities of the solutions of (3.15). The basic result is found in Ince [18, p. 319].

LEMMA 3.9. *Suppose $\Delta(u,\xi,\alpha)=0$ is an analytically parametrized family of ordinary differential equations for α in some domain Ω containing 0 as an interior point. If the general solution $u(\xi,\alpha)$ is uniform in ξ for $\alpha\in\Omega\sim\{0\}$, then it will be uniform for $\alpha=0$.*

In our case, let $\xi=\xi_0+\alpha\zeta$. Then if we consider u as a function of ζ, (3.15) becomes

$$\left(c_nD^n+\alpha c_{n-1}D^{n-1}+\cdots+\alpha^nc_0\right)\left[\frac{1}{2}(u-c)^2\right]+\alpha^na(u+d)=0,$$

where D now denotes $d/d\zeta$. For $\alpha=0$, this reduces to

$$D^n\left(\frac{1}{2}(u-c)^2\right)=0,$$

the solution of which is

$$u=c+\sqrt{P_n(\zeta)}$$

for an arbitrary polynomial P_n of degree $\leq n-1$. This, for appropriate P_n, has an algebraic branch-point at $\zeta=0$, so that, by the lemma, solutions of (3.15) must also have nonlogarithmic branch-points. (This involves a slight extension of the lemma above, but it is easy to infer its truth from the proof given by Ince.) If these solutions also satisfy decay or periodicity properties, Theorem 2.7 (together with Theorem 3.1) shows that model equations (3.12), (3.14) cannot be linearly completely integrable. In particular, Whitham's equation with $\mathfrak{D}=D^2-\nu^2$ is not integrable by inverse scattering.

3.4. The BBM equation. The equation

$$u_t+uu_x-u_{xxt}=0, \tag{3.16}$$

known as the BBM equation, was proposed by Benjamin, Bona and Mahony [5] as an alternative model to the KdV equation for the description of long waves in shallow water. In [29] it was shown to possess only three independent conservation laws, and therefore by the results of Gel′fand and Dikii cannot be completely integrable. Our consideration of this example runs into difficulties because the self-similar solutions do not satisfy any decay or periodicity properties, and the functions Q we can allow are limited, but we will indicate the method here.

First we note that (3.16) admits the symmetry group

$$G:(x,t,u)\to(x,\lambda^{-1}t,\lambda u), \qquad 0<\lambda\in\mathbf{R},$$

of scale transformations. Invariants of G are provided by x and $w=tu$, for $t>0$, and the reduced equation for $G=$invariant solutions is then

$$w''+ww'-w=0, \tag{3.17}$$

the primes denoting derivatives with respect to x. It can be readily checked, by the procedure in Ince [18], that (3.17) is not of Painlevé type. Indeed, it is of Ince's type i(b) [18, p. 330]. Applying the α-method as Ince does, one can readily check that branch-points appear, although possibly only logarithmic, and this, granted the existence of a suitable Banach space $\mathfrak{B}$, would show that the BBM equation is not Q-completely integrable for Q, say, the identity.

However, a closer investigation of the behavior of the real solutions of (3.17) is required. Since x does not appear, it can be integrated to yield

$$(1-w')e^{w'}=ce^{-w^2/2}. \tag{3.18}$$

In principle, this equation can again be integrated by solving for w' in terms of w. To investigate the solutions qualitatively, note that $w'=0$ if and only if $w^2=2\log c$, $c\geq 1$. The only double root is when $c=1$, and only in this case do solutions decay at $+\infty$ or $-\infty$. However, it is readily seen that a solution decaying at one endpoint cannot decay at the other, nor are periodic solutions possible. Thus we are unable to apply our results to this case.

3.5. Lax pairs of composite order. Gel′fand and Dikii [12], [13] succeeded in classifying all Lax pairs of differential operators of the following special type. Let

$$L_n=D^n+u_{n-2}D^{n-2}+\cdots+u_1D+u_0$$

be a scalar differential operator of order n with $u=(u_0,\cdots,u_{n-2})$ independent C^∞ functions, and $D=d/dx$. They showed that for each integer m not a multiple of n, there is a differential operator

$$P_m=D^m+p_{m,m-2}D^{m-2}+\cdots+p_{m,1}D+p_{m,0}$$

of order m, the $p_{m,i}$ being polynomials in the u_j and their derivatives, such that the Lax representation

$$\frac{\partial L_n}{\partial t}=[P_m,L_n]$$

is a nontrivial system of evolution equations

(3.19) $$u_t=K_m(u).$$

Moreover, the P_m are unique if we require the coefficients $p_{m,j}$ to have no constant term.

Consider the stationary solutions of the system (3.19), i.e., those in which u is independent of t. These satisfy the system $K_m(u)=0$, or equivalently, the "stationary Lax representation"

(3.20) $$[P_m,L_n]=0.$$

THEOREM 3.10. *If the orders n,m of the operators L_n,P_m in the Lax representation of* (3.19) *are not relatively prime integers, then stationary solutions of* (3.19) *with arbitrary singularities in the complex plane exist.*

Proof. Let $k>1$ be the greatest common divisor of m and n. Consider the operator

$$M_k=D^k+v_{k-2}D^{k-2}+\cdots+v_1D+v_0,$$

whose coefficients $v_j(x)$ are sufficiently differentiable for $x\in\mathbf{R}$ but are otherwise arbitrary functions. Then

$$L_{n,0}=(M_k)^{n/k},\qquad P_{m,0}=(M_k)^{m/k}$$

obviously satisfy the stationary Lax representation (3.20) and, moreover, using the formalism of Gel′fand and Dikii, it is easy to prove that $P_{m,0}$ is derivable from $L_{n,0}$ via the same formulae as gave P_m from L_n. Therefore each such M_k gives a stationary solution of the evolutionary system (3.19). □

Now suppose that L_n is any such operator, where n is a composite number. If there exists a Gel′fand–Levitan type of integral equation for solving the inverse problem for the operator L_n, then Theorem 2.7 would imply the meromorphic character of the group-invariant solutions of the evolutionary system (3.19), using similar arguments to those used in the integration of the Korteweg–de Vries equation. This, however, is in contradiction to Theorem 3.10 for the case of time-invariant solutions. (The relevant symmetry group is just translation in t.) This indicates that such a differential operator of composite order does not have an inverse-scattering formalism in the sense that the Schrödinger operator does—either no such Gel′fand–Levitan equation exists, or the assumptions regarding analyticity are not justified. Indeed, we know of no such Gel′fand–Levitan equation for any operator of composite order, e.g. for order $n=4$.

Appendix. Group-invariant solutions of differential equations. The general theory was developed by Lie and, more recently, Ovsjannikov. For details, the best references are [6], [27], [30]. Here we briefly review the relevant concepts.

Let

(A1) $$\Delta(x,u)=0,\qquad x\in\mathbf{R}^m,\quad u\in\mathbf{R}^n,$$

be a system of partial differential equations in m independent and n dependent variables. A *symmetry group* is a local Lie group of transformations acting on the space $\mathbf{R}^n \times \mathbf{R}^m$ which takes solutions of the system to other solutions. (The group acts on solutions by transforming their graphs. In the case of *projectable groups*, meaning that all transformations are of the form $(\tilde{x},\tilde{u})=(\alpha(x),\beta(x,u))$, a solution $u=f(x)$ will be transformed into the solution $\tilde{u}=\tilde{f}(\tilde{x})=\beta(\alpha^{-1}(\tilde{x}),f(\alpha^{-1}(\tilde{x})))$, provided α is invertible.)

The most helpful property of continuous symmetry groups is that for a given system they can *all* be found by systematic computations of an elementary character. The key step, which was Lie's fundamental discovery, is to look for the infinitesimal generators of the group, which are vector fields of the general form

$$v=\sum_{i=1}^{m} \xi^i(x,u)\frac{\partial}{\partial x_i}+\sum_{j=1}^{n} \varphi_j(x,u)\frac{\partial}{\partial u_j},$$

the group transformations themselves being recovered from the auxiliary ordinary differential equations governing the integration of the above vector field. This leads to the following infinitesimal criterion for a symmetry group of a given system [28].

THEOREM. *Let G be a connected local Lie group. Then G is a symmetry group of the system of partial differential equations $\Delta=0$ if and only if*

$$\text{(A2)} \qquad \operatorname{pr} v(\Delta)=0 \quad \textit{whenever}\ \Delta=0$$

for every infinitesimal generator v of G.

Here $\operatorname{pr} v$ refers to the "prolongation" of the vector field v, obtained as the infinitesimal generator of the action of the group G on the spaces of partial derivatives of u with respect to x induced by the action of G on functions $u=f(x)$. The point is that the condition (A2) leads to a large number of elementary partial differential equations for the coefficients ξ^i, φ_j of v, the general solution of which is the most general infinitesimal generator of a one-parameter symmetry group of the given system of differential equations. Examples of this computation can be found in the above-mentioned references.

Now, given a symmetry group G, a G-invariant (or self-similar) solution of (A1) is a solution which is unchanged by the transformations in G. The fundamental property of G-invariant solutions is that, roughly speaking, they may all be found via the integration of a system of partial differential equations in fewer independent variables. To make this precise, we must assume that G acts "regularly" in the sense of Palais [31] on an open subset $U\subset \mathbf{R}^m\times\mathbf{R}^n$. This requires, in U,

i) that all the orbits of G have the same dimension,

ii) that, for any point (x,u), there exist arbitrarily small neighborhoods N such that the intersection of any orbit O of G with N is a connected subset of O.

(The prototypical group actions excluded by the second requirement are the irrational flows on the torus.)

Under these two assumptions, it is well known that the quotient space $M=U/G$, whose points correspond to the orbits of G, can be naturally endowed with the structure of a smooth (although not always Hausdorff) manifold. Moreover, the G-invariant solutions of (A1) are all obtained by integrating a reduced system $\Delta/G=0$ of partial differential equations on M, which necessarily has fewer independent variables. Precise statements and proofs of these results may be found in [27].

For our purposes, the construction of the reduced system for the G-invariant solutions proceeds as follows: Local coordinate systems on the quotient manifold M are provided by a "complete set of functionally independent invariants of G", cf. [30]. If G is projectable, these are functions of the form

$$\xi^1(x),\cdots,\xi^{m-l}(x),w^1(x,u),\cdots,w^n(x,u),$$

which are unchanged under the action of G. The functional independence means that the Jacobian matrix

$$\begin{pmatrix} \partial\xi/\partial x & 0 \\ \partial w/\partial x & \partial w/\partial u \end{pmatrix}$$

is everywhere nonsingular. The reduced system $\Delta/G=0$ will then be found in terms of the new independent variables ξ^i and the new dependent variables w^j.

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