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Recursion Operators and Hamiltonian Systems

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Abstract

This paper reviews the basic concepts in the theory of recursion operators and their applications to infinite dimensional Hamiltonian systems. Connections with the Poisson complex are explained in detail. The theory is illustrated by new results for first order hyperbolic systems, including the equations of gas dynamics.

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1. Introduction.

Generalized symmetries first make their appearance the original paper of E. Noether on the correspondence between symmetries of variational problems and conservation laws of the associated Euler-Lagrange equations. The terminology *generalized* refers to the fact that the infinitesimal generators are allowed to depend on derivatives of the dependent variables, which makes the corresponding group transformations nonlocal. Recurson operators were first introduced in [9] to provide a mechanism for generating infinite families of generalized symmetries. A fundamental advance in the subject was the work of Magri, [7], who showed how recursion operators could be constructed for systems with two compatible Hamiltonian structures. In this paper, we develop the general theory of recursion operators and biHamiltonian systems, based on the important constuct of the Poisson complex. New Hamiltonian structures and recursion operators for systems of hyperbolic conservation laws, including the equations of gas dynamics and one-dimensional elasticity are found. Many of the topics in this article are more extensively developed in [11], [12], [14] to which we refer the interested reader for a more complete exposition of the theory, applications and history.

2. Generalized Symmetries, Recursion Operators and Conservation Laws.

Consider a system of partial differential equations

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \ldots, m, \tag{2.1}$$

defined on some open subset $M^{(n)}$ of the jet space, whose coordinates $(x,u^{(n)})$ consist of the independent variables $x = (x^1,\ldots,x^p)$, the dependent variables $u = (u^1,\ldots,u^q)$, and their partial derivatives $u^\alpha_J = \partial^J u^\alpha/\partial x^J$ up to order n. A *generalized vector field* is a partial differential operator of the form

$$\mathbf{v}_Q = \sum_{\alpha=1}^{q} Q_\alpha(x,u^{(k)}) \frac{\partial}{\partial u^\alpha}, \tag{2.2}$$

in which the *characteristic* $Q[u] = (Q_1,\ldots,Q_q)$ is a q-tuple of *differential functions*, meaning smooth functions of x, u, and derivatives of u. The vector field $\mathbf{v}_Q$ generates a one-

parameter group of transformations on a suitable space of functions. Specifically, if $u = f(x)$ is a prescribed function, then the transformed function $\tilde{f}_\varepsilon(x) = g_\varepsilon \cdot f(x) = f(x,\varepsilon)$ is found by evaluating the solution $u = f(x,\varepsilon)$ to the Cauchy problem

$$\frac{\partial u^\alpha}{\partial \varepsilon} = Q_\alpha(x,u^{(k)}), \qquad \alpha = 1, \ldots, q, \qquad u(x,\varepsilon = 0) = f(x). \tag{2.3}$$

at time ε. (We ignore complications involving the existence and uniqueness of solutions of the Cauchy problem (2.3).) The vector field $\mathbf{v}_Q$ is called an (infinitesimal) *generalized symmetry* of the system (2.1) if it takes solutions to solutions (at least formally), i.e. if $u = f(x)$ is a solution, so is $u = g_\varepsilon \cdot f(x)$. In particular, if the characteristic takes the special form

$$Q_\alpha(x, u^{(1)}) = \varphi_\alpha(x,u) - \sum_{i=1}^{p} \xi^i(x,u) \frac{\partial u^\alpha}{\partial x^i}, \qquad \alpha = 1, \ldots, q,$$

then the group corresponds to the geometrical group of transformations generated by the vector field

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi_\alpha(x,u) \frac{\partial}{\partial u^\alpha}.$$

Other generalized symmetries act non-locally on functions.

To obtain the infinitesimal invariance criterion for the system of differential equation (2.1), we prolong $\mathbf{v}_Q$ to the infinite jet space $M^{(\infty)}$, which we realize as the direct limit of the finite jet spaces $M^{(n)}$ as $n \to \infty$, leading to the partial differential operator

$$\mathrm{pr}\, \mathbf{v}_Q = \sum_{\alpha=1}^{q} \sum_{J} D_J Q_\alpha \frac{\partial}{\partial u^\alpha_J}.$$

Here $D_J = D_{j_1} \cdot \ldots \cdot D_{j_k}$ denotes the k^{th} order total derivative corresponding to the multi-index $J = (j_1, \ldots, j_k)$.

Theorem 1. Suppose the system of partial differential equations (2.1) is totally nondegenerate in the sense of [11; Definition 2.83]. Then $\mathbf{v}_Q$ is a generalized symmetry

of the system if and only if

$$\text{pr } \mathbf{v}_Q(\Delta_\nu) = 0, \qquad \nu = 1, \ldots, m, \tag{2.4}$$

for all solutions $u = f(x)$ to the system (2.1).

The nondegeneracy condition required for the validity of the theorem is very mild, and is satisfied by well-nigh every system of partial differential equations which arises in physical applications. The symmetry conditions (2.4) constitute a large, over-determined system of elementary partial differential equations, called the *determining equations*, for the characteristic Q of $\mathbf{v}_Q$. In practice, these can be systematically solved to determine all the generalized symmetries of the system (2.1).

Define the Fréchet derivative of $\Delta[u] = (\Delta_1, \ldots, \Delta_m)$ to be the differential operator

$$\mathbf{D}_\Delta(v) = \frac{d}{d\varepsilon}\Big|_{\varepsilon = 0} \Delta[u + \varepsilon v].$$

Note the elementary identity

$$\text{pr } \mathbf{v}_Q(\Delta) = \mathbf{D}_\Delta(Q),$$

so that the symmetry condition (2.4) can be rewritten as

$$\mathbf{D}_\Delta(Q) = 0 \qquad \text{whenever } u \text{ is a solution to } \Delta = 0. \tag{2.5}$$

There is a natural Lie bracket operation between generalized vector fields. Specifically, if $\mathbf{v}_Q$ and $\mathbf{v}_R$ are generalized vector fields, their Lie bracket $\mathbf{v}_S = [\mathbf{v}_Q, \mathbf{v}_R]$ is the generalized vector field with characteristic

$$S = \text{pr } \mathbf{v}_Q(R) - \text{pr } \mathbf{v}_R(Q). \tag{2.6}$$

The reader can verify the usual binlinearity, skew-symmetry and Jacobi identity for this bracket. In particular, if $\mathbf{v}_Q$ and $\mathbf{v}_R$ are generalized symmetries of the system (2.1), so is the vector field $\mathbf{v}_S = [\mathbf{v}_Q, \mathbf{v}_R]$.

By definition, a *recursion operator* $\mathfrak{R}$ for a system of differential equations is a linear operator which maps symmetries to symmetries; in other words if $\mathbf{v}_Q$ is a generalized symmetry, and $\tilde{Q} = \mathfrak{R} \cdot Q$, then $\mathbf{v}_{\tilde{Q}}$ is also a generalized symmetry. There is a

simple criterion for detemining when a given operator is a recursion operator. The proof is an elementary application of formula (2.5).

Theorem 2. A linear operator $\mathfrak{R}$ is a recursion operator for the system of differential equations $\Delta[u] = 0$ if there is a second linear operator $\tilde{\mathfrak{R}}$ such that the identity

$$\mathbf{D}_\Delta \cdot \mathfrak{R} = \tilde{\mathfrak{R}} \cdot \mathbf{D}_\Delta \tag{2.7}$$

holds on solutions to Δ.

Example 3. We shall illustrate our concepts using the elementary example of the *Riemann equation*

$$u_t = u\, u_x, \tag{2.8}$$

which is the simplest possible scalar nonlinear conservation law. Later we will see how many of the results for this very simple equation have direct counterparts in the case of two-dimensional hyperbolic systems, including the equations for polytropic gas dynamics and one-dimensional elasticity .

We begin by determining all the generalized symmetries of the Riemann equation. Since (2.5) is required to hold only on solutions to (2.8), we can always replace t-derivatives of u by equivalent expressions involving only x-derivatives. If $\Delta = u_t - uu_x$, then the Fréchet derivative operator is $\mathbf{D}_\Delta = D_t - u\cdot D_x - u_x$. Therefore, a differential function $Q[u] = Q(x,t,u,u_x,\dots,u_k)$, where $u_k \equiv d^k u/dx^k$, is the characteristic of a generalized symmetry if and only if Q is a solution to the first order partial differential equation

$$D_t Q = u\cdot D_x Q + u_x \cdot Q$$

whenever u solves (2.8). Expanding the total derivatives, and replacing t-derivatives of u by x-derivatives, we see that Q must be a solution to the first order partial differential equation

$$\mathbf{w}(Q) \equiv Q_t + u\cdot Q_x + u_x^2 \cdot Q_{u_x} + 3u_x \cdot u_{xx} \cdot Q_{u_{xx}} + \dots + \{D_x^k(u\cdot u_x) - u\cdot u_{k+1}\}\cdot Q_{u_k} = u_x \cdot Q, \tag{2.9}$$

subscripts on Q denoting partial derivatives. By the method of characteristics for a first order linear partial differential equation, it is easy to determine the general solution:

Theorem 4. Define the rational differential functions

$$I_0 = u, \quad I_1 = x - t\cdot u, \quad I_2 = \frac{u}{u_x} - x, \quad I_3 = \frac{u_{xx}}{u_x^3}, \quad I_{j+1} = \frac{1}{u_x} D_x I_j, \quad j \geq 3.$$

A differential function Q is the characteristic of a generalized symmetry $\mathbf{v}_Q = Q\cdot\partial_u$ of (2.8) if and only if

$$Q = u_x\cdot G(I_0, I_1, I_2, \ldots, I_{k+1}), \tag{2.10}$$

where G is an arbitrary smooth function of its arguments.

It turns out that there are several recursion operators for the Riemann equation. Two zeroth order ones are given by

$$\mathfrak{R}_1 = 2u + u_x\cdot D_x^{-1}, \qquad \mathfrak{R}_2 = u^2 + u\, u_x\cdot D_x^{-1}. \tag{2.11}$$

For instance, to prove (2.7) for $\mathfrak{R}_1$, we note that since $u\cdot D_x + u_x = D_x\cdot u$, the commutator

$$\begin{aligned}[\mathbf{D}_\Delta, \mathfrak{R}_1] &= [D_t - u\cdot D_x - u_x, 2u + u_x\cdot D_x^{-1}] \\ &= 2u_t + u_{xt}\cdot D_x^{-1} - 2u\cdot u_x - (u\cdot u_{xx} + u_x^2)\cdot D_x^{-1}\end{aligned}$$

vanishes on solutions, which proves (2.7), with $\tilde{\mathfrak{R}} = \mathfrak{R}_1$. The verification for $\mathfrak{R}_2$ is similar. Therefore, starting with the translational symmetry, with characteristic $Q_0 = u_x$, we generate a hierarchy of higher order symmetries with characteristics $Q_n = u^n\cdot u_x$; explicitly

$$\mathfrak{R}_1(Q_n) = \frac{2n+3}{n+1} Q_{n+1}, \qquad \mathfrak{R}_2(Q_n) = \frac{n+2}{n+1} Q_{n+2}.$$

(Interestingly, even though there are two independent recursion operators, the two hierarchies happen to coincide. However, this is special to the polynomial symmetries; on other symmetries, these recursion operators will act differently.)

The Riemann equation admits an additional first order recursion operator

$$\Re = D_x \cdot \frac{1}{u_x} . \tag{2.12}$$

This latter operator acts on the hierarchy Q_n according to

$$\Re(Q_n) = n\, Q_{n-1},$$

and so, up to multiple, "inverts" the first order recursion operator $\Re_1$. Again, this is special to the polynomial hierarchy. For instance, starting with the rational second order generalized symmetry with characteristic $\hat{Q}_2 = u_x \cdot I_3 = u_x^{-2} \cdot u_{xx}$, the recursion operator $\Re$ generates the additional hierarchy of higher order symmetries $\hat{Q}_k = u_x \cdot I_{k+1}$, $k = 2, 3, \dots$, whereas $\Re_1$ and $\Re_2$ lead to yet other second order symmetries.

Given a system of partial differential equations (2.1), a *conservation law* is a p-tuple of differential functions $P[u] = (P_1, \dots, P_p)$ whose divergence

$$\operatorname{Div} P = \sum_{i=1}^{p} D_i P_i = 0,$$

vanishes on all solutions to (2.1). For dynamic problems, the conservation law takes the form

$$D_t T + \operatorname{Div} X = 0.$$

(Div here refers to the spatial variables.) The t-component of such a conservation law is referred to as the conserved density, and, for suitable solutions, (in particular those for which the flux X vanishes on the boundary) the integral $\int T[u]\, dx$ provides a constant of the motion. A conserved density is called *trivial* if it is a (spatial) divergence $T = \operatorname{Div} Y$ on solutions. In the Lagrangian framework, Noether's Theorem, cf. [11; Theorem 5.42], provides a complete correspondence between generalized symmetries of a variational problem and conservation laws of the associated Euler-Lagrange equations. In the next section, we shall see how this extends to the Hamiltonian framework.

Example 5. For the Riemann equation (2.8), any conserved density T, which, without loss of generality, we can take to depend only on $x, t, u, \dots, u_n$, must satisfy

$$D_t T + D_x X = 0$$

on solutions to the equation for some flux X. Writing this out, and replacing t-derivatives by the corresponding expressions in terms of x-derivatives, we find

$$u \cdot D_x T + \mathbf{w}(T) + D_x X = 0,$$

where $\mathbf{w}$ is the vector field given in (2.9). Let $X = Y - u \cdot T$, so this becomes

$$\mathbf{w}(T) - u_x \cdot T + D_x Y = 0.$$

If we rewrite $T = u_x \cdot F(t, I_0, I_1, I_2, \ldots, I_{k+1})$, in terms of the invariants I_j of $\mathbf{w}$, and the single parametric variable t, then $\mathbf{w}(T) - u_x \cdot T = u_x \cdot G_t$, hence the functions

$$u_x \cdot F(t, I_0, I_1, I_2, \ldots, I_{k+1}) - u_x \cdot F(0, I_0, I_1, I_2, \ldots, I_{k+1}) = D_x \{ \int_0^t Y \, ds \}$$

differ by a trivial conserved density. Setting $t = 0$ in the formula for T, we conclude that it is equivalent to a conserved density of the form

$$T = u_x \cdot G(I_0, I_1, I_2, \ldots, I_k).$$

Thus, surprisingly, for the Riemann equation the expressions for symmetries and conserved densities are the same! In particular, we note the infinite sequence of zeroth order conserved densities

$$H_n(u) = u^n, \qquad n = 1,2,3, \ldots . \tag{2.13}$$

3. The Poisson Complex and Hamiltonian Systems.

We now present an approach to the theory of Hamiltonian systems based on the important Poisson complex, which plays as fundamental a role here as the deRham complex does in the theory of differential forms. The Poisson complex, though, involves the dual objects to differential forms, which are known as multi–vectors, or, in the infinite–dimensional case, functional multi–vectors. This complex, in the finite–dimensional case, is due to Lichnerowicz, [6], and was generalized to infinite dimensions in Olver, [10]. We begin by recalling the basic definitions; see [11] for many of the details.

On the infinite jet space, $M^{(\infty)}$, the space Λ^0_* of *functionals* is defined as the

cokernel of the total divergence operator, so that two differential functions $L[u]$ and $\bar{L}[u]$ define the same functional $\mathcal{L}[u] = \int L[u]\, dx$ if and only if they differ by a total divergence: $L = \bar{L} + \operatorname{Div} P$. More generally, define a *vertical k-form* to be a finite sum

$$\hat{\omega} = \sum P_J^\alpha[u]\, du_{J_1}^{\alpha_1} \wedge \ldots \wedge du_{J_k}^{\alpha_k},$$

where the coefficients P_J^α are arbitrary differential functions. The total derivatives D_i act as Lie derivatives on the vertical forms, and the space Λ_*^k of *functional k-forms* is analogously defined as the cokernel of the total divergence. In particular, an easy integration by parts argument shows that any functional one-form is uniquely equivalent to one in the form

$$\omega = \int \{ \sum_{\alpha=1}^{q} P_\alpha[u]\, du^\alpha \}\, dx = \int \{P \cdot du\}\, dx. \tag{3.1}$$

Similarly, it can be shown that any functional 2-form can be placed into canonical form

$$\Omega = \tfrac{1}{2} \int \{ \sum_{\alpha,\beta} du^\alpha \wedge \mathcal{E}_{\alpha\beta} \cdot du^\beta \}\, dx = \tfrac{1}{2} \int \{du \wedge \mathcal{E} \cdot du\}\, dx, \tag{3.2}$$

uniquely determined by the skew-adjoint matrix differential operator $\mathcal{E} = (\mathcal{E}_{\alpha\beta})$.

The *vertical differential* $\hat{d}$ takes a vertical k-form to a vertical (k+1)-form, and is induced by its action

$$\hat{d} P = \sum_{\alpha, J} \frac{\partial P}{\partial u_J^\alpha}\, du_J^\alpha$$

on differential functions. It can be shown that $\hat{d}$ commutes with each total derivative D_i, and hence induces a well-defined map

$$\delta : \Lambda_*^k \longrightarrow \Lambda_*^{k+1}$$

on the spaces of functional forms, called the *variational differential*. An easy argument based on the finite-dimensional Poincaré lemma shows that the *variational complex*

$$0 \longrightarrow \Lambda_*^0 \xrightarrow{\delta} \Lambda_*^1 \xrightarrow{\delta} \Lambda_*^2 \xrightarrow{\delta} \Lambda_*^3 \xrightarrow{\delta} \ldots .$$

is locally exact, meaning that, over suitable (star-shaped) subdomains $\delta\omega = 0$ if and only if $\omega = \delta\zeta$ for some functional form ζ. In particular, if $\mathcal{L}[u] = \int L[u]\, dx$ is a functional, its variational differential is the one-form

$$\delta\mathcal{L} = \int \{E(L) \cdot du\}\, dx \tag{3.3}$$

determined by the Euler-Lagrange expression $E(L)$ or variational derivative of $\mathcal{L}$. The exactness of the varational complex at the $\bigwedge^1_*$ - stage leads to the well-known Helmholtz conditions for a differential equation to be the Euler-Lagrange equation for some variational problem, [11; Theorem 5.68], namely $\Delta = E(L)$ for some Lagrangian L if and only if its Fréchet derivative is self-adjoint: $\mathbf{D}_\Delta = \mathbf{D}^*_\Delta$.

Our main interest here is not in the functional forms, but rather in the dual objects - the functional multi-vectors. By definition, a *functional k-vector* is an alternating, k-linear map from the space $\bigwedge^1_*$ of functional one-forms to the space $\bigwedge^0_*$ of functionals. It can be shown that each functional k-vector can be written in the form

$$\Theta = \int \Big\{ \sum R^\alpha_J[u]\, \theta^{\alpha_1}_{J_1} \wedge \ldots \wedge \theta^{\alpha_k}_{J_k} \Big\}\, dx,$$

where the θ^α_J form a basis for the *vertical vectors*, dual to the basis du^α_J of vertical forms. We find

$$\langle \Theta;\, \omega_1 \wedge \ldots \wedge \omega_k \rangle = \int \Big\{ \sum R^\alpha_J[u] \cdot \det(D_{J_i} P^j_{\alpha_i}) \Big\}\, dx,$$

where $\omega_j = \int \{ \sum P^j_\alpha[u]\, du^\alpha \}\, dx$ are functional one-forms written in canonical form (3.1). The total derivatives act as Lie derivatives on vertical multi-vectors, and so the space $\bigwedge^*_k$ of functional k-vectors is again the cokernel of the total divergence operator. In particular, integation by parts can be used to place any functional uni-vector (i.e. $k = 1$) in the canonical form

$$v_Q = \int \Big\{ \sum_{\alpha=1}^{q} Q_\alpha[u]\, \theta^\alpha \Big\}\, dx = \int \{Q \cdot \theta\}\, dx, \tag{3.4}$$

and the space $\bigwedge^*_1$ can be identified with the space of generalized vector fields. Similarly, any functional bi-vector, i.e. element of $\bigwedge^*_2$, can be placed in canonical form

$$\Theta = \Theta_{\mathcal{D}} = \frac{1}{2}\int \{\theta \wedge \mathcal{D}\theta\}\, dx, \tag{3.5}$$

determined by a unique $q \times q$ skew-adjoint matrix differential operator $\mathcal{D}$.

Warning: The space $\bigwedge^*_k$ of functional multi-vectors is *not* the dual space to the space $\bigwedge^k_*$ of functional k-forms. This is because the wedge product of two functional forms is not a well-defined functional form!

If $\mathbf{v}_Q$ is a generalized vector field or uni-vector, we can define its prolongation to act as a Lie derivative on the space of functional multi-vectors. The key formula is

$$\operatorname{pr} \mathbf{v}_Q(\theta) = \mathbf{D}_Q^* \cdot \theta,$$

where $\mathbf{D}_Q^*$ is the adjoint of the Fréchet derivative of Q, and θ denotes the column vector of basis uni-vectors θ^α, $\alpha = 1,\dots,q$. (See [10], [12] for a justification of this formula.) The prolongation $\operatorname{pr} \mathbf{v}_Q$ acts on differential functions as before, and the action is extended to the entire space by the usual rules of derivation and commutation with the total derivatives. In particular, as the reader can check, this definition of the Lie derivative recovers the correct form of the Lie bracket (2.6) between generalized vector fields.

If Θ is a functional k-vector, and $\omega_I \equiv \omega_{i_1} \wedge \dots \wedge \omega_{i_m}$, $m \le k$, a wedge product of functional one-forms, we define the *interior product* to be the functional $(k - m)$-vector $\omega_I \lrcorner \Theta$ determined by the formula

$$\langle \omega_I \lrcorner \Theta; \eta_1 \wedge \dots \wedge \eta_{k-m} \rangle = \langle \Theta; \omega_I \wedge \eta_1 \wedge \dots \wedge \eta_{k-m} \rangle . \tag{3.6}$$

In particular, if $m = k - 1$, then $\omega_I \lrcorner \Theta \in \bigwedge^*_1$, and hence can be viewed as an generalized vector field, as in (3.4).

The most important operation on functional multi–vectors is the Schouten bracket, which generalizes the Lie bracket between vector fields. If Φ is a k–vector and Ψ an ℓ–vector, then their Schouten bracket $[\Phi, \Psi]$, is a $(k+\ell-1)$–vector. It is uniquely defined by the following formula:

$$\langle [\Phi,\Psi]; \delta\mathcal{L}_1 \wedge \ldots \wedge \delta\mathcal{L}_{k+\ell-1} \rangle =$$

$$\frac{(-1)^{k\cdot\ell+\ell}}{\ell} \sum_I (\text{sgn I}) \langle \Phi; \{\delta\mathcal{L}_I \lrcorner \Psi\} \delta\mathcal{L}_{I'} \rangle + \frac{(-1)^k}{k} \sum_J (\text{sgn J}) \langle \Psi; \{\delta\mathcal{L}_J \lrcorner \Phi\} \delta\mathcal{L}_{J'} \rangle. \tag{3.7}$$

which must hold for every set of functionals $\mathcal{L}_1, \ldots, \mathcal{L}_{k+\ell-1}$, with variational derivatives $\delta\mathcal{L}_i$ given by (3.3). In (3.7), the first sum is over all multi-indices $I = (i_1, \ldots, i_{\ell-1})$, $1 \le i_1 < \ldots < i_{\ell-1} \le k+\ell-1$, with $I' = (i'_1, \ldots, i'_k)$ being the complementary multi-index, so $I \cup I' = \pi(1, \ldots, k+\ell-1)$ for some permutation π, and sgn I denoting the sign of the permutation π. Similarly, the second sum is over all multi-indices $J = (j_1, \ldots, j_{k-1})$, $1 \le j_1 < \ldots < j_{k-1} \le k+\ell-1$, with J' and sgn J being defined analogously. Note also that, according to the remark in the previous paragraph, the terms $\delta\mathcal{L}_I \lrcorner \Psi$ and $\delta\mathcal{L}_J \lrcorner \Phi$ are in $\bigwedge^*_1$, and hence determine generalized vector fields, which act on the remaining wedge products $\delta\mathcal{L}_{I'}$ and $\delta\mathcal{L}_{J'}$ as Lie derivatives. (This definition, first proposed in [10], has the advantage of being the only one I know of which works equally well in both finite and infinite dimensions.)

The Schouten bracket satisfies the following properties. Let $\Phi \in \bigwedge^*_k$, $\Psi \in \bigwedge^*_\ell$, $\Theta \in \bigwedge^*_m$ be functional multi-vectors.

i) *Bilinearity*: $[\Phi, \Psi]$ is an $\mathbb{R}$–bilinear function of Φ and Ψ.

ii) *Super–symmetry*: $[\Phi, \Psi] = (-1)^{k\cdot\ell} [\Psi, \Phi]$.

iii) *Super–Jacobi Identity*:

$$(-1)^{k\cdot m} [[\Phi,\Psi],\Theta] + (-1)^{\ell\cdot m} [[\Theta, \Phi],\Psi] + (-1)^{k\cdot\ell} [[\Psi,\Theta], \Phi] = 0.$$

iv) *Lie Derivative*: If $\mathbf{v}_Q$ is a generalized vector field or functional uni-vector, then the Schouten bracket $[\mathbf{v}_Q, \Phi]$ is also a functional k–vector, and coincides with the Lie derivative of Φ with respect to pr $\mathbf{v}_Q$. In particular, the Schouten bracket of two generalized vector fields is the same as their Lie bracket (2.6).

Each functional bi–vector $\Theta_{\mathcal{E}}$ determines an alternating, bilinear map on the space of one–forms, and hence a bilinear, skew–symmetric "bracket" on the space of real–valued function(al)s:

$$\{\mathcal{F}, \mathcal{H}\} \equiv \langle \Theta_{\mathcal{D}}; \delta\mathcal{F}, \delta\mathcal{H} \rangle.$$

Explicitly, using (3.5), we see that this bracket is given by the standard formula

$$\{\mathcal{F}, \mathcal{H}\} = \int E(F) \cdot \mathcal{D} \cdot E(H)\ dx, \tag{3.8}$$

where F and H are the integrands of the functionals $\mathcal{F}$, $\mathcal{H}$. The bracket automatically satisfies the Leibniz rule, and hence to be a genuine Poisson bracket must only satisfy the additional restriction imposed by the Jacobi identity. This can be easily expressed in invariant form using the tri–vector $[\Theta_{\mathcal{D}}, \Theta_{\mathcal{D}}]$ obtained by bracketing $\Theta_{\mathcal{D}}$ with itself:

$$\{\{F, H\}, P\} + \{\{P, F\}, H\} + \{\{H, P\}, F\} = \frac{2}{3} \langle [\Theta_{\mathcal{D}}, \Theta_{\mathcal{D}}]; dF, dH, dP \rangle.$$

Therefore a functional bi-vector $\Theta_{\mathcal{D}}$ determines a Poisson bracket if and only if it satisfies the extra condition

$$[\Theta_{\mathcal{D}}, \Theta_{\mathcal{D}}] = 0. \tag{3.9}$$

This condition is a nonlinear condition on the underlying differential operator $\mathcal{D}$. Any functional bi-vector satisfying (3.9) is called a Hamiltonian bi-vector; similarly, any skew-adjoint differential operator coming from a Hamiltonian bi-vector is called a *Hamiltonian operator.*

Given a Hamiltonian bi-vector, let $\vartheta = \vartheta_{\Theta}$ be the map taking functional k–vectors to functional (k+1)–vectors defined by bracketing with the Poisson bivector Θ:

$$\vartheta(\Psi) = [\Theta, \Psi]. \tag{3.10}$$

The determining property (3.9) along with the super Jacobi identity for the Schouten bracket immediately implies that

$$\vartheta(\vartheta(\Psi)) = [\Theta, [\Theta, \Psi]] = 0$$

for any multi–vector Ψ. Therefore the maps ϑ determine a complex, called the *Poisson complex* corresponding to the Poisson bivector Θ:

$$0 \longrightarrow \bigwedge\nolimits_0^* \xrightarrow{\vartheta} \bigwedge\nolimits_1^* \xrightarrow{\vartheta} \bigwedge\nolimits_2^* \xrightarrow{\vartheta} \bigwedge\nolimits_3^* \xrightarrow{\vartheta} \cdots.$$

The composition of two successive maps is always trivial: $\vartheta \circ \vartheta = 0$.

The first stage of the Poisson complex, $\vartheta : \Lambda^*_0 \to \Lambda^*_1$, maps functionals to generalized vector fields. Specifically, if $\mathcal{H}[u] = \int H[u]\, dx$ is a (Hamiltonian) functional, the corresponding generalized vector field

$$\hat{\mathbf{v}}_{\mathcal{H}} = \vartheta(\mathcal{H}) = [\Theta, \mathcal{H}] \tag{3.11}$$

is called the *Hamiltonian vector field* determined by $\mathcal{H}$. Explicitly, it is readily seen that $\hat{\mathbf{v}}_{\mathcal{H}}$ has characteristic $Q = \mathcal{D}\cdot E(H)$, where $\mathcal{D}$ is the Hamiltonian operator determined by Θ. The corresponding Hamiltonian flow, cf. (2.3), is governed by the Hamiltonian system of evolution equations

$$u_t = \mathcal{D}\cdot E(H). \tag{3.12}$$

We note the standard formula

$$\{\mathcal{H}, \mathcal{F}\} = -\operatorname{pr} \hat{\mathbf{v}}_{\mathcal{H}}(\mathcal{F}) = \operatorname{pr} \hat{\mathbf{v}}_{\mathcal{F}}(\mathcal{H}) \tag{3.13}$$

for any pair of functionals $\mathcal{H}, \mathcal{F}$, which proves that a functional $\mathcal{F}$ determines a conserved density for the Hamiltonian system (3.12) if and only if $\{\mathcal{H}, \mathcal{F}\} = 0$. Therefore, every conserved density $\mathcal{F}$ determines a generalized (Hamiltonian) symmetry $\hat{\mathbf{v}}_{\mathcal{F}}$ of the Hamiltonian system (3.12).

Conversely, if an generalized vector field $\mathbf{v}_Q$ is a Hamiltonian vector field, so $Q = \mathcal{D}\cdot E(H)$ for some differential function H, then closure of the Poisson complex at the Λ^*_1–stage implies that

$$\vartheta(\mathbf{v}_Q) = [\Theta_{\mathcal{D}}, \mathbf{v}_Q] = \operatorname{pr} \mathbf{v}_Q(\Theta_{\mathcal{D}}) = 0. \tag{3.14}$$

If the Poisson complex is *exact* at the Λ^*_1–stage, then (3.14) is both necessary and sufficient for $\mathbf{v}_Q$ to be a Hamiltonian vector field. In this case, (3.14) provides a simple and readily verifiable condition that will tell whether or not a given vector field is Hamiltonian with respect to the given Poisson bracket. Writing out (3.14) explicitly leads to the following characterization of Hamiltonian vector fields, [12]; see also [5] for a similar result.

Proposition 6. Let $\Theta_{\mathcal{H}}$ be a Hamiltonian bivector, with $\mathcal{H}$ the corresponding Hamiltonian differential operator. If the evolutionary vector field $\mathbf{v}_Q$ is Hamiltonian, meaning that $Q = \mathcal{H}\cdot E(H)$ for some differential function H, then

$$D_Q\cdot\mathcal{H} + \mathcal{H}\cdot D_Q^* = \mathbf{v}_Q(\mathcal{H}). \tag{3.15}$$

Conversely, if the Poisson complex corresponding to $\Theta_{\mathcal{H}}$ is exact at the $\wedge_1^*$–stage, then (3.15) is both necessary and sufficient for Q to be of the Hamiltonian form (3.12).

Explicit conditions on the Hamiltonian bi-vector that imply exactness of the corresponding Poisson complex are not known in general; this is one of the main open problems in the subject. However, in the case of constant coefficient skew-adjoint differential operators (which are always automatically Hamiltonian), one can prove the following theorem on exactness of the Poisson complex at the initial stage:

Theorem 7. Let $\mathcal{H}$ be a nondegenerate, skew–adjoint, constant coefficient $q\times q$ matrix differential operator. Then, except for a finite dimensional space of linear differential functions, the Poisson complex for $\Theta_{\mathcal{H}}$ is exact at the $\wedge_1^*$–stage. More specifically, there exist linear differential functions $Q_1[u],\dots,Q_n[u]$ such that an generalized vector field $\mathbf{v}_Q$ satisfies the condition

$$D_Q\cdot D_x + D_x\cdot D_Q^* = 0,$$

if and only if

$$Q = \mathcal{H}\cdot E(H) + c_1Q_1 + \dots + c_nQ_n$$

for some Hamiltonian $H[u]$, and some constants $c_1, \dots, c_n$. (In fact, even the linear functions Q_j are Hamiltonian provided one admits nonlocal Hamiltonian functionals depending on the potential w.)

Example 8. For the Hamiltonian operator D_x, the Poisson complex is exact at the $\wedge_1^*$–stage. This is equivalent to the statement that $Q[u] = D_xE(H)$ for some differential function H if and only if $D_Q\cdot D_x + D_x\cdot D_Q^* = 0$. Exactness fails for the third order operator D_x^3, but there are just two linear counterexamples: $Q_1 = u_x$, and $Q_2 = xu_x + u$. Thus we have the analogous result that $D_Q\cdot D_x^3 + D_x^3\cdot D_Q^* = 0$ if and only if $Q[u] = D_x^3E(H) + c_1Q_1 + c_2Q_2$, for some constants c_1, c_2.

A similar exactness result should hold for the later $\bigwedge^*_k$–stages, $k > 1$, of the Poisson complex, but I have not tried to construct a proof. The only field dependent Hamiltonian operator for which exactness at the $\bigwedge^*_1$–stage is known is the second Korteweg–deVries Hamiltonian operator $\mathcal{D} = D_x^3 + 2uD_x + u_x$, cf. [12], although the Darboux Theorems of [2], [13] greatly extend the range of application of the constant coefficient results. I believe a detainled investigation into the Poisson complex corresponding to a Hamiltonian operator will lead to significant results in infinite-dimensional geometry and differential algebra.

4. Bi-Hamiltonian Systems.

The most productive way to derive recursion operators is through the theory of biHamiltonian systems, first proposed by Magri, [7].

Definition. Two functional bi-vectors $\Theta_{\mathcal{D}}$ and $\Theta_{\mathcal{E}}$ are said to form a *Hamiltonian pair* if every linear combination $c\cdot\Theta_{\mathcal{D}} + d\cdot\Theta_{\mathcal{E}}$, $c, d \in \mathbb{R}$, is a Hamiltonian bi-vector.

Since the condition (3.9) is quadratic, to check whether $\Theta_{\mathcal{D}}$ and $\Theta_{\mathcal{E}}$ form a Hamiltonian pair it suffices to prove that the three functional bi-vectors $\Theta_{\mathcal{D}}$, $\Theta_{\mathcal{E}}$, and $\Theta_{\mathcal{D}} + \Theta_{\mathcal{E}}$ are Hamiltonian. Equivalently, we need check

$$[\Theta_{\mathcal{D}}, \Theta_{\mathcal{D}}] = [\Theta_{\mathcal{E}}, \Theta_{\mathcal{E}}] = 0$$

and the additional compatibility condition

$$[\Theta_{\mathcal{D}}, \Theta_{\mathcal{E}}] = 0. \tag{4.1}$$

A system of differential equations is said to be *biHamiltonian* with respect to a Hamiltonian pair $\Theta_{\mathcal{D}}$ and $\Theta_{\mathcal{E}}$ if it can be written in the two Hamiltonian forms

$$u_t = \mathcal{D}\cdot\delta\mathcal{H}_1 = \mathcal{E}\cdot\delta\mathcal{H}_0.$$

for some Hamiltonian functionals $\mathcal{H}_0$ and $\mathcal{H}_1$. We now state a version of Magri's theorem on the "complete integrability" of biHamiltonian systems, and give an elementary

proof based on the exactness of the Poisson complex.

Theorem 9. Let $\Theta_{\mathcal{D}}$ and $\Theta_{\mathcal{E}}$ form a Hamiltonian pair, and assume that the Poisson complex for the bi-vector $\Theta_{\mathcal{D}}$ is exact at the $\bigwedge^*_1$–stage. Let

$$u_t = Q_1[u] = \mathcal{D}\cdot\delta\mathcal{H}_1 = \mathcal{E}\cdot\delta\mathcal{H}_0 \tag{4.2}$$

be an associated biHamiltonian system. Then the operator $\mathcal{R} = \mathcal{E}\cdot\mathcal{D}^{-1}$ is a recursion operator, and leads to a hierarchy of differential functions

$$Q_{n+1}[u] = \mathcal{R}\cdot Q_n[u].$$

Each of the corresponding evolutionary vector fields $\mathbf{v}_n \equiv \mathbf{v}_{Q_n}$ is also biHamiltonian

$$u_t = Q_n[u] = \mathcal{D}\cdot\delta\mathcal{H}_n = \mathcal{E}\cdot\delta\mathcal{H}_{n-1}. \tag{4.3}$$

The functionals $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \ldots$ are in involution with respect to either Poisson bracket:

$$\{\mathcal{H}_n, \mathcal{H}_m\}_{\mathcal{D}} = 0 = \{\mathcal{H}_n, \mathcal{H}_m\}_{\mathcal{E}}.$$

Thus, for any m, n, $\mathcal{H}_m$ is a conservation law for the evolutionary system governed by the vector field $\mathbf{v}_n$, and $\mathbf{v}_m$ is the corresponding generalized symmetry.

Note that the theorem automatically implies the invertibility of the Hamiltonian operator $\mathcal{D}$ on the hierarchy $Q_n[u]$, which shows the advantage of the Poisson complex approach.

Proof.

The proof that $\mathcal{R}$ is recursion operator in general can be found in [11; Theorem 7.27]; interestingly, this does not require the compatibility of the two bivectors. Here we just prove the properties about the hierarchy Q_n. Proceeding by induction on n, according to (3.11), the condition that the evolution equation (4.3) be biHamiltonian is equivalent to the fact that the corresponding generalized vector field $\mathbf{v}_n \equiv \mathbf{v}_{Q_n}$ can be written in the two forms

$$\mathbf{v}_n = [\Theta_{\mathcal{D}}, \mathcal{H}_n] = [\Theta_{\mathcal{E}}, \mathcal{H}_{n-1}].$$

Let

$$\mathbf{v}_{n+1} = [\Theta_{\mathcal{E}}, \mathcal{H}_n]$$

be the next evolutionary vector field in the presumed hierarchy. The main task is to prove that $\mathbf{v}_{n+1}$ is a Hamiltonian vector field for the operator $\mathcal{D}$, i.e.

$$\mathbf{v}_{n+1} = [\Theta_{\mathcal{D}}, \mathcal{H}_{n+1}]$$

for some functional $\mathcal{H}_{n+1}$. By exactness of the Poisson complex for $\Theta_{\mathcal{D}}$, we need only verify that $\mathbf{v}_{n+1}$ is closed, i.e.

$$[\Theta_{\mathcal{D}}, \mathbf{v}_{n+1}] = 0.$$

To verify this latter condition, we use the super–Jacobi identity and the compatibility condition (4.1) for the Hamiltonian pair:

$$[\Theta_{\mathcal{D}}, \mathbf{v}_{n+1}] = [\Theta_{\mathcal{D}}, [\Theta_{\mathcal{E}}, \mathcal{H}_n]] = -[\Theta_{\mathcal{E}}, [\Theta_{\mathcal{D}}, \mathcal{H}_n]] = -[\Theta_{\mathcal{E}}, [\Theta_{\mathcal{E}}, \mathcal{H}_{n-1}]] = 0,$$

the last equality being a consequence of the closure of the $\mathcal{E}$–Poisson complex. (Note that we do not require exactness of the $\mathcal{E}$–Poisson complex.)

To prove the involutiveness of the resulting sequence of functionals note that according to (3.12),

$$\{\mathcal{H}_n, \mathcal{H}_m\}_{\mathcal{D}} = \operatorname{pr} \mathbf{v}_n(\mathcal{H}_m), \qquad \{\mathcal{H}_n, \mathcal{H}_m\}_{\mathcal{E}} = \operatorname{pr} \mathbf{v}_{n-1}(\mathcal{H}_m),$$

hence

$$\{\mathcal{H}_n, \mathcal{H}_m\}_{\mathcal{D}} = \{\mathcal{H}_n, \mathcal{H}_{m-1}\}_{\mathcal{E}}.$$

We now employ the skew-symmetry of the Poisson bracket to work our way down:

$$\{\mathcal{H}_n, \mathcal{H}_m\}_{\mathcal{D}} = \{\mathcal{H}_n, \mathcal{H}_{m-1}\}_{\mathcal{E}} = \{\mathcal{H}_{n+1}, \mathcal{H}_{m-1}\}_{\mathcal{D}} = \cdots = \{\mathcal{H}_k, \mathcal{H}_k\} = 0,$$

where k is the integer part of $\frac{m-n}{2}$ and the final Poisson bracket is the $\mathcal{D}$-Poisson bracket if $m-n$ is even, the $\mathcal{E}$-Poisson bracket if $m-n$ is odd. This completes the proof.

We also note that, for a biHamiltonian system, there is a hierarchy of higher order

Hamiltonian operators, cf. [5]. The only problem with this result is that these operators are usually, but not always, integro-differential operators.

Proposition 10. If $\mathcal{D}$, $\mathcal{E}$ are the Hamiltonian operators for a Hamiltonian pair, and $\mathcal{R} = \mathcal{E}\cdot\mathcal{D}^{-1}$ the associated recursion operator, then the operators

$$\mathcal{R}^k\cdot\mathcal{E} = \mathcal{E}\cdot\mathcal{D}^{-1}\cdot\mathcal{E}\cdot\mathcal{D}^{-1}\cdot \dots \cdot\mathcal{D}^{-1}\cdot\mathcal{E} \tag{4.4}$$

are Hamiltonian operators.

Example 11. The Riemann equation is a "quadri-Hamiltonian system", meaning that it can be written in Hamiltonian form in four distinct ways; however not all pairs of Hamiltonian operators are compatible. The three first order Hamiltonian operators

$$\mathcal{D}_0 = D_x, \qquad \mathcal{D}_1 = 2u\cdot D_x + u_x, \qquad \mathcal{D}_2 = u^2\cdot D_x + u\, u_x, \tag{4.5}$$

are all compatible, i.e. any two of them form a Hamiltonian pair. We find that the Riemann equation (2.8) can be written in the three Hamiltonian forms

$$u_t = \mathcal{D}_0\cdot E(\tfrac{1}{6}H_3) = \mathcal{D}_1\cdot E(\tfrac{1}{6}H_2) = \mathcal{D}_2\cdot E(H_1),$$

using the hierarchy of zeroth order conserved densities (2.13). Moreover, these Hamiltonian operators are not trivially related by (4.4). The resulting recursion operators $\mathcal{R}_1 = \mathcal{D}_1\cdot\mathcal{D}_0^{-1}$, $\mathcal{R}_2 = \mathcal{D}_2\cdot\mathcal{D}_0^{-1}$ are as given in (2.11), while $\mathcal{R}_3 = \mathcal{D}_2\cdot\mathcal{D}_1^{-1}$ is trivially related by the equation $\mathcal{R}_2 = \mathcal{R}_3\cdot\mathcal{R}_1$. The infinite hierarchy of commuting Hamiltonian flows generated by Theorem 9 consists of the generalized vector fields $\mathbf{v}_n = Q_n[u]\cdot\partial_u$, with characteristics $Q_n[u] = u^n\cdot u_x$. The nth flow is also tri-Hamiltonian:

$$\begin{aligned} u_t &= Q_n[u] = u^n\cdot u_x \\ &= \mathcal{D}_0\cdot E\Big(\frac{1}{(n+2)(n+1)}H_{n+2}\Big) = \mathcal{D}_1\cdot E\Big(\frac{1}{(2n+1)(n+1)}H_{n+1}\Big) = \mathcal{D}_2\cdot E\Big(\frac{1}{n^2}H_n\Big). \end{aligned}$$

Each of these Hamiltonian systems admits an additional third order Hamiltonian operator

$$\mathcal{E} = D_x\cdot\frac{1}{u_x}\cdot D_x\cdot\frac{1}{u_x}\cdot D_x, \tag{4.6}$$

and can be written in yet another Hamiltonian form

$$u_t = u^n u_x = \mathfrak{E}\cdot E\Big(\frac{1}{(n+1)(n+2)(n+3)(n+4)} H_{n+4} \Big). \tag{4.7}$$

The Hamiltonian operators $\mathfrak{D}_0$ and $\mathfrak{E}$ are compatible; however, $\mathfrak{E}$ is not compatible with either $\mathfrak{D}_1$ or $\mathfrak{D}_2$. The consequent recursion operator $\hat{\mathfrak{R}} = \mathfrak{E}\cdot\mathfrak{D}_0^{-1} = \mathfrak{R}^2$ is the square of the simpler first order recursion operator (2.12). Note also that by proposition 10, there is an entire hierarchy of Hamiltonian differential operators $\left(D_x \cdot \frac{1}{u_x} \right)^{2k} \cdot D_x$ and each of the equations (4.7) can be written in Hamiltonian form using any one of these higher order Hamiltonian differential operators!

The third order generalized symmetry $\hat{\mathbf{v}}_3$, with characteristic $\hat{Q}_3 = u_x\cdot I_4$, cf. (2.10) is bi-Hamiltonian

$$u_t = \mathfrak{D}_0\cdot E(\hat{H}_1) = \mathfrak{E}\cdot E(\hat{H}_0),$$

using the conserved densities $\hat{H}_0 = u_x\cdot I_1$, $\hat{H}_1 = u_x\cdot I_3$. Consequently, the odd order vector fields $\hat{\mathbf{v}}_{2n+1}$ are also bi-Hamiltonian, corresponding to the higher order rational conserved densities $\hat{H}_n = u_x\cdot I_{n+2}$. For any solution to the general first order flow (4.7) corresponding to the vector field $\mathbf{v}_n$, each of the higher order quantities $\int \hat{H}_m[u]\,dx$ is a linear function of t, (provided that the integral converges). In fact, if H(u) is any zeroth order Hamiltonian, with flow $u_t = D_x E(H)$, then $\hat{H}_m + t\,\frac{\partial^{2m+1} H}{\partial u^{2m+1}}$ is a conserved density. In particular, $\hat{H}_m$ is a conserved density for $\mathbf{v}_n$ whenever $2m \geq n - 1$.

5. Gas Dynamics

In the final section, I will describe some recent joint work with Y. Nutku on Hamiltonian structure, symmetries and conservation laws for quasi-linear hyperbolic systems, including the classical equations of gas dynamics, [14]. The general form of a two-component hyperbolic system of conservation laws of Hamiltonian type is

$$u_t = D_x\cdot H_v, \qquad v_t = D_x\cdot H_u,$$

where the Hamiltonian density H(u,v) depends only on u, v. The Hamiltonian operator is the matrix differential operator $\mathfrak{D}_0 = \sigma_1\cdot D_x$, where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The system can be written in the convenient matrix form

$$\mathbf{u}_t = \mathfrak{S}_0 E[H] = \mathbf{H}\cdot\mathbf{u}_x, \tag{5.1}$$

where

$$\mathbf{H} = \sigma_1 \cdot D^2 H = \begin{pmatrix} H_{uv} & H_{vv} \\ H_{uu} & H_{uv} \end{pmatrix}.$$

In the terminology of Dubrovin and Novikov, [4], [15] these systems are of "hydrodynamic type".

There are three important examples. The equations of gas dynamics are of this form, where u represents the velocity, and v the density of the fluid, and the Hamiltonian is $H(u,v) = -\left(\frac{1}{2}u^2 v + F(v)\right)$; the function F is related to the physical pressure P according to the equation $P'(v) = v\cdot F''(v)$, cf. [19; Chapter 6]. The equations of polytropic gas dynamics correspond to the choice $F(v) = \dfrac{v^\gamma}{\gamma(\gamma-1)}$ for some $\gamma \neq 0,1$; the case $\gamma = 2$ also arises in shallow water theory, [8], [19; p.84]. A second important example is provided by the Hamiltonian $H = \frac{u}{v} + \frac{v}{u}$, in which case (1.2) is equivalent to the Born-Infeld equation from nonlinear electrodynamics, [1], [19; p. 579]. Finally, the Hamiltonian density $H(u,v) = \frac{1}{2}u^2 + F(v)$ gives many simple models for a one-dimensional nonlinear elastic media; the derivative F' being a monotone function of v corresponds to an ideal fluid or elastic solid; nonmonotone functions provide simple models of phase transitions, [16], [19; p. 123]. The case $F(v) = (1+v)^{-\gamma}$ corresponds to the Euler equation arising in nonlinear acoustics, cf. [8].

We begin by characterizing all the zeroth order conserved densities for the Hamiltonian system (5.1), cf. [15].

Proposition 12. A functional $\mathcal{F}[u] = \int F(u,v)\,dx$ is a conservation law for the hyperbolic system (5.1) if and only if F is a solution to the second order linear partial differential equation

$$A(u,v)\,F_{uu} = B(u,v)\,F_{vv}, \tag{5.2}$$

with $A = H_{vv}$, $B = H_{uu}$.

The most important class of Hamiltonian systems (5.1) are the *separable* systems, for which the corresponding partial differential equation (5.2) admits a separation of variables in the rectangular (u,v)-coordinates, meaning that $\frac{H_{vv}}{H_{uu}} = \frac{\mu(v)}{\lambda(u)}$. The special case when $\lambda \equiv 1$, which includes gas dynamics and the elastic models, has added importance; such systems are said to be of *generalized gas dynamics type*. For simplicity, we will restrict our attention to generalized gas dynamics systems here, although extensions to more general separable systems can be found in [14].

For a gas dynamics system, there are two fundamental hierarchies of conserved densities, each of the form

$$H_{2m+\varepsilon}(u,v) = \sum_{k=0}^{m+\varepsilon} \frac{u^{2k+\varepsilon}}{(2k+\varepsilon)!} \cdot G_k(v) ,$$

where ε is 0 or 1, depending on whether $n = 2m + \varepsilon$ is even or odd, and the functions G_k are generated recursively by

$$G_{k+1}(v) = \int_0^v (v - w)\cdot\mu(w)\cdot G_k(w)\, dw,$$

where $\mu(v) = H_{vv}/H_{uu}$. The two hierarchies depend on the initial selection of G_0; the first takes $G_0 = 1$, while the second has $G_0 = v$. Thus, we have the explicit conserved densities

$$\begin{aligned}
&H_0 = 1, && \tilde{H}_0 = v,\\
&H_1 = u, && \tilde{H}_1 = uv,\\
&H_2 = \tfrac{1}{2}u^2 + G_1(v), && \tilde{H}_2 = \tfrac{1}{2}u^2 v + \tilde{G}_1(v),\\
&H_3 = \tfrac{1}{6}u^3 + u\, G_1(v), && \tilde{H}_3 = \tfrac{1}{6}u^3 v + u\, \tilde{G}_1(v),\\
&H_4 = \tfrac{1}{24}u^4 + \tfrac{1}{2}u^2 G_1(v) + G_2(v), && \tilde{H}_4 = \tfrac{1}{24}u^4 v + u^2 \tilde{G}_1(v) + \tilde{G}_2(v),
\end{aligned} \qquad (5.3)$$

etc. Note that the elastic Hamiltonian appears in the first hierarchy as H_2, whereas the gas

dynamics Hamiltonian appears in the alternative hierarchy as $-\tilde{H}_2$.

Each of these Hamiltonian functions generates a Hamiltonian flow, governed by the corresponding evolutionary system. We let

$$\mathbf{Q}_n = \mathcal{D}_0 E[H_n] = \mathbf{H}_n \cdot \mathbf{u}_x, \tag{5.4}$$

cf. (5.1), denote the right hand side of this equation, which is also the characteristic for the symmetry vector field $\mathbf{v}_n = \mathbf{Q}_n \cdot \partial_{\mathbf{u}}$. We define $\tilde{\mathbf{H}}_n$, $\tilde{\mathbf{Q}}_n$ and $\tilde{\mathbf{v}}_n$ for the alternative hierarchy $\tilde{H}_n$ similarly. All the Hamiltonians H_n and $\tilde{H}_n$ are in involution with respect to the Poisson bracket determined by the Hamiltonian operator $\mathcal{D}_0$.

In the case of polytropic gas dynamics, there are two additional first order Hamiltonian structures. Using the Hamiltonian hierarchies, we find that we can write the polytropic gas dynamics equations in the alternative Hamiltonian forms

$$u_t = \mathcal{D}_1 \cdot E\left(\frac{1}{\gamma}\tilde{H}_1\right) = \mathcal{D}_2 \cdot E(\tilde{H}_0),$$

with the two Hamiltonian operators, [8],

$$\mathcal{D}_1 = \begin{pmatrix} v^{\gamma-2}\cdot D_x + D_x \cdot v^{\gamma-2} & (\gamma-1)u\cdot D_x + u_x \\ (\gamma-1)u\cdot D_x + (\gamma-2)u_x & v\cdot D_x + D_x \cdot v \end{pmatrix},$$

$$\mathcal{D}_2 = \begin{pmatrix} uv^{\gamma-2}\cdot D_x + D_x \cdot uv^{\gamma-2} & \left\{\frac{1}{2}(\gamma-1)u^2 + 2\frac{v^{\gamma-1}}{\gamma-1}\right\}\cdot D_x + uu_x + v^{\gamma-2}v_x \\ \left\{\frac{1}{2}(\gamma-1)u^2 + 2\frac{v^{\gamma-1}}{\gamma-1}\right\}\cdot D_x + (\gamma-2)uu_x + v^{\gamma-2}v_x & uv\cdot D_x + D_x \cdot uv \end{pmatrix}.$$

The Hamiltonian operators $\mathcal{D}_0$, $\mathcal{D}_1$, $\mathcal{D}_2$ are mutually compatible, leading to three distinct Hamiltonian pairs. The corresponding recursion operators

$$\mathcal{R}_1 = \mathcal{D}_1 \cdot \mathcal{D}_0^{-1}, \qquad \mathcal{R}_2 = \mathcal{D}_2 \cdot \mathcal{D}_0^{-1}, \qquad \mathcal{R}_3 = \mathcal{D}_2 \cdot \mathcal{D}_1^{-1},$$

are trivially related by the identity $\mathcal{R}_2 = \mathcal{R}_3 \cdot \mathcal{R}_1$, but are otherwise distinct. Nevertheless, they both give rise to the same series of gas dynamics Hamiltonians, since

$$\mathfrak{R}_1(Q_n) = Q_{n+1}, \qquad \mathfrak{R}_2(Q_n) = Q_{n+2},$$

and similarly for the alternative hierarchy $\tilde{\mathbf{v}}_n$. Strangely, there does not appear to be a counterpart of these two recursion operators in the general non-polytropic case, i.e. when the pressure is not proportional to a power of the density.

There is, in addition, a third order Hamiltonian operator, analogous to (4.6), for any generalized gas dynamics system. Let $M(v) = \int_0^v \mu(s)\, ds$, and define the matrix variable

$$U(u,v) = \begin{pmatrix} u & M(v) \\ v & u \end{pmatrix}.$$

We use the notation

$$U_x = \begin{pmatrix} u_x & \mu(v)v_x \\ v_x & u_x \end{pmatrix} \qquad \text{and} \qquad U_x^{-1} = \frac{1}{\delta}\begin{pmatrix} u_x & -\mu(v)v_x \\ -v_x & u_x \end{pmatrix},$$

where $\delta = u_x^2 - \mu(v)\cdot v_x^2$, for the total x derivative of the matrix U and its matrix inverse. A nontrivial calculation proves that the operator

$$\mathfrak{E} = D_x\cdot U_x^{-1}\cdot D_x\cdot U_x^{-1}\cdot \sigma_1\cdot D_x = D_x\cdot U_x^{-1}\cdot D_x\cdot \sigma_1\cdot U_x^{-T}\cdot D_x$$

is Hamiltonian. The operators $\mathfrak{E}$ and $\mathfrak{D}_0$ form a Hamiltonian pair; however, for polytropic gas dynamics, the Hamiltonian operators $\mathfrak{E}$ and $\mathfrak{D}_1$ are *not* compatible, nor are the Hamiltonian operators $\mathfrak{E}$ and $\mathfrak{D}_2$.

Theorem 13. Let $H(u,v)$ be any generalized gas dynamics Hamiltonian density. Then there exists a second zeroth order conserved density H^* such that the corresponding Hamiltonian system (5.1) can be written in biHamiltonian form

$$\mathbf{u}_t = \mathfrak{D}_0 E[H] = \mathfrak{E}\, E[H^*].$$

If the Hamiltonian density H in Theorem 13 is one of the densities H_n in the hierarchy (5.3), then it is not hard to see that the corresponding density $H^*(u,v)$ can be taken to be the density H_{n+2}; similarly, if $H = \tilde{H}_n$, then $H^* = \tilde{H}_{n+2}$.

According to Theorem 9, the operator $\hat{\mathcal{R}} = \mathcal{E}\cdot\mathcal{D}_0^{-1} = D_x\cdot U_x^{-1}\cdot D_x\cdot U_x^{-1}$ is a recursion operator which, just as in the one-dimensional case, is the square of a simpler recursion operator $\mathcal{R} = D_x\cdot U_x^{-1}$. On the zeroth order symmetries, $\mathcal{R}(\mathbf{Q}_n) = \mathbf{Q}_{n-1}$, $\mathcal{R}(\tilde{\mathbf{Q}}_n) = \tilde{\mathbf{Q}}_{n-1}$. In the polytropic case, $\mathcal{R}$ is the "inverse" to the recursion operator $\mathcal{R}_1$ on the hierarchies (5.3), although as always, this is special to these particular hierarchies. There is also an additional hierarchy of higher order symmetries.

Theorem 14. Let $H = H_n$ be one of the n^{th} order generalized gas dynamics Hamiltonians, and let $\mathbf{v}_n$ be the corresponding first order Hamiltonian flow. Let $\hat{\mathbf{v}}_m$ denote the generalized vector field of order m with characteristic

$$\hat{\mathbf{Q}}_m = \mathcal{R}^m(x\mathbf{u}_x) = \mathcal{R}^{m-1}\cdot\begin{pmatrix}1\\0\end{pmatrix}.$$

Then $\hat{\mathbf{v}}_m$ is a symmetry for the flow generated by $\mathbf{v}_n$ provided $m \geq n-1$. Similarly, $\hat{\mathbf{v}}_m$ is a symmetry for the flow generated by $\tilde{\mathbf{v}}_n$ corresponding to the Hamiltonian $H = \tilde{H}_n$ provided $m \geq n$.

In polytropic gas dynamics, we can construct additional recursion operators by combining the Hamiltonian operator $\mathcal{E}$ with the operators $\mathcal{D}_1$, $\mathcal{D}_2$, even though they are not compatible. However, the resulting higher order symmetries appear to always be non-local since we cannot explicitly invert $\mathcal{D}_1$ or $\mathcal{D}_2$.

Finally, we indicate how to construct higher order conservation laws for any generalized gas dynamics system. First note that, if (5.1) is of gas dynamics type, it is equivalent to the matrix evolution equation $U_t = \mathbf{H}\cdot U_x$. This, and the fact that the matrices $\mathbf{H}$ and U_x commute, immediately leads to the important matrix identity

$$D_t\cdot(U_x^{-1}) - D_x(\mathbf{H}\cdot U_x^{-1}) = -(\mathbf{H}_x\cdot U_x^{-1} + U_x^{-1}\cdot\mathbf{H}_x), \tag{5.5}$$

which holds on solutions to the system (5.1). In particular, the (2,1)-entry of (5.5) reads

$$D_t\left(\frac{v_x}{\delta}\right) + D_x\left(\frac{H_{uu}\cdot v_x - H_{uv}\cdot u_x}{\delta}\right) = -2\,H_{uuu}\cdot$$

For classical gas dynamics, $H_{uuu} \equiv 0$, and we recover the conserved density

$$\hat{H}_1[u,v] = \frac{v_x}{\delta} = \frac{v_x}{u_x^2 - \mu(v)\cdot v_x^2} . \tag{5.6}$$

due to Verosky, [17]. For more general gas dynamics Hamiltonians, H_{uuu} will no longer be 0, and $\delta^{-1}\cdot v_x$ will no longer be a conserved density; however, we can simply modify it to get a time-dependent conservation law with density $\hat{H}_1^* = \delta^{-1}\cdot v_x + 2t\, H_{uuu}$. Equivalently, the integral $\hat{\mathcal{H}}_1 = \int \delta^{-1}\cdot v_x \, dx$, when it converges, is a linear function of t.

The first order conserved density $\hat{H}_1 = \delta^{-1}\cdot v_x$ leads to a Hamiltonian flow using the basic Hamiltonian operator $\mathcal{D}_0$. This will allow us to apply Theorem 9 to the Hamiltonian pair $\mathcal{E}$ and $\mathcal{D}_0$, and thereby generate a new hierarchy of higher order conservation laws in gas dynamics. We find that, as with the Riemann equation, the symmetry $\hat{v}_3$ is Hamiltonian with respect to the Hamiltonian pair $\mathcal{D}_0$, $\mathcal{E}$ and the corresponding conserved density is -2 times Verosky's density (5.6). Therefore, there is a hierarchy of m^{th} order Hamiltonian densities $\hat{H}_m$, $m = 1,2,\ldots$, and corresponding commuting biHamiltonian systems

$$u_t = \hat{Q}_{2m+1} = \mathcal{D}_0 E[\hat{H}_m] = \mathcal{E}\, E[\hat{H}_{m-1}], \qquad m \geq 1.$$

These Hamiltonians are in involution with respect to both the $\mathcal{D}_0$ and $\mathcal{E}$ Poisson brackets. More generally, if $H = H_n$ is a Hamiltonian density in the first generalized gas dynamics hierarchy, then the higher order density $\hat{H}_m$ is conserved for the Hamiltonian system (5.4) provided $n \leq 2m+1$. If $H = \tilde{H}_n$ is in the second generalized gas dynamics hierarchy, then $\hat{H}_m$ is conserved provided $n \leq 2m+2$.

Extensions to higher dimensional hyperbolic systems, cf. [18], and applications to discontinuous solutions and shock waves, as in [3], are under investigation.

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