Convergence of Normal Form Power Series for Infinite-Dimensional Lie Pseudo-Group Actions

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Abstract

We prove the convergence of normal form series for suitably nonsingular analytic submanifolds under a broad class of infinite-dimensional Lie pseudo-group actions. Our theorem is illustrated by a number of examples, and includes, as a particular case, Chern and Moser's celebrated convergence theorem for normal forms of real hypersurfaces with trivial isotropy. The construction of normal forms relies on the equivariant moving frame method, while the convergence proof is based on the realization that the normal form can be recovered as part of the solution to an initial value problem for an involutive system of differential equations, whose analyticity is guaranteed by the Cartan–Kähler Theorem.

Keywords: Involutive system of differential equations, Lie pseudo-group actions, moving frames, normal form power series.

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1 Introduction

In general, a *normal form*, also known as a *canonical form*, is defined as a simple representative element chosen from an equivalence class of objects. The identification of a normal form serves to simplify the treatment of such objects, and also solves the equivalence problem; namely, two objects are equivalent if and only if they have the same normal form. A simple algebraic example is the Jordan canonical form, which represents the similarity class of a square matrix, [58]. In dynamical systems, [45, 46], normal forms are extensively used to study bifurcations, classify singular points, and determine the behavior of solutions.

In the present paper, we focus on the problem of determining normal forms of analytic p-dimensional submanifolds under the action of a Lie pseudo-group, which includes the case of Lie group actions. Such problems arise in a wide range of applications, including classical invariant theory, [49], ordinary differential equations, [17,29,62], partial differential equations, [7, 44], differential operators, [31], the calculus of variations, [30, 32, 34], control theory, [22], nonholonomic geometry, [15], image processing, [4, 8, 26], and many more. Normal forms can be algorithmically constructed using the method of equivariant moving frames, [18, 41, 52, 54], which produces formal power series whose non-constant Taylor coefficients provide a complete set of independent differential invariants of the pseudo-group action on submanifolds. Our main result is a theorem guaranteeing the convergence of such formal power series under rather general conditions on the Lie pseudo-group action in the infinite-dimensional case —

since convergence in the case of finite-dimensional Lie group actions is immediate — and on the cross-section used for the moving frame construction.

Our results were inspired by the seminal paper of Chern and Moser, [12], that constructed normal form power series for nonsingular analytic real hypersurfaces in complex manifolds, and then proved their convergence. This particular equivalence problem was first studied by Poincaré in [59], who gave two heuristic counting arguments that indicated that two real hypersurfaces in $\mathbb{C}^2$ are, in general, biholomorphically inequivalent, and raised the question of finding the invariants that distinguish real hypersurfaces. This question was then solved in the two-dimensional case by Cartan, [9], and, subsequently, in higher dimensions by Chern and Moser, [12]. Their analysis was based on an ingenious combination of Cartan's equivalence method and an innovative convergence proof, based on their method of chains, which relies on the observation that the successive transformations mapping a regular hypersurface to its normal form can be characterized as solutions to ordinary differential equations, and are therefore analytic. On the other hand, Kolář, [35], produced examples of singular hypersurfaces whose normal form power series are divergent, thus indicating the subtlety of the convergence issue. The normal form approach promoted by Chern and Moser has inspired many developments in CR geometry, [3, 36, 39, 66], and has also been applied to differential equations, [20, 38, 43, 61], control systems, [60], and the geometry of submanifolds, [5, 13, 24, 25, 52]. In the authors' previous paper [57], Chern and Moser's analysis was extended to construct normal form power series for singular hypersurfaces by applying the equivariant method of moving frames for Lie pseudo-groups, [54]. However, the convergence of the resulting power series continued to rely on the Chern–Moser chain-based arguments that only apply to a limited range of problems; see, for instance, [16,20,36–39,42]. The present paper grew out of our inability at the time to provide an independent proof of convergence.

In this paper we establish a new proof of convergence, that applies in great generality, and includes Chern and Moser's convergence theorem for normal forms of real hypersurfaces with trivial isotropy as a special case. Our result is based on characterizing the normal form as part of the solution to a suitable involutive initial value problem, whose solutions are analytic as a consequence of the Cartan–Kähler existence theorem. Since the theory of involutive differential equations is at the heart of our proof, we begin the paper with an introduction to the general theory, as developed in [19, 63]. We will use a purely partial differential equation version of the Cartan–Kähler existence theorem, which thus circumvents all the differential form constructions that appear in most other treatments, e.g., [6, 48].

There are four key steps in our argument. The first is to recall that every analytic Lie pseudo-group is characterized by an involutive system of partial differential equations known as its determining equations, [53, 64], in that every local diffeomorphism belonging to the pseudo-group is a solution to the determining equations and conversely. One of our key innovations is to formulate a system of partial differential equations satisfied by the restriction of the pseudo-group transformations to a prescribed submanifold, which we call the *reduced determining equations*. If the pseudo-group satisfies a certain reducibility condition, we show, using the Cartan–Kuranishi Prolongation Theorem, that the reduced determining equations are involutive, and, moreover, their first p Cartan characters are equal to those of the Lie pseudo-group determining equations. Fortunately, a very wide range of Lie pseudo-groups are reducible, including all those that act eventually freely on an open subset of the submanifold jet space, which are precisely the pseudo-groups that can be handled by the equivariant method of moving frames, [54, 56]. The next step is to rewrite the reduced determining equations in an equivalent form, which we call the *normal form determining equations*, which have the property that part of their solution is the normal form of the submanifold upon which we based the reduction. Since the rewriting is simply a change of variables, involutivity of the normal form determining equations is assured. The final step is to apply the method of equivariant moving frames to prescribe formally well-posed initial conditions for the normal form determining equations. These initial conditions are given by what we call a *well-posed cross-section* defining the moving frame. A well-posed cross-section is a refinement of the notion of algebraic cross-section introduced in [55], the key difference being that a well-posed cross-section is determined with respect to a Pommaret basis, while an algebraic cross-section is established using a Gröbner basis. In doing so, we show that once the reduced pseudo-group action becomes free at order n_f, the moving frame construction, and thus the prescribed initial conditions, is compatible with the involutivity of the normal form determining equations starting at order $n_f + 1$. This enables us to appeal to the Cartan–Kähler Theorem to demonstrate that the solution to the normal form determining equations is analytic, which, in particular, implies the analyticity of the normal form and hence convergence of the power series constructed by the moving frame algorithm. Moreover, our proof sheds new light on Chern and Moser's notion of chains used to prove the convergence of the normal forms constructed within their paper, [12]; see also [16, 42].

To the best of our knowledge, our convergence theorem provides the most general result available in the literature. All related works on the subject prove the convergence of power series normal forms within a specific context, [16,20,36–38]. In CR geometry, one of the most general convergence result recently appeared in the work of Lamel and Stolovitch, [39], where the convergence of normal form power series for a class of nondegenerate CR submanifolds subject to certain constraints on the normal form was established. Their results, however, only apply to a very particular family of pseudo-group actions.

The equivariant approach to moving frames [18, 41, 52, 54] that underlies the final stage of our construction generalizes the classical method due to Cartan, [10, 14], and is completely algorithmic. The version used here differs from the original implementations introduced in [18] for general Lie group actions and [54, 55] for infinite-dimensional Lie pseudo-groups, in that it is based on the action of the reduced pseudo-group instead of the original pseudo-group. That said, both implementations yield the same differential invariants. In [52], the moving frame construction was reinterpreted as the specification of a normal form for submanifolds under the pseudo-group action. The paper [54] ends with two examples of the normal form construction for relatively simple infinite-dimensional Lie pseudo-group actions, although, being concerned with the algebraic formulation of the method, the resulting power series were only formal, and the question of convergence was not considered. We note that Arnaldsson, [1, 2], has recently combined equivariant moving frames with Cartan's equivalence method for solving equivalence problems, basing his method on involutive bases for polynomial ideals.

Remark 1.1. Our results are demonstrated by a running example also considered in [53–55]. In the final section, we present a number of further examples illustrating our methods and results, including revisiting the Chern-Moser example of nonsingular real hypersurfaces. Applications to additional substantial pseudo-group actions will be the subject of subsequent papers.

2 Jet Bundles and Partial Differential Equations.

In this section we review the standard geometric language of jet spaces for studying systems of differential equations, and present the basic operations of prolongation and projection. While many of our considerations hold in more general contexts, we work in the analytic category throughout as we will rely on the Cartan–Kähler Theorem to prove the convergence of normal form power series.

Let $\mathcal{X}$ be an analytic p-dimensional manifold, and $\pi\colon M \to \mathcal{X}$ an analytic fiber bundle with q-dimensional fibers. Locally, the total space M is isomorphic to the Cartesian product $\mathcal{X} \times \mathcal{U} \subset \mathbb{R}^p \times \mathbb{R}^q$. Accordingly, we introduce the local coordinates $z = (x, u) \in M$ with $x = (x^1, \ldots, x^p) \in \mathcal{X}$ parametrizing the base space, and so will play the role of independent variables, and $u = (u^1, \ldots, u^q) \in \mathcal{U}$ the fibers, which will be the dependent variables in our system of differential equations. In the following, we let $m = p + q$ denote the dimension of the total space M.

In general, given two analytic manifolds, say $\mathcal{X}$ and $\mathcal{U}$, and an integer $0 \leq n < \infty$, we let $\mathrm{J}^n = \mathrm{J}^n(\mathcal{X},\mathcal{U})$ denote the n-th order *jet space*, whose points (jets) represent equivalence classes of local functions $u\colon \mathcal{X} \to \mathcal{U}$ up to n-th order contact, or, equivalently, possessing the same order n Taylor series at the base point x, [48]. In particular $\mathrm{J}^0 = \mathrm{J}^0 M = M$. In the above framework, we can identify such functions with local sections of $M \to \mathcal{X}$, and $\mathrm{J}^n(\mathcal{X},\mathcal{U}) \subset \mathrm{J}^n M$ is an open subset (coordinate chart) of the jet bundle $\mathrm{J}^n M$ of sections of the fiber bundle. Even more generally, the graphs of sections form p-dimensional submanifolds of M that are transverse to the fibers, and thus $\mathrm{J}^n M \subset \mathrm{J}^n(M, p)$ is an open dense submanifold of the (extended) submanifold jet bundle, [47]. However, since all our considerations are local, we can concentrate on $\mathrm{J}^n = \mathrm{J}^n(\mathcal{X},\mathcal{U})$ throughout. For any $0 \leq k < n$, we have the jet projection

$$\pi^n_k\colon \mathrm{J}^n \to \mathrm{J}^k, \tag{2.1}$$

together with the base projection

$$\pi^n\colon \mathrm{J}^n \to \mathcal{X} \qquad \text{given by} \qquad \pi^n = \pi \circ \pi^n_0.$$

The induced coordinates on the n-th order jet space $\mathrm{J}^n \simeq \mathcal{X} \times \mathcal{U}^{(n)}$ are given by $z^{(n)} = (x, u^{(n)})$ where $x \in \mathcal{X}$ and $u^{(n)} \in \mathcal{U}^{(n)}$. Separating the jet coordinates by order,

$$\mathcal{U}^{(n)} = \mathcal{U}^0 \times \mathcal{U}^1 \times \cdots \times \mathcal{U}^n, \qquad 0 \leq n < \infty,$$

where

$$\mathcal{U}^k = \{(\ldots, u^\alpha_J, \ldots) \,:\, |J| = k,\ \alpha = 1, \ldots, q\}, \qquad 0 \leq k \leq n,$$

denotes the space coordinatized by all k-th order derivatives of the dependent variables, which has dimension

$$\mathsf{t}_k = \dim \mathcal{U}^k = q \binom{p+k-1}{k}.$$

Throughout the paper we use the symmetric multi-index notation for derivatives. Thus, $J = (j_1, \ldots, j_k)$, with $1 \leq j_\nu \leq p$, corresponds to the k-th order derivative $\partial_J = \partial^k / \partial x^{j_1} \cdots \partial x^{j_k}$, and the jet coordinate u^α_J represents the J-th derivative of $u^\alpha(x)$ at the base point x. We also use the concatenation notation $J, i = (j_1, \ldots, j_k, i)$ to denote the symmetric multi-index obtained by appending i to J. Inversely, we use $J \setminus k$ to denote the multi-index obtained by removing $k \in J$ from J.

As noted above, we can identify finite order jets of sections with Taylor polynomials. Explicitly, for $0 \leq n < \infty$, we identify a jet $z^{(n)} = (x, u^{(n)}) \in \mathrm{J}^n$ with the q-tuple of polynomials of degrees $\leq n$ whose entries are

$$P^\alpha_n(y) = \sum_{0 \leq |J| \leq n} \frac{u^\alpha_J}{J\,!}\,(y-x)^J, \qquad \alpha = 1, \ldots, q. \tag{2.2}$$

If $(x, u^{(n)})$ is the n-jet of a section $u(x)$, so u^α_J represents the J-th partial derivative of its component $u^\alpha(x)$ at x, then $P^\alpha_n(y)$ is the corresponding Taylor polynomial of degree n.

There are two inequivalent ways to define the infinite order jet bundle. The usual method is to define J^∞ as the projective (or inverse) limit of the finite order jet bundles J^n under the projection maps (2.1). Thus, an infinite jet has local coordinates x^i, u^α_J for all $i = 1, \ldots, p$, $\alpha = 1, \ldots, q$, and all multi-indices $|J| \geq 0$. We can identify such an infinite jet with a collection of formal power series

$$P^\alpha(y) = \sum_{|J| \geq 0} \frac{u^\alpha_J}{J\,!}\,(y-x)^J, \qquad \alpha = 1, \ldots, q. \tag{2.3}$$

Since the coefficients u^α_J are arbitrary, there is no guarantee that (2.3) converges.

An alternative approach is, in analogy with the finite order case, to define infinite jets as equivalence classes of sections up to infinite order contact, which is equivalent to the condition that their Taylor series (2.3) agree at the base point. Since we restrict to analytic sections, the corresponding Taylor series converge and, indeed, uniquely determine the section. Since the coefficients u^α_J must now define a convergent series, with a non-zero radius of convergence, they are no longer allowed to be arbitrary. Thus, the result of the latter construction is a subbundle $\mathrm{A}^\infty \subset \mathrm{J}^\infty$ of the preceding infinite jet bundle, which consists of infinite jets that produce convergent Taylor series, as in (2.3). We will call A^∞ the *analytic infinite jet bundle*.

Traditionally, the equivariant moving frame calculus takes place in the ordinary infinite jet bundle J^∞, without regard to convergence. Thus, the goal of this paper is to provide conditions, on both the pseudo-group action and the cross-section defining the normalizations, that guarantee that the normal form determined by the moving frame normalizations belongs to the analytic infinite jet bundle A^∞.

A system of n-th order *differential equations* is given by a system of equations

$$\Delta(x, u^{(n)}) = \big(\,\Delta_1(x, u^{(n)}), \ldots, \Delta_l(x, u^{(n)})\,\big) = 0 \tag{2.4}$$

involving the n-th order jet space coordinates. To avoid singularities, the defining functions $\Delta\colon \mathrm{J}^n \to \mathbb{R}^l$ are assumed to be analytic, the corresponding subvariety

$$\mathcal{R}^{(n)} = \{(x, u^{(n)}) \,|\, \Delta(x, u^{(n)}) = 0\} \subset \mathrm{J}^n \tag{2.5}$$

forms an analytic fibered submanifold of the fiber bundle $\pi^n\colon \mathrm{J}^n \to \mathcal{X}$, and the Jacobian matrix of the defining functions is of maximal rank on $\mathcal{R}^{(n)}$, as in [47].

Prolongation and *projection* are two natural operations on differential equations. The former lifts the system of differential equations to higher orders by differentiation, while the latter lowers the order by keeping only the equations (if any) of a specified lower order. The prolongation of (2.4) to order $n+k$ is the fibered submanifold $\mathcal{R}^{(n+k)} \subset \mathrm{J}^{n+k}$ locally described by the system of equations

$$\mathcal{R}^{(n+k)} = \left\{ \begin{matrix} \Delta_\nu(x, u^{(n)}) = 0, & 1 \leq \nu \leq l \\ \mathrm{D}_x^J \Delta_\nu(x, u^{(n)}) = 0, & 1 \leq |J| \leq k \end{matrix} \right\},$$

where

$$\mathrm{D}_{x^i} = \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{|J| \geq 0} u^\alpha_{J,i} \frac{\partial}{\partial u^\alpha_J}, \qquad i = 1, \ldots, p, \tag{2.6}$$

are the usual total derivative operators, which mutually commute, and $\mathrm{D}_x^J = \mathrm{D}_{x^{j_1}} \cdots \mathrm{D}_{x^{j_k}}$, for $J = (j_1, \ldots, j_k)$ a symmetric multi-index, are their higher order iterations. On the other hand, the projection of the n-th order differential equation $\mathcal{R}^{(n)}$ to a differential equation of order $n-k$, with $0 \leq k \leq n$, which encodes the relations (if any) among derivatives of order $\leq n-k$, is given by

$$\pi^n_{n-k}(\mathcal{R}^{(n)}) \subseteq \mathrm{J}^{n-k}.$$

To construct a local representation of $\pi^n_{n-k}(\mathcal{R}^{(n)})$ one starts with (2.4) and eliminates, using only algebraic operations, all derivatives of order greater than $n-k$ in as many equations as possible. If there are no equations of order $\leq n-k$, then, at least locally, $\pi^n_{n-k}(\mathcal{R}^{(n)}) = \mathrm{J}^{n-k}$. As in [63], we assume that the systems of differential equations are *regular* so that, to avoid dealing with singular points and subsets, all projections and prolongations are assumed to be fibered submanifolds.

The k-th prolongation and projection of a system of differential equations $\mathcal{R}^{(n)}$ is given by

$$\pi^{n+k}_n(\mathcal{R}^{(n+k)}) \subseteq \mathcal{R}^{(n)}.$$

This process may not return the original system $\mathcal{R}^{(n)}$ due to the existence of integrability conditions. A system of differential equations $\mathcal{R}^{(n)}$ is said to be *formally integrable* if for all $k \geq 0$, the equality

$$\pi^{n+k+1}_{n+k}(\mathcal{R}^{(n+k+1)}) = \mathcal{R}^{(n+k)} \tag{2.7}$$

holds. In other words, a system of differential equations is formally integrable if, no matter the order at which the system is prolonged, no additional integrability conditions arise.

3 Involutivity.

Formal integrability does not in itself suffice to guarantee the existence of solutions to a system of differential equations, and, for this purpose, we need to introduce the notion of involutivity. To this end, we summarize the theory of involutive systems of partial differential equations, in the form presented by Seiler in his book [63]; see also [19].

We begin with the linearization of a system of partial differential equations. Consider the tangent bundle $T\mathrm{J}^n \to \mathrm{J}^n$ parametrized by $(x, u^{(n)}, \xi, \psi^{(n)})$. Any vector field (section of $T\mathrm{J}^n$)

is locally represented by

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{0\le |J|\le n} \sum_{\alpha=1}^{q} \psi_J^\alpha \frac{\partial}{\partial u_J^\alpha},$$

where the coefficients ξ_i, ψ_J^α depend on $z^{(n)} = (x, u^{(n)})$. We introduce the vertical (fiber) projection $\pi_V : T\mathrm{J}^n|_{z^{(n)}} \to T\mathcal{U}^{(n)}|_{z^{(n)}}$ given by removing the horizontal component[1]:

$$\pi_V(\mathbf{v}) = \sum_{0\le |J|\le n} \sum_{\alpha=1}^{q} \psi_J^\alpha \frac{\partial}{\partial u_J^\alpha}.$$

The (*vertical*) *linearization* $\mathcal{L}_{\mathcal{R}}^{(n)}|_{z^{(n)}} \subset T\mathcal{U}^{(n)}|_{z^{(n)}}$ of the system of differential equations $\mathcal{R}^{(n)}$ given by (2.4) at a point $z^{(n)} \in \mathcal{R}^{(n)}$ consists of the system of linear equations

$$\mathcal{L}_{\mathcal{R}}^{(n)} = \pi_V(\mathbf{v})\,\Delta = \left\{ \sum_{0\le |J|\le n} \sum_{\alpha=1}^{q} \frac{\partial \Delta_\nu}{\partial u_J^\alpha} \psi_J^\alpha = 0, \quad \nu = 1, \dots, l \right\}. \tag{3.1}$$

We further introduce the *highest order term map* $\mathbf{H} : T\mathcal{U}^{(n)}|_{z^{(n)}} \to T\mathcal{U}^n|_{z^{(n)}}$ which only retains the terms ψ_J^α of order $|J| = n$ in (3.1). The resulting system of linear equations

$$\Sigma_{\mathcal{R}}^n = \mathbf{H}(\mathcal{L}_{\mathcal{R}}^{(n)}) = \left\{ \sum_{|J|=n} \sum_{\alpha=1}^{q} \frac{\partial \Delta_\nu}{\partial u_J^\alpha} \psi_J^\alpha = 0, \quad \nu = 1, \dots, l \right\}.$$

is called the *symbol* of the differential equation $\mathcal{R}^{(n)}$. Its $l \times q \binom{p+n-1}{p-1}$ coefficient matrix

$$M_{\mathcal{R}}^n = \left(\frac{\partial \Delta_\nu}{\partial u_J^\alpha} \right)$$

is called the *n-th order symbol matrix*. In line with the common regularity assumption, we suppose in the following that all algebraic properties of the symbol, e.g., rank, etc., are independent of the point $z^{(n)} \in \mathcal{R}^{(n)}$ under consideration.

The columns of the symbol matrix $M_{\mathcal{R}}^n$ correspond to the unknowns ψ_J^α of order $|J| = n$. In order to formulate the involutivity and solvability of the system of partial differential equations, we need to order the columns in an intelligent manner; our preferred ordering will be prescribed by the notion of the class of a multi-index, which relies on a choice of ordering of the independent variables. For general arguments, we use the natural ordering $x^1 \prec x^2 \prec \cdots \prec x^p$ throughout. With this choice of ordering, the definition of class is as follows.

Definition 3.1. The *class* of a multi-index $J = (j_1, \dots, j_k)$ is the smallest index that appears in J:

$$\operatorname{cls} J = \min\{j_1, \dots, j_k\}.$$

[1] We will often suppress the dependence on $z^{(n)}$ to avoid cluttering formulas.

Note that, in the set $\mathcal{U}^k$ of jet coordinates u_J^α of order $|J| = k$, there are

$$\mathsf{t}_k^{(i)} = q \binom{p+k-i-1}{k-1}$$

multi-indices J of class i. Thus,

$$\mathsf{t}_k = \mathsf{t}_k^{(1)} + \cdots + \mathsf{t}_k^{(p)}$$

is the total number of jet coordinates of order exactly k.

We sort the columns of the symbol matrix $M_\mathcal{R}^n$ using a class respecting term ordering so that if $\operatorname{cls} J > \operatorname{cls} K$, then the column corresponding to the unknown ψ_J^α must be to the *left* of the column corresponding to the unknown ψ_K^β. Within a fixed class, one is free to choose any ordering of the columns. For example, if $p = 3$ and we order $x \prec y \prec z$, then one possible ordering of the order $n = 2$ columns of a symbol matrix is $\psi_{zz}, \psi_{yz}, \psi_{yy}, \psi_{xz}, \psi_{xy}, \psi_{xx}$, so the first column has class 3, the next two, which can be switched, have class 2 and the final three, again in any order, are of class 1.

With this column ordering, let $M_{\mathcal{R},\mathrm{REF}}^n$ be the row reduction of $M_\mathcal{R}^n$ to its row-echelon form, cf. [58]. Every unknown ψ_J^α corresponding to the first non-vanishing entry of each row in $M_{\mathcal{R},\mathrm{REF}}^n$, i.e., its pivot, is called the *leader* of the row. We will use r_n to denote the rank of the symbol matrix $M_\mathcal{R}^n$, i.e., the number of leaders/pivots.

The jet coordinates u_J^α of order $|J| = n$ that correspond to the leader columns of the symbol matrix $M_\mathcal{R}^n$ are known as *principal derivatives*. It follows that the number of principal derivatives of order n is

$$\mathsf{r}_n = \operatorname{rank} M_\mathcal{R}^n,$$

which also equals the number of independent differential equations of order n in the system. The other jet coordinates of order n corresponding to the non-pivot columns are known as *parametric derivatives*. The number of parametric derivatives of order n is given by

$$\mathsf{d}_n = \mathsf{t}_n - \mathsf{r}_n.$$

We let

$$\mathsf{r}^{(n)} = \sum_{k=0}^{n} \mathsf{r}_k$$

denote the total number of principal derivatives of order $\leq n$, and

$$\mathsf{d}^{(n)} = q \binom{p+n}{n} - \mathsf{r}^{(n)} = \sum_{k=0}^{n} \mathsf{d}_k,$$

the total number of parametric derivatives of order $\leq n$. By the Implicit Function Theorem and our regularity assumptions, $\mathsf{d}^{(n)}$ equals the fiber dimension of the n-th order system (2.5).

An n-th order system of partial differential equations is said to be in *Cartan normal form*, if its symbol matrices of order $0 \leq k \leq n$ are either empty or in reduced row-echelon form with respect to the above class-respecting ordering of the columns. We further say that it is in *reduced Cartan normal form* if, in addition, the entire symbol matrix

$$M_\mathcal{R}^{(n)} = \begin{pmatrix} M_\mathcal{R}^0 & M_\mathcal{R}^1 & \cdots & M_\mathcal{R}^n \end{pmatrix}^T$$

is in reduced row-echelon form, [19]. Thus, the differential equations are in the reduced Cartan normal form when they take the form

$$u_J^\alpha = \Delta_J^\alpha(x, \ldots, u_K^\beta, \ldots), \tag{3.2}$$

where u_J^α are the principal derivatives, and all the jet coordinates u_K^β appearing on the right hand side are parametric and correspond to the columns that have *nonzero* entries in the corresponding row of the reduced row echelon form of the entire symbol matrix. At order $|K| = |J|$, these are all parametric derivatives that appear after u_J^α in the class respecting term ordering, that is $\mathrm{cls}\, K \leq \mathrm{cls}\, J$. As a consequence of the Implicit Function Theorem, any regular system of differential equations of order n can be placed in reduced Cartan normal form.

Definition 3.2. The number of leaders of class $1 \leq k \leq p$ in the row-echelon symbol matrix $M^n_{\mathcal{R},\mathrm{REF}}$ is denoted by $\mathsf{b}_n^{(k)}$. The resulting $\mathsf{b}_n^{(1)}, \ldots, \mathsf{b}_n^{(p)}$ are called the *indices* of the n-th order symbol $\Sigma^n_{\mathcal{R}}$.

We are now able to state the key definition of an involutive symbol.

Definition 3.3. The symbol $\Sigma^n_{\mathcal{R}}$ with indices $\mathsf{b}_n^{(k)}$ is said to be *involutive* if the symbol matrix $M^{n+1}_{\mathcal{R}}$ of the prolonged symbol $\Sigma^{n+1}_{\mathcal{R}}$ satisfies

$$\mathsf{r}_{n+1} = \mathrm{rank}\, M^{n+1}_{\mathcal{R}} = \sum_{k=1}^{p} k\, \mathsf{b}_n^{(k)}. \tag{3.3}$$

Remark 3.4. We observe that the class of a derivative is not invariant under coordinate transformations. The notion of a *δ-regular coordinate chart* is characterized by the fact that the sum on the right hand side of (3.3) takes its maximal value under all possible (linear) changes of coordinates. In particular, a necessary condition for δ-regularity is that the highest index $\mathsf{b}_n^{(p)}$ takes its maximal value. For a first order system of differential equations, this means that a maximal number of equations must be solvable for an x^p-derivative, and hence the surface $x^p = 0$ cannot be characteristic. Clearly, the involutivity condition (3.3) requires that we work in a δ-regular coordinate system. Indeed, we will assume throughout that we are always working in δ-regular coordinates, noting that generic coordinate systems are δ-regular, [23,63]. In some of our examples, the most natural coordinate system for the system is not δ-regular, and so the involutivity criterion (3.3) is not satisfied, and we must impose a suitable change of variables before conducting the analysis.

Definition 3.5. A system of differential equations $\mathcal{R}^{(n)}$ is *involutive* if it is formally integrable and its symbol $\Sigma^n_{\mathcal{R}}$ is involutive.

Formal integrability requires verifying (2.7) for all $k \geq 0$. The next result states that, when the system is involutive, it suffices to check integrability when $k = 0$. A proof can be found in [63].

Theorem 3.6. *A system of differential equations $\mathcal{R}^{(n)}$ is involutive if and only if its symbol $\Sigma^n_{\mathcal{R}}$ is involutive and $\pi^{n+1}_n(\mathcal{R}^{(n+1)}) = \mathcal{R}^{(n)}$.*

Thus, to check involutivity at order n, one needs to make sure that the coordinate chart is δ-regular, then verify the algebraic involutivity condition (3.3) for the indices of order n, and finally check that there are no integrability conditions at order $n+1$.

The indices $\mathsf{b}_n^{(k)}$ determine the number of principal derivatives of order n and of class k in the system of differential equations $\mathcal{R}^{(n)}$. On the other hand, the number of parametric derivatives of order n and class k is given by the *Cartan character* $\mathsf{c}_n^{(k)}$, which is related to the corresponding index via the equation

$$\mathsf{c}_n^{(k)} = \mathsf{t}_n^{(k)} - \mathsf{b}_n^{(k)}, \qquad 1 \leq k \leq p. \tag{3.4}$$

We note that, by [63, Proposition 8.2.2], involutivity implies that the Cartan characters are non-increasing:

$$\mathsf{c}_n^{(1)} \geq \mathsf{c}_n^{(2)} \geq \ \cdots \ \geq \mathsf{c}_n^{(p)} \geq 0. \tag{3.5}$$

Remark 3.7. Owing to their direct relationship (3.4), when formulating results or illustrative examples, one can work either just with the indices or just with the Cartan characters, depending upon one's preference. We have chosen to display both in order to suit readers of either persuasion.

Remark 3.8. If $\mathsf{c}_n^{(k)} > \mathsf{c}_n^{(k+1)} = 0$ is the last nonzero Cartan character of an involutive system of differential equations, then the general solution to the system depends on $\mathsf{c}_n^{(k)}$ functions of k variables, which can be identified with the initial conditions of order k. On the other hand, the number of functions of less than k variables required to express a general solution is not well-defined; see also [6, 11, 48, 63].

Any system of differential equations (2.4) can be written as a first-order system of differential equations by setting the jet coordinates u_J^α of order $|J| \leq n-1$ to be new dependent variables. To write down this new system of equations, we introduce the del notation

$$\partial_i u_J^\alpha = \frac{\partial u_J^\alpha}{\partial x^i}$$

to denote differentiation. Then a first order representation $\widetilde{\mathcal{R}}^{(1)}$ of $\mathcal{R}^{(n)}$ is given by

$$\widetilde{\mathcal{R}}^{(1)} = \left\{ \begin{array}{lll} \widetilde{\Delta}_\nu(x, (u^{(n-1)})^{(1)}) = 0, & 1 \leq \nu \leq l & \\ \partial_i u_J^\alpha = u_{J,i}^\alpha, & |J| < n-1, & 1 \leq i \leq p \\ \partial_i u_J^\alpha = \partial_k u_{J,i\setminus k}^\alpha, & |J| = n-1, & k = \operatorname{cls} J < i \leq p \end{array} \right\}.$$

The function $\widetilde{\Delta}_\nu$ is not uniquely defined, as there are in general several possibilities to express a higher-order derivative u_J^α in terms of the new coordinates. To easily compute the indices of the symbol $\Sigma^1_{\widetilde{\mathcal{R}}}$, we use the mapping

$$u_J^\alpha = \begin{cases} u_J^\alpha, & |J| \leq n-1, \\ \partial_k u_{J\setminus k}^\alpha, & |J| = n, \quad \operatorname{cls} J = k. \end{cases} \tag{3.6}$$

Proposition 3.9. *Let $\widetilde{\mathsf{c}}_1^{(1)}, \dots, \widetilde{\mathsf{c}}_1^{(p)}$ be the Cartan characters of the first order representation $\widetilde{\mathcal{R}}^{(1)}$ and $\mathsf{c}_n^{(1)}, \dots, \mathsf{c}_n^{(p)}$ those of the original system of differential equations $\mathcal{R}^{(n)}$. Then*

$$\widetilde{\mathsf{c}}_1^{(k)} = \mathsf{c}_n^{(k)}, \qquad 1 \le k \le p,$$

and the differential equation $\mathcal{R}^{(n)}$ is involutive if and only if its first order representation $\widetilde{\mathcal{R}}^{(1)}$ is involutive.

The proof of Proposition 3.9 may be found in [63, Appendix A.3]. For a first-order system of differential equations $\mathcal{R}^{(1)}$, the reduced Cartan normal form is

$$\begin{aligned} u_p^\alpha &= \Delta_p^\alpha(x, \dots, u_k^\beta, \dots), & 1 \le \alpha \le \mathsf{b}_1^{(p)}, \\ u_{p-1}^\alpha &= \Delta_{p-1}^\alpha(x, \dots, u_k^\beta, \dots), & 1 \le \alpha \le \mathsf{b}_1^{(p-1)}, \\ &\vdots \\ u_1^\alpha &= \Delta_1^\alpha(x, \dots, u_k^\beta, \dots), & 1 \le \alpha \le \mathsf{b}_1^{(1)}, \\ u^\alpha &= \Delta^\alpha(x, u^\delta), & 1 \le \alpha \le \mathsf{b}_0, \end{aligned} \tag{3.7}$$

with

$$0 \le \mathsf{b}_0 \le \mathsf{b}_1^{(1)} \le \dots \le \mathsf{b}_1^{(p-1)} \le \mathsf{b}_1^{(p)} \le q,$$

and where all the derivatives appearing on the right hand side of each equation are parametric of class smaller than or equal to the class of the principal derivative occurring on the left hand side of the equation. If $\mathsf{b}_0 = 0$ there are no algebraic equations. If $\mathsf{b}_0 > 0$, since the equations are in reduced Cartan normal form, no derivatives of order 0 or 1 of the parametric u^α's appear on the right hand side of any of the equations.

Formally well-posed initial value conditions for the first-order system of differential equations in Cartan normal form (3.7) are prescribed by

$$\begin{aligned} u^\alpha(x^1, \dots, x^p) &= f^\alpha(x^1, \dots, x^p), & \mathsf{b}_1^{(p)} < \alpha \le q, \\ u^\alpha(x^1, \dots, x^{p-1}, 0) &= f^\alpha(x^1, \dots, x^{p-1}), & \mathsf{b}_1^{(p-1)} < \alpha \le \mathsf{b}_1^{(p)}, \\ &\vdots \\ u^\alpha(x^1, 0, \dots, 0) &= f^\alpha(x^1), & \mathsf{b}_1^{(1)} < \alpha \le \mathsf{b}_1^{(2)}, \\ u^\alpha(0, \dots, 0) &= f^\alpha, & \mathsf{b}_0 < \alpha \le \mathsf{b}_1^{(1)}. \end{aligned} \tag{3.8}$$

Remark 3.10. In (3.8) we use the convention that if, for example, $\mathsf{b}_1^{(p)} = q$, then the first set of equations in the initial conditions (3.8) are vacuous, and similarly for the other equations.

As they should, the initial conditions (3.8) specify the parametric jets occurring on the right hand side of the system of differential equations (3.7). For example, the parametric derivative of class 1 are determined by differentiating the equations $u^\alpha(x^1, 0, \dots, 0) = f^\alpha(x^1)$ for $\mathsf{b}_1^{(1)} < \alpha \le \mathsf{b}_1^{(2)}$. The parametric derivatives of class 2 are obtained from the initial conditions on the plane $\{(x^1, x^2, 0, \dots, 0)\}$, and so on.

We are now able to state the Cartan–Kähler Theorem for first order involutive systems of differential equations, which are placed in reduced Cartan normal form (3.7). This fundamental existence theorem is a consequence and generalization of the basic Cauchy–Kovalevskaya existence theorem for analytic system of partial differential equations, [48, 63].

Theorem 3.11. *Let the functions Δ_k^α and f^α in (3.7) and (3.8) be real-analytic at the origin. If the system (3.7) is involutive, then it possesses one and only one solution that is analytic at the origin and satisfies the initial conditions (3.8).*

4 Lie Pseudo-Groups.

In this section, we apply the preceding constructions to the differential equations defining Lie pseudo-group actions, referring to [53] for details. Let $\mathcal{D} = \mathcal{D}(M)$ denote the Lie pseudo-group of all local diffeomorphisms[2] $\varphi\colon M \to M$. We will employ Cartan's convenient notational convention and use lower case letters to denote source coordinates and the corresponding capital letters to denote target coordinates. Thus, given a local diffeomorphism $\varphi \in \mathcal{D}$, its local coordinate formula will be written $Z = \varphi(z)$ with $Z = (Z^1, \ldots, Z^m)$ and $z = (z^1, \ldots, z^m)$.

Given $0 \le n < \infty$, let $\mathcal{D}^{(n)} \subset \mathrm{J}^n(M, M)$ be the space of n-th order jets of local diffeomorphisms of M, which forms a groupoid under composition. We let $\mathcal{D}^{(\infty)} \subset \mathrm{J}^\infty(M, M)$ denote the corresponding space of infinite order jets of diffeomorphisms, and $\mathcal{A}^{(\infty)} \subset \mathcal{D}^{(\infty)}$ the subspace of analytic diffeomorphism jets, i.e., those that define convergent Taylor series.

Given a regular analytic Lie pseudo-group $\mathcal{G} \subset \mathcal{D}$, let $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ denote the subgroupoid consisting of n-th order jets of pseudo-group diffeomorphisms, which we can identify with the n-th order determining equations of $\mathcal{G}$, whose solutions are the pseudo-group elements. Note that, by analyticity, $\mathcal{G}^{(\infty)} \subset \mathcal{A}^{(\infty)}$. According to Theorem 3.4 of [28], there exists an order $n_\star \in \mathbb{N}$ such that, for all finite $n \ge n_\star$, the determining equations

$$\mathcal{G}^{(n)} = \left\{ \Delta_\nu(z, Z^{(n)}) = 0, \quad \nu = 1, \ldots, l_n \right\} \tag{4.1}$$

are involutive. Separating the pseudo-group jet coordinates by order, we let

$$\begin{aligned} \mathcal{D}^{(n)} &\simeq M \times D^{(n)} = M \times D^0 \times D^1 \times \cdots \times D^n, \\ \mathcal{G}^{(n)} &\simeq M \times G^{(n)} = M \times G^0 \times G^1 \times \cdots \times G^n, \end{aligned}$$

where

$$D^k = \{\, (\ldots, Z^a_B, \ldots) \mid \ |B| = k, \quad a = 1, \ldots, m \,\}$$

denotes the space of k-th order derivatives of the local diffeomorphism $Z = \varphi(z) \in \mathcal{D}$, and similarly for G^k. We then have

$$\mathsf{t}_k = \dim\, D^k = m \binom{m+k-1}{k},$$

while the number of derivatives of order $k \ge 1$ and of class a is

$$\mathsf{t}_k^{(a)} = m \binom{m+k-a-1}{k-1}, \qquad 1 \le a \le m.$$

[2]In general, the notation allows φ to only be defined on an open subset of M.

For $k \geq 1$, we note the relations

$$\sum_{a=1}^{m} \mathsf{t}_k^{(a)} = \mathsf{t}_k, \qquad \sum_{a=1}^{m} a\, \mathsf{t}_k^{(a)} = \mathsf{t}_{k+1}, \qquad \sum_{k=0}^{n} \mathsf{t}_k = \mathsf{t}^{(n)} = \dim D^{(n)}.$$

For the Lie pseudo-group $\mathcal{G}$, and each $0 \leq n < \infty$, we let $\mathsf{d}^{(n)} = \dim G^{(n)}$ denote the fiber dimension of the projection $\pi^n \colon \mathcal{G}^{(n)} \to M$. For $0 \leq k \leq n$, let $\mathsf{d}_k = \dim G^k$ denote the number of parametric pseudo-group parameters of order k so that $\mathsf{d}^{(n)} = \mathsf{d}_0 + \mathsf{d}_1 + \cdots + \mathsf{d}_n$. The number of principal pseudo-group parameters of order k is then given by $\mathsf{r}_k = \mathsf{t}_k - \mathsf{d}_k$.

Let

$$\mathsf{j}_n \mathbf{V} = \sum_{a=1}^{m} \sum_{0 \leq |B| \leq n} \zeta_B^a \frac{\partial}{\partial Z_B^a}$$

denote a vertical vector field in $TD^{(n)}$. The linearization of the pseudo-group determining equations (4.1) at the identity jet $\mathbb{1}_z^{(n)}$ are the *linearized determining equations*

$$\mathcal{L}_{\mathcal{G}}^{(n)} = \left\{ L_\nu(z, \zeta^{(n)}) = \sum_{0 \leq |B| \leq n} \sum_{a=1}^{m} \frac{\partial \Delta_\nu}{\partial Z_B^a} \bigg|_{\mathbb{1}_z^{(n)}} \zeta_B^a = 0, \quad \nu = 1, \ldots, l_n \right\}, \tag{4.2}$$

which serve to define the Lie algebroid associated with the Lie pseudo-group groupoid $\mathcal{G}^{(n)}$, [53]. As in the previous section, we introduce the highest order term map $\mathbf{H} \colon T\mathcal{D}^{(n)}|_{\mathbb{1}_z^{(n)}} \to TD^n|_{\mathbb{1}_z^{(n)}}$ which only keeps the linear terms of order n in (4.2) to obtain the n-th order pseudo-group *symbol*

$$\Sigma_{\mathcal{G}}^n = \mathbf{H}(\mathcal{L}_{\mathcal{G}}^{(n)}) = \left\{ \sum_{|B|=n} \sum_{a=1}^{m} \frac{\partial \Delta_\nu}{\partial Z_B^a} \bigg|_{\mathbb{1}_z^{(n)}} \zeta_B^a = 0, \qquad \nu = 1, \ldots, l_n \right\}.$$

Our regularity assumption on $\mathcal{G}$ requires that the algebraic properties of the symbol are independent of the point $(z, Z^{(n)}) \in \mathcal{G}^{(n)}$.

For $n \geq n_\star$, the order of involutivity, the indices and Cartan characters of the determining equations $\mathcal{G}^{(n)}$ satisfy

$$\sum_{a=1}^{m} \mathsf{b}_n^{(a)} = \mathsf{r}_n \qquad \text{and} \qquad \sum_{a=1}^{m} \mathsf{c}_n^{(a)} = \mathsf{d}_n, \tag{4.3}$$

and, since the equations are involutive,

$$\sum_{a=1}^{m} a\, \mathsf{b}_n^{(a)} = \mathsf{r}_{n+1} \qquad \text{and} \qquad \sum_{a=1}^{m} a\, \mathsf{c}_n^{(a)} = \mathsf{d}_{n+1}.$$

Example 4.1. The following well-studied Lie pseudo-group

$$X = f(x), \qquad Y = f_x(x)\, y + g(x), \qquad U = u + \frac{f_{xx}(x)\, y + g_x(x)}{f_x(x)}, \tag{4.4}$$

where $f \in \mathcal{D}(\mathbb{R})$, and $g \in C^\infty(\mathbb{R})$, will serve as our running example illustrating the constructions. The determining equations $\mathcal{G}^{(2)}$ of order two (in Cartan normal form) are

$$\begin{gathered} X_y = X_u = 0, \quad Y_x = (U-u)X_x, \quad Y_y = X_x, \quad Y_u = 0, \quad U_u = 1, \\ X_{xx} = U_y X_x, \quad X_{xy} = X_{xu} = X_{yy} = X_{yu} = X_{uu} = 0, \quad Y_{xx} = \big(U_x + (U-u)U_y\big) X_x, \\ Y_{xy} = U_y X_x, \quad Y_{xu} = Y_{yy} = Y_{yu} = Y_{uu} = 0, \quad U_{xu} = U_{yy} = U_{yu} = U_{uu} = 0. \end{gathered} \tag{4.5}$$

Thus, the parametric jet variables parametrizing $\mathcal{G}^{(2)}$ are

$$X, \quad Y, \quad U, \quad X_x, \quad U_x, \quad U_y, \quad U_{xx}, \quad U_{xy}; \tag{4.6}$$

all the other second order jet coordinates, i.e., those appearing on the left hand side of the determining equations (4.5), are principal. We observe that $\mathsf{d}_0 = \mathsf{d}_1 = 3, \ \mathsf{d}_2 = 2$, and so $\mathsf{d}^{(0)} = 3, \ \mathsf{d}^{(1)} = 6, \ \mathsf{d}^{(2)} = 8$. It is not hard to see that, in general, the order $n \geq 2$ parametric variables are $U_{x^n}, U_{x^{n-1}y}$, hence $\mathsf{d}_n = 2$ and $\mathsf{d}^{(n)} = 2n + 4$. Using the notation

$$\mathrm{j}_\infty \mathbf{V} = \sum_{|B| \geq 0} \xi_B \frac{\partial}{\partial X_B} + \eta_B \frac{\partial}{\partial Y_B} + \phi_B \frac{\partial}{\partial U_B}$$

to denote a vertical vector field, the corresponding linearized determining equations $\mathcal{L}^{(2)}_{\mathcal{G}}$ of order two are obtained by applying $\mathbf{v}$ to the determining equations (4.5) and then evaluating the result at the identity jet; namely, set $X = x, \ Y = y, \ U = u, \ X_x = Y_y = U_u = 1$, and all other jet coordinates to 0; the result is

$$\begin{gathered}
\xi_y = \xi_u = 0, \qquad \eta_x = \phi, \qquad \eta_y = \xi_x, \qquad \eta_u = 0, \qquad \phi_u = 0, \\
\xi_{xx} = \phi_y, \qquad \xi_{xy} = \xi_{xu} = \xi_{yy} = \xi_{yu} = \xi_{uu} = 0, \\
\eta_{xx} = \phi_x, \qquad \eta_{xy} = \phi_y, \qquad \eta_{xu} = \eta_{yy} = \eta_{yu} = \eta_{uu} = \phi_{xu} = \phi_{yy} = \phi_{yu} = \phi_{uu} = 0.
\end{gathered}$$

The symbol $\Sigma^2_{\mathcal{G}}$ is given by the equations

$$\begin{gathered}
\xi_{xx} = \xi_{xy} = \xi_{yy} = \xi_{xu} = \xi_{yu} = \xi_{uu} = 0, \qquad \eta_{xx} = \eta_{xy} = \eta_{yy} = \eta_{xu} = \eta_{yu} = \eta_{uu} = 0, \\
\phi_{xu} = \phi_{yy} = \phi_{yu} = \phi_{uu} = 0.
\end{gathered}$$

Using the term ordering $x \prec y \prec u$, the indices are

$$\mathsf{b}_2^{(1)} = 7, \qquad \mathsf{b}_2^{(2)} = 6, \qquad \mathsf{b}_2^{(3)} = 3,$$

while the Cartan characters are

$$\mathsf{c}_2^{(1)} = 2, \qquad \mathsf{c}_2^{(2)} = \mathsf{c}_2^{(3)} = 0. \tag{4.7}$$

On the other hand, the determining equations of order three are obtained by differentiating those of order 2 and then replacing any principal derivatives using the preceding equations, producing

$$\begin{gathered}
X_{xxx} = (U_{xy} + U_y^2) X_x, \qquad X_{xxy} = X_{xxu} = X_{xyy} = X_{xyu} = X_{xuu} = X_{yyy} = 0, \\
X_{yyu} = X_{yuu} = X_{uuu} = 0, \qquad Y_{xxx} = \big(U_{xx} + (U - u)(U_{xy} + U_y^2) + 2 U_x U_y\big) X_x, \\
Y_{xxy} = (U_{xy} + U_y^2) X_x, \qquad Y_{xxu} = Y_{xyy} = Y_{xyu} = Y_{xuu} = Y_{yyy} = Y_{yyu} = Y_{yuu} = Y_{uuu} = 0, \\
U_{xxu} = U_{xyy} = U_{xyu} = U_{xuu} = U_{yyy} = U_{yyu} = U_{yuu} = U_{uuu} = 0,
\end{gathered}$$

from which we see that the algebraic involutivity constraint

$$\mathsf{b}_2^{(1)} + 2\,\mathsf{b}_2^{(2)} + 3\,\mathsf{b}_2^{(3)} = \mathsf{r}_3 = 28$$

is satisfied. Alternatively, in terms of the Cartan characters

$$\mathsf{c}_2^{(1)} + 2\,\mathsf{c}_2^{(2)} + 3\,\mathsf{c}_2^{(3)} = \mathsf{d}_3 = 2.$$

Since $\pi^3_2(\mathcal{G}^{(3)}) = \mathcal{G}^{(2)}$, the determining equations (4.5) are involutive. Based on the Cartan characters (4.7), the solution depends on two functions of one variable, as was already clear from the original formula (4.4) for the pseudo-group transformations.

5 Reduction of Lie Pseudo-Group Actions.

We are now interested in the action of a Lie pseudo-group on p-dimensional submanifolds of the total space M. To work in local coordinates, we assume that the submanifolds are transverse to the fibers, and thus form local sections of $M \to \mathcal{X}$. In this section, we formulate the reduced determining equations for the action of pseudo-group elements on sections, and prove that they form an involutive system of differential equations. This construction is a key intermediate step towards our formulation of the system of differential equations satisfied by the normal forms of submanifolds.

As in [54, 55], let $\mathcal{E}^{(n)} \to \mathrm{J}^n$ denote the *lifted bundle* obtained by pulling back the diffeomorphism jet bundle $\mathcal{D}^{(n)} \to M$ to the submanifold jet space via the standard projection $\pi_0^n \colon \mathrm{J}^n \to M$. Local coordinates on $\mathcal{E}^{(n)}$ are given by $(z^{(n)}, Z^{(n)}) = (x, u^{(n)}, X^{(n)}, U^{(n)})$, where $z^{(n)} = (x, u^{(n)})$ are coordinates on the submanifold jet bundle J^n while $Z^{(n)} = (X^{(n)}, U^{(n)})$ are the fiber coordinates of the diffeomorphism jet bundle $\mathcal{D}^{(n)}$. The lifted bundle has the structure of a groupoid using the double fibration with source map $\boldsymbol{\sigma}^{(n)}(z^{(n)}, Z^{(n)}) = z^{(n)}$ and target map $\boldsymbol{\tau}^{(n)}(z^{(n)}, Z^{(n)}) = Z^{(n)} \cdot z^{(n)}$ prescribed by the prolonged action of the diffeomorphisms on submanifold jets.

When writing out the action of a pseudo-group transformation on a submanifold, we will use, in accordance with Cartan's convention, lower case letters, so $u = u(x)$, for the source submanifold and its jet coordinates. However, to avoid notational confusion, especially when distinguishing submanifold jets from diffeomorphism jets, we will use hats on the dependent variable and its derivatives to denote the target submanifold, which we thus write as $\widehat{U} = \widehat{U}(X)$ with the order zero jet being simply $\widehat{U} = U$. Later, once the reader becomes used to which symbol denotes which type of jet coordinate, the hats can be dropped to clean up the formulas, and, indeed, we shall do so in the examples in the final section.

Example 5.1. In the case of planar curves, given the action of a diffeomorphism of $\mathbb{R}^2$ on curves, the source curve is the graph of a scalar function $u = u(x)$ for $x, u \in \mathbb{R}$, while the target is the graph of a scalar function, which, in accordance with the above-stated convention, is written as $\widehat{U} = \widehat{U}(X)$ for $X, \widehat{U} \in \mathbb{R}$, and its jet coordinates are $\widehat{U}, \widehat{U}_X, \widehat{U}_{XX}, \ldots$. The coordinates on the lifted bundle $\mathcal{E}^{(n)}$ are thus given by

$$\begin{aligned}(z^{(n)}, Z^{(n)}) &= (x, u^{(n)}, X^{(n)}, U^{(n)}) \\ &= (x, u, u_x, u_{xx}, \ldots, X, U, X_x, X_u, U_x, U_u, X_{xx}, X_{xu}, X_{uu}, U_{xx}, U_{xu}, U_{uu}, \ldots).\end{aligned}$$

where u, u_x, u_{xx}, $\ldots$ are the curve jet coordinates and X, U, X_x, X_u, U_x, U_u, $\ldots$ are the diffeomorphism jet coordinates. The source and target maps are

$$\begin{aligned}&\boldsymbol{\sigma}^{(n)}(z^{(n)}, Z^{(n)}) = (x, u, u_x, u_{xx}, \ldots), \\ &\boldsymbol{\tau}^{(n)}(z^{(n)}, Z^{(n)}) = (X, \widehat{U}, \widehat{U}_X, \widehat{U}_{XX}, \ldots) \\ &\quad = \left(X, U, \frac{U_x + u_x U_u}{X_x + u_x X_u}, \frac{\begin{array}{c}\big[(X_x + u_x X_u)(U_{xx} + 2\,u_x U_{xu} + u_x^2 U_{uu} + u_{xx} U_u) \\ \qquad - (U_x + u_x U_u)(X_{xx} + 2\,u_x X_{xu} + u_x^2 X_{uu} + u_{xx} X_u)\big]\end{array}}{(X_x + u_x X_u)^3}, \ldots \right),\end{aligned}$$

where the higher order target jets are obtained by repeatedly applying the operator of implicit differentiation

$$\mathrm{D}_X = \frac{1}{\mathrm{D}_x X}\,\mathrm{D}_x = \frac{1}{X_x + u_x X_u}\,\mathrm{D}_x$$

to $\widehat{U}$; see also (6.1) below.

The *horizontal total derivative operators* on $\mathcal{E}^{(\infty)}$ are

$$\mathrm{D}_{x^j} = \mathbb{D}_{x^j} + \sum_{\alpha=1}^{q} \left(u^\alpha_j\,\mathbb{D}_{u^\alpha} + \sum_{|J|\geq 1} u^\alpha_{J,j} \frac{\partial}{\partial u^\alpha_J} \right), \qquad j = 1,\ldots,p, \tag{5.1}$$

where[3]

$$\mathbb{D}_{z^a} = \frac{\partial}{\partial z^a} + \sum_{b=1}^{m} \sum_{|A|\geq 0} Z^b_{A,a} \frac{\partial}{\partial Z^b_A}, \qquad a = 1,\ldots,m,$$

are the total derivative operators on the diffeomorphism jet bundle $\mathcal{D}^{(\infty)}$. We use the same notation (2.6) and (5.1) for the total derivative operators on J^∞ and $\mathcal{E}^{(\infty)}$, respectively, since they coincide when $F(z^{(n)}) = F(x,u^{(n)})$ does not depend on the diffeomorphism jet coordinates.

Given a local section $f\colon \mathcal{X} \to M$, whose graph defines a p-dimensional submanifold $s = f(\mathcal{X})$, and a local diffeomorphism $\varphi \in \mathcal{D}(M)$, with $s \subset \operatorname{dom}\varphi$, we will call the composition $\overline{\varphi} = \varphi \circ f$ the *reduction* of φ to the submanifold s. The reduced map $\overline{\varphi}\colon \mathcal{X} \to M$ is in general not a section of M since $\varphi \circ f(x)$ does not necessarily belong to the fiber of M over $x \in \mathcal{X}$. On the other hand, its image, namely $S = \varphi\big[\, f(\mathcal{X})\,\big] = \overline{\varphi}(s)$ is an equivalent submanifold. If we assume that the image S is transversal to the fibers of M, we can locally identify it with the graph of a local section $F\colon \mathcal{X} \to M$, so $S = F(\mathcal{X})$.

Remark 5.2. We will use overbars to denote reduced maps and jet coordinates. As with the hats, these can also dropped once the reader becomes used to which symbol denotes which jet coordinate, and, indeed, we shall do so in the final section.

Let $0 \leq n < \infty$. The reduced action of local diffeomorphisms on submanifolds is encoded by the *reduction map* $\mathfrak{r}^{(n)}\colon \mathcal{E}^{(n)} \to \mathrm{J}^n(\mathcal{X},\mathcal{U}\times M)$ given by

$$\mathfrak{r}^{(n)}(x,u^{(n)},X^{(n)},U^{(n)}) = \mathfrak{r}^{(n)}(z^{(n)},Z^{(n)}) = (z^{(n)},\overline{Z}^{(n)}) = (x,u^{(n)},\overline{X}^{(n)},\overline{U}^{(n)}), \tag{5.2}$$

where $\overline{Z}^{(n)} = (\overline{X}^{(n)},\overline{U}^{(n)})$ has components

$$\overline{Z}^a_J = \mathrm{D}^J_x Z^a \qquad \text{for} \qquad 0 \leq |J| \leq n, \quad a = 1,\ldots,m,$$

which are obtained by successively applying the total derivative operators (5.1) to the diffeomorphism target coordinates $Z = (X,U)$. We call $\overline{Z}^a_J$ the *reduced jet coordinates*. The reduction map is compatible with the reduction of diffeomorphisms to submanifolds. Namely, given a diffeomorphism φ and a section $s = f(x) = (x,u(x))$ contained in its domain, let $(x,u^{(n)},X^{(n)},U^{(n)}) \in \mathcal{E}^{(n)}$ be given by their combined jets, so that $(x,u^{(n)}) = \mathrm{j}_n f|_x$ and $(x,u,X^{(n)},U^{(n)}) = \mathrm{j}_n\varphi|_{(x,u)}$, then $\mathrm{j}_n(\varphi\circ f) = \mathfrak{r}^{(n)}(x,u^{(n)},X^{(n)},U^{(n)})$.

[3]Here z^a can be either x^j or u^α.

Example 5.3. Let $M = \mathbb{R}^2$ and $\mathcal{X} = \mathbb{R}$, corresponding to plane curves $s = \{(x, u(x))\}$. Then the reduction map (5.2) is computed by successively applying the total differential operator

$$\mathrm{D}_x = \mathbb{D}_x + u_x\,\mathbb{D}_u + u_{xx}\frac{\partial}{\partial u_x} + u_{xxx}\frac{\partial}{\partial u_{xx}} + \ \cdots \ , \tag{5.3}$$

where

$$\begin{aligned}
\mathbb{D}_x &= \frac{\partial}{\partial x} + X_x\frac{\partial}{\partial X} + U_x\frac{\partial}{\partial U} + X_{xx}\frac{\partial}{\partial X_x} + X_{xu}\frac{\partial}{\partial X_u} + U_{xx}\frac{\partial}{\partial U_x} + U_{xu}\frac{\partial}{\partial U_u} + \ \cdots \ ,\\
\mathbb{D}_u &= \frac{\partial}{\partial u} + X_u\frac{\partial}{\partial X} + U_u\frac{\partial}{\partial U} + X_{xu}\frac{\partial}{\partial X_x} + X_{uu}\frac{\partial}{\partial X_u} + U_{xu}\frac{\partial}{\partial U_x} + U_{uu}\frac{\partial}{\partial U_u} + \ \cdots \ ,
\end{aligned} \tag{5.4}$$

to $\overline{X}, \overline{U}$, producing, at order $n = 2$,

$$\begin{aligned}
&\mathfrak{r}^{(2)}(x, u, u_x, u_{xx}, X, U, X_x, X_u, U_x, U_u, X_{xx}, X_{xu}, X_{uu}, U_{xx}, U_{xu}, U_{uu})\\
&\qquad = (x, u, u_x, u_{xx}, \overline{X}, \overline{U}, \overline{X}_x, \overline{U}_x, \overline{X}_{xx}, \overline{U}_{xx})\\
&\qquad = (x, u, u_x, u_{xx}, X, U, \mathrm{D}_x X, \mathrm{D}_x U, \mathrm{D}_x^2 X, \mathrm{D}_x^2 U)\\
&\qquad = (x, u, u_x, u_{xx}, X, U, X_x + u_x X_u, U_x + u_x U_u,\\
&\qquad\qquad\qquad X_{xx} + 2\,u_x X_{xu} + u_x^2 X_{uu} + u_{xx} X_u, U_{xx} + 2\,u_x X_{xu} + u_x^2 U_{uu} + u_{xx} U_u).
\end{aligned}$$

Observe that the expressions for the reduced jet coordinates are obtained by total differentiation of $\overline{X} = \overline{X}(x, u), \overline{U} = \overline{X}(x, u)$, treating u as a function of x.

We will regard $\mathrm{J}^n(\mathcal{X}, \mathcal{U} \times M) \to \mathrm{J}^n(\mathcal{X}, \mathcal{U}) = \mathrm{J}^n$ as a fiber bundle over the submanifold jet bundle, so that the reduced jet coordinates $\overline{Z}^{(n)} = (\ \ldots\ \overline{Z}^a_J\ \ldots\ )$ are its fiber coordinates.

5.1 The Reduced Determining Equations.

Just as the original pseudo-group jets satisfy a system of differential equations $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$, so do the reduced pseudo-group jets. To construct this system, first define the *lifted subgroupoid* $\mathcal{H}^{(n)} \subset \mathcal{E}^{(n)}$ to be the pullback of $\mathcal{G}^{(n)}$ to J^n. We then define the n-th order *reduced pseudo-group jet bundle* by applying the reduction map (5.2) to the lifted subgroupoid:

$$\overline{\mathcal{G}}^{(n)} = \mathfrak{r}^{(n)}(\mathcal{H}^{(n)}) \subset \mathrm{J}^n(\mathcal{X}, \mathcal{U} \times M), \tag{5.5}$$

which can be written as a system of equations of the form

$$\overline{\mathcal{G}}^{(n)} = \big\{\, \overline{\Delta}_\nu(z^{(n)}, \overline{Z}^{(n)}) = 0, \quad \nu = 1, \ldots, \bar{l}_n \big\}. \tag{5.6}$$

If we fix a section $s = \{(x, u(x))\}$ with jet $z^{(n)} = \mathrm{j}_n s|_x = (x, u^{(n)}(x))$, then (5.6) can be viewed as an n-th order system of differential equations for the reduced diffeomorphism $\overline{Z} = \overline{\varphi}(x)$, that we call the *reduced determining equations*, whose properties will be investigated shortly.

In local coordinates, the reduced determining equations encode all the algebraic relations existing among the reduced jets $\overline{Z}^{(n)}$. These are obtained by writing out the formulas for the reduced jet coordinates in terms of the parametric pseudo-group jet coordinates, and then eliminating the latter from the resulting algebraic expressions, i.e., implicitizing the resulting parametric formulae, thereby producing the identities involving only the submanifold jet coordinates and the reduced jet coordinates.

Example 5.4. Recalling the determining equations (4.5) of the Lie pseudo-group (4.4), we now deduce the reduced determining equations, assuming that $u = u(x, y)$. The pseudo-group jet coordinates parametrizing $\mathcal{G}^{(2)}$ are given in (4.6). At order zero, we trivially have

$$\overline{X} = X, \qquad \overline{Y} = Y, \qquad \overline{U} = U.$$

Next, at order one,

$$\begin{aligned}
&\overline{X}_x = X_x + X_u u_x = X_x, && \overline{X}_y = X_y + X_u u_y = 0,\\
&\overline{Y}_x = Y_x + Y_u u_x = Y_x = (U-u)X_x, && \overline{Y}_y = Y_y + Y_u u_y = X_x,\\
&\overline{U}_x = U_x + U_u u_x = U_x + u_x, && \overline{U}_y = U_y + U_u u_y = U_y + u_y.
\end{aligned}$$

Differentiating again, and skipping details of the computations, at order two we obtain

$$\begin{aligned}
&\overline{X}_{xx} = U_y X_x, && \overline{X}_{xy} = 0, && \overline{X}_{yy} = 0,\\
&\overline{Y}_{xx} = \big(U_x + (U-u)U_y\big)X_x, && \overline{Y}_{xy} = U_y X_x, && \overline{Y}_{yy} = 0,\\
&\overline{U}_{xx} = U_{xx} + u_{xx}, && \overline{U}_{xy} = U_{xy} + u_{xy}, && \overline{U}_{yy} = u_{yy}.
\end{aligned}$$

Implicitization, i.e., eliminating the parametric variables $X, Y, U, X_x, U_x, U_y, U_{xx}, U_{xy}$, we find that, up to order two, the relations among the reduced pseudo-group jets are

$$\begin{gathered}
\overline{X}_y = 0, \qquad \overline{Y}_x = (\overline{U} - u)\overline{X}_x, \qquad \overline{Y}_y = \overline{X}_x, \qquad \overline{X}_{xx} = (\overline{U}_y - u_y)\overline{X}_x,\\
\overline{X}_{xy} = \overline{X}_{yy} = 0, \qquad \overline{Y}_{xx} = \big(\,\overline{U}_x - u_x + (\overline{U} - u)(\overline{U}_y - u_y)\,\big)\overline{X}_x,\\
\overline{Y}_{xy} = (\overline{U}_y - u_y)\overline{X}_x, \qquad \overline{Y}_{yy} = 0, \qquad \overline{U}_{yy} = u_{yy},
\end{gathered} \tag{5.7}$$

which thus form the second order reduced determining equations. We note that the parametric variables are $\overline{X}, \overline{Y}, \overline{U}, \overline{X}_x, \overline{U}_x, \overline{U}_y, \overline{U}_{xx}, \overline{U}_{xy}$.

A key observation that we will need in Section 7 is that the reduced determining equations must become identities when the pseudo-group element is the identity map, and hence the two sections coincide. Algebraically, this specialization amounts to equating

$$\begin{gathered}
\overline{X}^i = x^i, \qquad \overline{X}^i_i = 1, \qquad \overline{X}^i_J = 0, \qquad i = 1, \ldots, p, \qquad J \neq i, \qquad |J| \geq 1,\\
\overline{U}^\alpha_K = u^\alpha_K, \qquad \alpha = 1, \ldots, q, \qquad 0 \leq |K| \leq n.
\end{gathered} \tag{5.8}$$

The equations defining $\overline{\mathcal{G}}^{(n)}$ must vanish identically on the affine subvariety defined by (5.8). Thus, in the case of the pseudo-group in Example 5.4, every reduced determining equation in (5.7) vanishes identically when

$$\begin{gathered}
\overline{X}_x = \overline{Y}_y = 1, \qquad \overline{X}_y = \overline{Y}_x = \overline{X}_{xx} = \overline{X}_{xy} = \overline{X}_{yy} = \overline{Y}_{xx} = \overline{Y}_{xy} = \overline{Y}_{yy} = 0,\\
\overline{U} = u, \qquad \overline{U}_x = u_x, \qquad \overline{U}_y = u_y, \qquad \overline{U}_{yy} = u_{yy}.
\end{gathered} \tag{5.9}$$

According to [47, Proposition 2.10], this implies that the equations can be expressed as a linear combination

$$\Delta_\nu = \sum_{i=1}^{p} \left[A^i_\nu(\overline{X}^i - x^i) + A^{i,i}_\nu(\overline{X}^i_i - 1) + \sum_{\substack{J \neq i\\ 1 \leq |J| \leq n}} A^{i,J}_\nu \overline{X}^i_J \right] + \sum_{\alpha=1}^{q} \sum_{0 \leq |K| \leq n} B^{\alpha,K}_\nu (\overline{U}^\alpha_K - u^\alpha_K), \tag{5.10}$$

where the coefficient functions $A_\nu^i, A_\nu^{i,i}, A_\nu^{i,J}, B_\nu^{\alpha,K}$ are analytic.

We now state the key condition to be imposed on the pseudo-group actions to be considered in this paper.

Definition 5.5. The pseudo-group $\mathcal{G}$ is order n *reducible* on the local section $s\colon \mathcal{X} \to M$ if, for all $x \in \operatorname{dom} s$ with $z^{(n)} = \mathrm{j}_n s|_x$, the reduction map $\mathfrak{r}^{(n)}\colon \mathcal{H}^{(n)}|_{z^{(n)}} \to \overline{\mathcal{G}}^{(n)}|_{z^{(n)}}$ is one-to-one on the indicated fibers. The pseudo-group $\mathcal{G}$ is *reducible* on s if it is reducible for all sufficiently large $n \geq n_\natural$. The integer $n_\natural$ is called the *order of reducibility*.

As we will see in Theorem 6.5 below, all pseudo-groups for which the moving frame calculus is applicable automatically satisfy this condition on generic sections. In particular, this implies that any finite-dimensional Lie group action is reducible.

Definition 5.6. A section $s\colon \mathcal{X} \to M$ is called *regular* if $\mathcal{G}$ is reducible on it.

In what follows, we will only deal with regular sections. In particular, the reduced determining equations are to be evaluated only on regular sections. Assuming analyticity, if the pseudo-group is regular on one section, regularity holds on generic sections.

Let $\overline{\mathsf{d}}^{(n)}$ denote the fiber dimension of the reduced determining equations (5.6), which can be identified as the number of parametric reduced pseudo-group parameters of order $\leq n$. A simple property of reducible Lie pseudo-groups is given in the following result.

Lemma 5.7. *Let $\mathcal{G}$ be a reducible Lie pseudo-group with order of reducibility $n_\natural$. Then for all $n \geq n_\natural$, the number of independent reduced pseudo-group parameters equals the number of pseudo-group parameters. That is,*

$$\mathsf{d}^{(n)} = \overline{\mathsf{d}}^{(n)}. \tag{5.11}$$

In other words, reducibility requires that the reduction map does not change the fiber dimensions at sufficiently high orders. Since

$$0 \leq \mathsf{d}^{(n)} \leq (p+q)\binom{p+q+n}{n} \qquad \text{and} \qquad 0 \leq \overline{\mathsf{d}}^{(n)} \leq (p+q)\binom{p+n}{n}, \tag{5.12}$$

we see that reducibility imposes constraints on the size of the pseudo-group $\mathcal{G}$, in that it cannot be too large; see Lemma 5.11 below. For example, $\mathcal{G}$ cannot be the diffeomorphism pseudo-group $\mathcal{D}$, which maximizes the inequality for $\mathsf{d}^{(n)}$.

Example 5.8. Returning to Example 5.4, in view of (5.7) and its prolongations, it follows that the parametric reduced pseudo-group jets are

$$\overline{X},\quad \overline{Y},\quad \overline{U},\quad \overline{X}_x,\quad \overline{U}_{x^k},\quad \overline{U}_{x^{k-1}y},\quad k \geq 1. \tag{5.13}$$

Thus, the reduced dimensions satisfy

$$\overline{\mathsf{d}}^{(1)} = 6 = \mathsf{d}^{(1)}, \qquad \overline{\mathsf{d}}^{(2)} = 8 = \mathsf{d}^{(2)}, \qquad \text{and, in general,} \qquad \overline{\mathsf{d}}^{(n)} = 2n+4 = \mathsf{d}^{(n)},$$

thus proving that this pseudo-group is reducible.

Example 5.9. An example where $n_\natural > 1$ in Definition 5.5 is provided by the 5-dimensional Lie group action

$$X = a\,x + b, \qquad U = c\,u + d\,x + e,$$

where $a, c \neq 0$ and $b, d, e \in \mathbb{R}$. Up to order two, the determining equations are

$$X_u = X_{xx} = X_{xu} = X_{uu} = 0, \qquad U_{xx} = U_{xu} = U_{uu} = 0.$$

Prolonging, we deduce that, as expected,

$$\mathsf{d}^{(n)} = 5 \qquad \text{for all} \qquad n \in \mathbb{N}.$$

On the other hand, assuming the regularity condition $u_{xx} \neq 0$, the reduced determining equations, up to order three, are

$$\overline{X}_{xx} = \overline{X}_{xxx} = 0, \qquad \overline{U}_{xxx} = \frac{u_{xxx}}{u_{xx}}\,\overline{U}_{xx},$$

and $\overline{\mathsf{d}}^{(1)} = 4$, while $\overline{\mathsf{d}}^{(n)} = 5$ for $n \geq 2$. Thus, $\mathsf{d}^{(n)} = \overline{\mathsf{d}}^{(n)}$ for all $n \geq n_\natural = 2$.

Example 5.10. Consider the Lie pseudo-group

$$X = x, \qquad U = f(x, u).$$

In this case,

$$\mathsf{d}^{(n)} = \binom{n+2}{2} \qquad \text{while} \qquad \overline{\mathsf{d}}^{(n)} = n,$$

and hence the pseudo-group is not reducible, basically because it has a one-dimensional base but the transformations depend upon a function of two variables. It is also easily seen that the prolonged action on $\mathrm{J}^n(\mathbb{R}, \mathbb{R})$ is never free.

The last example can be easily generalized, proving that a reducible pseudo-group cannot depend on functions of $\geq p+1$ variables. We state this fact in terms of its Cartan characters.

Lemma 5.11. *If the pseudo-group is reducible, then* $\mathsf{c}_{n_\star}^{(p+\alpha)} = 0$ *for* $\alpha = 1, \ldots, q$.

Proof. For the purpose of contradiction, assume there is a Cartan character $\mathsf{c}_{n_\star}^{(p+\alpha)} \neq 0$ for some $\alpha = 1, \ldots, q$. The pseudo-group thus admits at least one function depending on at least $p+1$ variables, and hence

$$\mathsf{d}^{(n)} \geq \mathsf{a}_n = \binom{p+n+1}{n} = \frac{(p+2)\cdots(p+n+1)}{n!},$$

where a_n is the number of jet components of order $0 \leq |J| \leq n$ associated with a function $f(z^1, \ldots, z^{p+1})$ of $p+1$ variables.

On the other hand, according to (5.12),

$$\overline{\mathsf{d}}^{(n)} \leq (p+q)\binom{p+n}{n} = \frac{(p+q)\,(p+1)\,(p+2)\cdots(p+n)}{n!} = \frac{(p+1)(p+q)}{p+n+1}\,\mathsf{a}_n < \mathsf{a}_n \leq \mathsf{d}^{(n)},$$

whenever $n \geq \max\{n_\star, n_\natural, p^2 + pq + q\}$, and hence the reducibility condition (5.11) cannot hold when n is sufficiently large. *Q.E.D.*

5.2 The Linearized Reduced Determining Equations.

Linearizing the reduced determining equations (5.6) at the reduced identity pseudo-group jet yields the *linearized reduced determining equations*

$$\mathcal{L}^{(n)}_{\overline{\mathcal{G}}} = \big\{ \overline{L}_\nu(z^{(n)}, \overline{\zeta}^{(n)}) = 0, \quad \nu = 1, \ldots, \overline{l}_n \big\}. \tag{5.14}$$

Keeping only the highest order terms, we obtain the *reduced symbol*

$$\Sigma^n_{\overline{\mathcal{G}}} = \mathbf{H}(\mathcal{L}^{(n)}_{\overline{\mathcal{G}}}), \tag{5.15}$$

where, again, $\mathbf{H}$ is the highest order term map which only keeps the order n terms in the linearized reduced determining equations (5.14). On the other hand, the coefficient matrix of the reduced symbol (5.15) yields the n-th order reduced symbol matrix $M^n_{\overline{\mathcal{G}}}$, from which we can compute the reduced indices $\overline{\mathsf{b}}^{(i)}_n$ and reduced Cartan characters $\overline{\mathsf{c}}^{(i)}_n$ for $i = 1, \ldots, p$.

As in the previous section, we separate the reduced pseudo-group jets by order and let

$$\overline{\mathcal{D}}^{(n)} \simeq \mathrm{J}^n \times \overline{D}^{(n)} = \mathrm{J}^n \times \overline{D}^0 \times \overline{D}^1 \times \cdots \times \overline{D}^n,$$
$$\overline{\mathcal{G}}^{(n)} \simeq \mathrm{J}^n \times \overline{G}^{(n)} = \mathrm{J}^n \times \overline{G}^0 \times \overline{G}^1 \times \cdots \times \overline{G}^n,$$

where

$$\overline{D}^k = \{ (\ldots, \overline{Z}^a_B, \ldots) \,:\, |B| = k, \quad a = 1, \ldots, m \}$$

denotes the space of k-th order derivatives of reduced local diffeomorphisms and similarly for $\overline{G}^k$. The number of derivatives of order k is

$$\overline{\mathsf{t}}_k = \dim\, \overline{D}^k = m \binom{p+k-1}{k}.$$

Of those, the number of derivatives of class $1 \leq i \leq p$ is

$$\overline{\mathsf{t}}^{(i)}_k = m \binom{p+k-i-1}{k-1},$$

so that

$$\sum_{i=1}^{p} \overline{\mathsf{t}}^{(i)}_k = \overline{\mathsf{t}}_k, \qquad \sum_{i=1}^{p} i\, \overline{\mathsf{t}}^{(i)}_k = \overline{\mathsf{t}}_{k+1}, \qquad \sum_{k=1}^{n} \overline{\mathsf{t}}_k = \overline{\mathsf{t}}^{(n)} = \dim \overline{D}^{(n)}.$$

For the reduced Lie pseudo-group $\overline{\mathcal{G}}$, we let $\overline{\mathsf{d}}_k = \dim \overline{G}^k$ denote the number of parametric reduced pseudo-group parameters of order k, so that $\overline{\mathsf{d}}_0 + \cdots + \overline{\mathsf{d}}_n = \overline{\mathsf{d}}^{(n)} = \dim \overline{G}^{(n)}$. The number of principal reduced pseudo-group parameters of order k is then given by

$$\overline{\mathsf{r}}_k = \overline{\mathsf{t}}_k - \overline{\mathsf{d}}_k.$$

Finally, the indices and Cartan characters of the reduced determining equations (5.6) satisfy

$$\overline{\mathsf{b}}^{(i)}_n + \overline{\mathsf{c}}^{(i)}_n = \overline{\mathsf{t}}^{(i)}_n, \qquad i = 1, \ldots, p, \tag{5.16}$$

with

$$\sum_{i=1}^{p} \overline{\mathsf{b}}^{(i)}_n = \overline{\mathsf{r}}_n = \operatorname{rank} M^n_{\overline{\mathcal{G}}} \qquad \text{and} \qquad \sum_{i=1}^{p} \overline{\mathsf{c}}^{(i)}_n = \overline{\mathsf{d}}_n = \dim \Sigma^n_{\overline{\mathcal{G}}}.$$

5.3 Involutivity of the Reduced Determining System.

The aim of this section is to prove that the reduced determining system (5.6) is involutive. Moreover, the first p Cartan characters of the determining system and its reduction coincide.

Theorem 5.12. *Let $\mathcal{G}$ be reducible. Then there exists $\overline{n}_\star \in \mathbb{N}$ such that for all $n \geq \overline{n}_\star \geq \max\{n_\star, n_\natural\}$,*

$$\mathsf{c}_n^{(i)} = \overline{\mathsf{c}}_n^{(i)}, \qquad i = 1, \ldots, p, \qquad\qquad \mathsf{c}_n^{(p+\alpha)} = 0, \qquad \alpha = 1, \ldots, q. \tag{5.17}$$

In particular, the involutivity of the n-th order determining equations $\mathcal{G}^{(n)}$ implies the involutivity of the n-th order reduced determining equations $\overline{\mathcal{G}}^{(n)}$.

Proof. First of all, the second set of equalities in (5.17) follows from Lemma 5.11 with $n \geq n_\star$.

Since $\mathcal{G}$ is reducible, consider the reduced determining equations $\overline{\mathcal{G}}^{(n_\natural)}$, where $n_\natural$ is the order of reducibility. By the Cartan–Kuranishi Theorem [63, Theorem 7.4.1], after prolongation and projection, there exists $\overline{n}_\star \geq n_\natural$ such that the reduced determining equations $\overline{\mathcal{G}}^{(\overline{n}_\star)}$ are involutive.

Let $n \geq \overline{n}_\star \geq \max\{n_\star, n_\natural\}$ and $k \geq 1$. By the definition (3.4) of the Cartan characters

$$\mathsf{d}^{(n+k)} = \mathsf{d}^{(n+k-1)} + \sum_{i=1}^{p} \mathsf{c}_{n+k}^{(i)} \qquad \text{and} \qquad \overline{\mathsf{d}}^{(n+k)} = \overline{\mathsf{d}}^{(n+k-1)} + \sum_{i=1}^{p} \overline{\mathsf{c}}_{n+k}^{(i)}.$$

Reducibility of the pseudo-group implies that $\mathsf{d}^{(n+k)} = \overline{\mathsf{d}}^{(n+k)}$ and $\mathsf{d}^{(n+k-1)} = \overline{\mathsf{d}}^{(n+k-1)}$, which requires

$$\sum_{i=1}^{p} \left(\mathsf{c}_{n+k}^{(i)} - \overline{\mathsf{c}}_{n+k}^{(i)}\right) = 0. \tag{5.18}$$

Using [63, Eq. (8.8a)], the higher order Cartan characters are related via the equation

$$\mathsf{c}_{n+k}^{(i)} = \sum_{j=i}^{p} \binom{k+j-i-1}{k-1} \mathsf{c}_n^{(j)}, \qquad i = 1, \ldots, p. \tag{5.19}$$

Thus,

$$\begin{aligned} \sum_{i=1}^{p} \mathsf{c}_{n+k}^{(i)} &= \sum_{i=1}^{p} \sum_{j=i}^{p} \binom{k+j-i-1}{k-1} \mathsf{c}_n^{(j)} = \sum_{j=1}^{p} \sum_{i=1}^{j} \binom{k+j-i-1}{k-1} \mathsf{c}_n^{(j)} \\ &= \sum_{j=1}^{p} \binom{k-1+j}{k} \mathsf{c}_n^{(j)} = \mathsf{c}_n^{(1)} + (k+1)\mathsf{c}_n^{(2)} + \cdots + \frac{(k+1)\cdots(k+p-1)}{(p-1)!} \mathsf{c}_n^{(p)}. \end{aligned}$$

Substituting the last expression and its reduced version into (5.18), we obtain

$$\left(\mathsf{c}_n^{(1)} - \overline{\mathsf{c}}_n^{(1)}\right) + (k+1)\left(\mathsf{c}_n^{(2)} - \overline{\mathsf{c}}_n^{(2)}\right) + \cdots + \frac{(k+1)\cdots(k+p-1)}{(p-1)!}\left(\mathsf{c}_n^{(p)} - \overline{\mathsf{c}}_n^{(p)}\right) = 0.$$

Viewing this expression as a degree $p-1$ polynomial in the variable k which vanishes for all $k \in \mathbb{N}$, we conclude that $\mathsf{c}_n^{(i)} - \overline{\mathsf{c}}_n^{(i)} = 0$ for $i = 1, \ldots, p$. *Q.E.D.*

Remark 5.13. Theorem 5.12 implies that if $\mathcal{G}$ is reducible, then, at a sufficiently high order, the determining equations of the pseudo-group and the reduced determining system contain the same number of parametric pseudo-group jets and, furthermore, their first p Cartan characters are the same.

Example 5.14. Continuing Example 5.4, we linearize the second order reduced determining equations (5.7) at the reduced identity jet and obtain

$$\begin{gathered} \overline{\xi}_y = 0, \qquad \overline{\eta}_x = \overline{\phi}, \qquad \overline{\eta}_y = \overline{\xi}_x, \\ \overline{\xi}_{xx} = \overline{\phi}_y, \qquad \overline{\xi}_{xy} = \overline{\xi}_{yy} = 0, \qquad \overline{\eta}_{xx} = \overline{\phi}_x, \qquad \overline{\eta}_{xy} = \overline{\phi}_y, \qquad \overline{\eta}_{yy} = \overline{\phi}_{yy} = 0. \end{gathered} \tag{5.20}$$

The order two reduced symbol is then given by the equations

$$\overline{\xi}_{xx} = \overline{\xi}_{xy} = \overline{\xi}_{yy} = \overline{\eta}_{xx} = \overline{\eta}_{xy} = \overline{\eta}_{yy} = \overline{\phi}_{yy} = 0$$

so that the reduced indices and Cartan characters are

$$\overline{\mathsf{b}}_2^{(1)} = 4, \qquad \overline{\mathsf{b}}_2^{(2)} = 3, \qquad \overline{\mathsf{c}}_2^{(1)} = 2, \qquad \overline{\mathsf{c}}_2^{(2)} = 0.$$

On the other hand, order 3 reduced determining equations are

$$\begin{gathered} \overline{X}_{xxx} = \big((\overline{U}_y - u_y)^2 + (\overline{U}_{xy} - u_{xy}) \big) \overline{X}_x, \qquad \overline{X}_{xxy} = \overline{X}_{xyy} = \overline{X}_{yyy} = 0, \\ \overline{Y}_{xxx} = \big(\overline{U}_{xx} - u_{xx} + (\overline{U} - u) \big(\overline{U}_{xy} - u_{xy} + (\overline{U}_y - u_u)^2 \big) + 2 (\overline{U}_y - u_y)(\overline{U}_x - u_x) \big) \overline{X}_x, \\ \overline{Y}_{xxy} = \big(\overline{U}_{xy} - u_{xy} + (\overline{U}_y - u_y)^2 \big) \overline{X}_x, \quad \overline{Y}_{xyy} = \overline{Y}_{yyy} = 0, \quad \overline{U}_{xyy} = u_{xyy}, \quad \overline{U}_{yyy} = u_{yyy}, \end{gathered}$$

from which we see that $\overline{\mathsf{r}}_3 = 10$, $\overline{\mathsf{d}}_3 = 2$, and $\pi_2^3(\overline{\mathcal{G}}^{(3)}) = \overline{\mathcal{G}}^{(2)}$. Since

$$\overline{\mathsf{b}}_2^{(1)} + 2\,\overline{\mathsf{b}}_2^{(2)} = \overline{\mathsf{r}}_3 \qquad \text{or, equivalently,} \qquad \overline{\mathsf{c}}_2^{(1)} + 2\,\overline{\mathsf{c}}_2^{(2)} = \overline{\mathsf{d}}_3,$$

the reduced determining equations (5.7) of order $\overline{n}_\star = 2$ are involutive.

Remark 5.15. In the previous example, the order at which the reduced determining equations became involutive, was the same as the order of the original determining equations (recall Example 4.1) , i.e. $\overline{n}_\star = n_\star = 2$. The next example shows that this does not always hold, and that, in general, $\overline{n}_\star \geq n_\star$.

Example 5.16. To illustrate the second half of the prededing remark, consider the Lie pseudo-group

$$X = x + a, \qquad Y = y + b, \qquad U = f(x)\, u + g(x)\, y + h(x),$$

with $f \in \mathcal{D}(\mathbb{R})$, $g, h \in C^\infty(\mathbb{R})$, and $a, b \in \mathbb{R}$. The determining equations, up to order two, are

$$\begin{gathered} X_x = Y_y = 1, \qquad X_y = X_u = Y_x = Y_u = 0, \\ X_{xx} = X_{xy} = X_{yy} = X_{xu} = X_{yu} = X_{uu} = 0, \\ Y_{xx} = Y_{xy} = Y_{yy} = Y_{xu} = Y_{yu} = Y_{uu} = 0, \qquad U_{yy} = U_{yu} = U_{uu} = 0. \end{gathered}$$

The corresponding indices and Cartan characters are

$$\mathsf{b}_2^{(1)} = \mathsf{b}_2^{(2)} = 6, \quad \mathsf{b}_2^{(3)} = 3, \qquad \mathsf{c}_2^{(1)} = 3, \quad \mathsf{c}_2^{(2)} = \mathsf{c}_2^{(3)} = 0.$$

Computing the order three determining equations, we obtain

$$X_{xxx} = X_{xxy} = X_{xxu} = X_{xyy} = X_{xyu} = X_{xuu} = X_{yyy} = X_{yyu} = X_{yuu} = X_{uuu} = 0,$$
$$Y_{xxx} = Y_{xxy} = Y_{xxu} = Y_{xyy} = Y_{xyu} = Y_{xuu} = Y_{yyy} = Y_{yyu} = Y_{yuu} = Y_{uuu} = 0,$$
$$U_{xyy} = U_{yyy} = U_{yyu} = U_{xyu} = U_{yuu} = U_{xuu} = U_{uuu} = 0.$$

Since there are no integrability conditions, and moreover,

$$\mathsf{b}_2^{(1)} + 2\,\mathsf{b}_2^{(2)} + 3\,\mathsf{b}_2^{(3)} = 27 = \mathsf{r}_3, \qquad \mathsf{c}_2^{(1)} + 2\,\mathsf{c}_2^{(2)} + 3\,\mathsf{c}_2^{(3)} = 3 = \mathsf{d}_3,$$

this proves involutivity at order $n_\star = 2$. On the other hand, the reduced determining equations, up to order two, are

$$\overline{X}_x = \overline{Y}_y = 1, \qquad \overline{X}_y = \overline{Y}_x = 0, \qquad \overline{X}_{xx} = \overline{X}_{xy} = \overline{X}_{yy} = \overline{Y}_{xx} = \overline{Y}_{xy} = \overline{Y}_{yy} = 0, \tag{5.21}$$

with reduced indices $\overline{\mathsf{b}}_2^{(1)} = 4$, $\overline{\mathsf{b}}_2^{(2)} = 2$, and reduced Cartan characters $\overline{\mathsf{c}}_2^{(1)} = 2$, $\overline{\mathsf{c}}_2^{(2)} = 1$. Furthermore, provided the regularity condition $u_{yy} \neq 0$ holds, the order 3 reduced determining equations are

$$\overline{X}_{xxx} = \overline{X}_{xxy} = \overline{X}_{xyy} = \overline{X}_{yyy} = 0, \qquad \overline{Y}_{xxx} = \overline{Y}_{xxy} = \overline{Y}_{xyy} = \overline{Y}_{yyy} = 0, \qquad \overline{U}_{yyy} = \frac{u_{yyy}}{u_{yy}}\,\overline{U}_{yy}.$$

Therefore, the involutivity test $\overline{\mathsf{b}}_2^{(1)} + 2\,\mathsf{b}_2^{(2)} = 8 \neq \overline{\mathsf{r}}_3 = 9$ fails, as does $\overline{\mathsf{c}}_2^{(1)} + 2\,\overline{\mathsf{c}}_2^{(2)} = 4 \neq \overline{\mathsf{d}}_3 = 3$. On the other hand, the reduced determining equations become involutive at order $\overline{n}_\star = 3$ with

$$\overline{\mathsf{b}}_3^{(1)} = 6, \quad \overline{\mathsf{b}}_3^{(2)} = 3, \quad \overline{\mathsf{c}}_3^{(1)} = 3, \quad \overline{\mathsf{c}}_3^{(2)} = 0, \quad \overline{\mathsf{b}}_3^{(1)} + 2\,\overline{\mathsf{b}}_3^{(2)} = 12 = \overline{\mathsf{r}}_4, \quad \overline{\mathsf{c}}_3^{(1)} + 2\,\overline{\mathsf{c}}_3^{(2)} = 3 = \overline{\mathsf{d}}_4.$$

Remark 5.17. According to Theorem 5.12, the conditions (5.17) on the Cartan characters eventually hold whenever the Lie pseudo-group is reducible. We note that (5.17) may also hold for some non-reducible pseudo-groups, and that these equalities imply the involutivity of the associated determining equations. Indeed, assume (5.17) holds for all $n \geq n_\diamond \geq n_\star$, for some natural number $n_\diamond$. First, (4.3) and (5.16), together with (5.17), imply $\mathsf{d}_n = \overline{\mathsf{d}}_n$. Similarly, at order $n+1$ we have $\mathsf{d}_{n+1} = \overline{\mathsf{d}}_{n+1}$. Combining the last equality with (5.17), we conclude that

$$\sum_{i=1}^{p} i\,\overline{\mathsf{c}}_n^{(i)} = \sum_{a=1}^{m} a\,\mathsf{c}_n^{(a)} = \mathsf{d}_{n+1} = \overline{\mathsf{d}}_{n+1}.$$

Thus, the reduced determining equations $\overline{\mathcal{G}}^{(n)}$ satisfy the algebraic involutivity test. Moreover, since $\mathcal{G}^{(n)}$ is involutive, $\pi_n^{n+1}(\mathcal{G}^{(n+1)}) = \mathcal{G}^{(n)}$, which implies $\pi_n^{n+1}(\mathcal{H}^{(n+1)}) = \mathcal{H}^{(n)}$. Then, using (5.5),

$$\overline{\pi}_n^{n+1}\big(\overline{\mathcal{G}}^{(n+1)}\big) = \overline{\pi}_n^{n+1}\big(\mathfrak{r}^{n+1}(\mathcal{H}^{(n+1)})\big) = \mathfrak{r}^{(n)}\big(\pi_n^{n+1}(\mathcal{H}^{(n+1)})\big) = \mathfrak{r}^{(n)}(\mathcal{H}^{(n)}) = \overline{\mathcal{G}}^{(n)},$$

which thereby proves involutivity of the reduced determining equations $\overline{\mathcal{G}}^{(n)}$.

We now illustrate the remark with an example.

Example 5.18. Consider the pseudo-group action

$$X = x + a, \qquad U = \lambda\, u + f(x),$$

where $a, \lambda \in \mathbb{R}$, with $\lambda \neq 0$, and $f(x) \in C^\infty(\mathbb{R})$. Up to order two, the determining equations are

$$X_x = 1, \quad X_u = 0, \qquad X_{xx} = X_{xu} = X_{uu} = U_{xu} = U_{uu} = 0.$$

These equations are involutive with indices and Cartan characters

$$\mathsf{b}_2^{(1)} = 3, \qquad \mathsf{b}_2^{(2)} = 2, \qquad \mathsf{c}_2^{(1)} = 1, \qquad \mathsf{c}_2^{(2)} = 0. \tag{5.22}$$

The number of parametric pseudo-group jets of order $\leq k \in \mathbb{N}$ is $\mathsf{d}^{(k)} = k + 3$. On the other hand, assuming $u = u(x)$, the reduced determining equations of order ≤ 2 are

$$\overline{X}_x = 1, \qquad \overline{X}_{xx} = 0.$$

At order two, we have the reduced index and reduced Cartan character

$$\overline{\mathsf{b}}_2^{(1)} = 1, \qquad \overline{\mathsf{c}}_2^{(1)} = 1, \tag{5.23}$$

while the dimension of the reduced pseudo-group jet bundles are $\overline{\mathsf{d}}^{(k)} = k+2$. Since $\overline{\mathsf{d}}^{(k)} < \mathsf{d}^{(k)}$, the pseudo-group is non-reducible. But (5.22) and (5.23) satisfy (5.17) when $n \geq 2$.

6 Reduced Moving Frames and Normal Forms.

In this section, we review the moving frame construction for infinite-dimensional Lie pseudo-groups, as originally introduced in [54]. Restricting ourselves to reducible Lie pseudo-groups, we work with the reduced pseudo-group jets rather than the original jets.

Let $\mathcal{G}$ be a reducible Lie pseudo-group acting on (local) sections $s = \{(x, u(x))\}$ of the bundle $\pi\colon M \to \mathcal{X}$. For transformations near the identity $\mathbb{1}_M$, the transformed submanifold $S = \varphi(s)$ remains a section. The prolonged action on the n-th order submanifold jet space J^n is obtained by applying the *implicit total derivative operators*

$$\mathrm{D}_{X^i} = \sum_{j=1}^{p} W_i^j\, \mathrm{D}_{x^j}, \tag{6.1}$$

where $(W_i^j) = (\overline{X}_j^i)^{-1}$ denotes the entries of the inverse reduced total Jacobian matrix (which can be simplified using the determining equations), to the reduced target dependent variables $\overline{U}^\alpha$:

$$\widehat{U}_J^\alpha = \mathrm{D}_X^J \overline{U}^\alpha = \mathrm{D}_{X^{j_1}} \cdots \mathrm{D}_{X^{j_k}} \overline{U}^\alpha. \tag{6.2}$$

If $\overline{g}^{(n)}$ denotes the parametric reduced pseudo-group parameters of $\overline{\mathcal{G}}^{(n)}$, then, as a consequence of the formula (6.1) for the implicit total derivative operators, the prolonged action (6.2) can be written in terms of the submanifold jet coordinates $(x, u^{(n)})$ and the *parametric* pseudo-group jets $\overline{g}^{(n)}$:

$$(\overline{X}, \widehat{U}^{(n)}) = P^{(n)}(x, u^{(n)}, \overline{g}^{(n)}). \tag{6.3}$$

Example 6.1. We compute the prolonged action for the Lie pseudo-group (4.4) acting on surfaces $u = u(x,y)$. In doing so, we take into account the reduced determining equations (5.7). In particular, we recall that the reduced parametric pseudo-group jets are given in (5.13). Thus, the lifted total derivative operators (6.1) are

$$\mathrm{D}_X = \frac{1}{\overline{X}_x}\mathrm{D}_x - \frac{\overline{Y}_x}{\overline{X}_x^2}\,\mathrm{D}_y = \frac{1}{\overline{X}_x}\big[\,\mathrm{D}_x + (u-\overline{U})\,\mathrm{D}_y\,\big], \qquad \mathrm{D}_Y = \frac{1}{\overline{X}_x}\,\mathrm{D}_y.$$

The coordinate expressions for the prolonged action at order three are

$$\begin{aligned}
\widehat{U}_X &= \frac{\overline{U}_x + (u-\overline{U})\overline{U}_y}{\overline{X}_x}, & \widehat{U}_Y &= \frac{\overline{U}_y}{\overline{X}_x},\\
\widehat{U}_{XX} &= \frac{\overline{U}_{xx} + (u_y-\overline{U}_y)\overline{U}_x + (u_x-\overline{U}_x)\overline{U}_y + (u-\overline{U})\big(2\overline{U}_{xy} + 2(u-\overline{U})u_{yy} + (u_y-\overline{U}_y)\overline{U}_x\big)}{\overline{X}_x^2}, \\
\widehat{U}_{XY} &= \frac{\overline{U}_{xy} + (u_y-\overline{U}_y)\overline{U}_y + (u-\overline{U})u_{yy}}{\overline{X}_x^2}, & \widehat{U}_{YY} &= \frac{u_{yy}}{\overline{X}_x^2},\\
\widehat{U}_{XYY} &= \frac{u_{xyy} + 2(u_y-\overline{U}_y)u_{yy} + (u-\overline{U})u_{yyy}}{\overline{X}_x^3}, & \widehat{U}_{YYY} &= \frac{u_{yyy}}{\overline{X}_x^3}.
\end{aligned} \tag{6.4}$$

Observe that, as stated in (6.3), the resulting formulas only depend on the reduced parametric pseudo-group jets and the submanifold jets.

Definition 6.2. . Let $\overline{\mathcal{H}}^{(n)} \to \mathrm{J}^n$ denote the lifted subgroupoid obtained by pulling back $\overline{\mathcal{G}}^{(n)} \to M$ to J^n. A *reduced moving frame* $\overline{\rho}^{(n)}$ of order n is a $\overline{\mathcal{G}}^{(n)}$ equivariant local section: $\overline{\rho}^{(n)} \colon \mathrm{J}^n \to \overline{\mathcal{H}}^{(n)}$.

Remark 6.3. The moving frame introduced in Definition 6.2 differs from the original definition given in [54] since it is based on the prolonged action of the reduced pseudo-group $\overline{\mathcal{G}}$ rather than the original pseudo-group $\mathcal{G}$. For non-reducible Lie pseudo-group actions, the two notions differ, whereas, as we now explain, for reducible pseudo-groups they are equivalent. We will discuss the explicit construction of a reduced moving frame through the choice of a cross-section to the pseudo-group orbits in Section 6.2 below.

In the original implementation, a moving frame exists at order n provided the prolonged action is regular and (locally) free, see [54] for more details.

Definition 6.4. The pseudo-group $\mathcal{G}$ acts *freely* at $z^{(n)} \in \mathrm{J}^n$ if its isotropy group $\mathcal{G}^{(n)}_{z^{(n)}} = \{\, g^{(n)} \in \mathcal{G}^{(n)} \,|\, g^{(n)} \cdot z^{(n)} = z^{(n)} \,\}$ is trivial: $\mathcal{G}^{(n)}_{z^{(n)}} = \{\, \mathbb{1}^{(n)}_z \,|\, \pi^n_0(z^{(n)}) = z \,\}$, i.e., the only pseudo-group jet fixing $z^{(n)}$ is the identity. The pseudo-group acts *locally freely* at $z^{(n)}$ if $\mathcal{G}^{(n)}_{z^{(n)}}$ is a discrete group.

Once the pseudo-group acts freely[4] on an open subset $V^{(n)} \subset \mathrm{J}^n$ for some n, persistence of freeness, [54, 56], implies that $\mathcal{G}$ acts freely on the open subset $V^{(n+k)} = (\pi^{n+k})^{-1} V^{(n)}$. We now observe that freeness of the prolonged action implies the reducibility of the Lie pseudo-group action.

[4]In general, one expects singular jets in J^n where the action is not free.

Theorem 6.5. *If $\mathcal{G}$ acts freely on the open subset $V^{(n)} \subset \mathrm{J}^n(\mathcal{X}, M)$ then it is order n reducible on any section whose jet lies in $V^{(n)}$.*

Proof. Note that the identity reduced jet $\overline{\mathbb{1}}_z^{(n)}$ fixes any jet $z^{(n)} \in \mathrm{J}^n$, where $z = \pi_0^n(z^{(n)})$. Thus, because the action of $\mathcal{G}$ on J^n factors through the reduced action, each element of

$$(\mathfrak{r}^{(n)})^{-1}\{\overline{\mathbb{1}}_z^{(n)}\} \cap \mathcal{H}^{(n)} \tag{6.5}$$

fixes $z^{(n)}$. If the action is not reducible, the subset (6.5) will contain non-identity jets, and hence the isotropy subgroup of $z^{(n)}$ will be non-trivial. *Q.E.D.*

Theorem 6.5 implies that once the prolonged action becomes free, the reduced prolonged action is also free, that is, the isotropy group $\overline{\mathcal{G}}_{z^{(n)}}^{(n)}$ is trivial. For a reducible Lie pseudo-group, the converse is also true.

Theorem 6.6. *Let $\mathcal{G}$ be reducible on $z^{(n)}$. If the prolonged action of the reduced pseudo-group $\overline{\mathcal{G}}$ is free at $z^{(n)}$, then $\mathcal{G}^{(n)}$ acts freely at $z^{(n)}$.*

Proof. Since $\mathcal{G}$ is reducible, the isotropy group $\mathcal{G}_{z^{(n)}}^{(n)}$ must also contain a single jet. Since $\mathbb{1}_z^{(n)} \in \mathcal{G}_{z^{(n)}}^{(n)}$, it follows that $\mathcal{G}_{z^{(n)}}^{(n)} = \{\mathbb{1}_z^{(n)}\}$. *Q.E.D.*

Theorems 6.5 and 6.6 imply that for reducible Lie pseudo-groups we can go back and forth between the construction of a moving frame for the original pseudo-group $\mathcal{G}$ and the reduced pseudo-group $\overline{\mathcal{G}}$.

6.1 Isotropy.

According to the preceding discussion, there are two types of isotropy of a submanifold jet — those where the reduced action fixes the jet, and, more restrictively, those with trivial reduced action. Let us characterize them for better understanding. Note that the observations in this subsection are not used in the subsequent developments, and can thus be skipped without loss of continuity.

Given the submanifold jet $z^{(n)} \in \mathrm{J}^n$, let $\mathcal{D}_{z^{(n)}}^{(n)} \subset \mathcal{D}^{(n)}$ denote its isotropy subgroup of order n, i.e., the set of n-jets of local diffeomorphisms which fix $z^{(n)}$. Let $\mathcal{T}_{z^{(n)}}^{(n)} \subset \mathcal{D}_{z^{(n)}}^{(n)}$ be those isotropy elements which have trivial reduction. We can thus identify $\mathcal{T}_{z^{(n)}}^{(n)} \simeq (\mathfrak{r}^{(n)})^{-1}\{\overline{\mathbb{1}}_z^{(n)}\}$ where we are now applying the reduction map $\mathfrak{r}^{(n)}$ — see (5.2) — to an arbitrary diffeomorphism jet. Let $\mathcal{Q}_{z^{(n)}}^{(n)} = \mathcal{D}_{z^{(n)}}^{(n)} / \mathcal{T}_{z^{(n)}}^{(n)}$ denote the quotient space.

We now investigate $\mathcal{D}_{z^{(n)}}^{(n)}$, $\mathcal{T}_{z^{(n)}}^{(n)}$, and $\mathcal{Q}_{z^{(n)}}^{(n)}$. By applying a suitable diffeomorphism, we can, without loss of generality, assume that our section s is, locally, the trivial zero section, $u(x) \equiv 0$, with zero n jet, so $z^{(n)} = 0^{(n)}$. In this setting, a diffeomorphism 1-jet $Z^{(1)} = (X^{(1)}, U^{(1)})$ belongs to $\mathcal{T}_{0^{(1)}}^{(1)}$ if and only if

$$X = U = 0, \qquad \text{and} \qquad \begin{aligned} \delta_j^i &= \overline{X}_j^i = X_{x^j}^i + \sum_{\beta=1}^q u_j^\beta X_{u^\beta}^i = X_{x^j}^i, \\ 0 &= u_j^\alpha = \overline{U}_j^\alpha = U_{x^j}^\alpha + \sum_{\beta=1}^q u_j^\beta U_{u^\beta}^\alpha = U_{x^j}^\alpha, \end{aligned} \qquad \begin{aligned} i, j &= 1, \ldots, p, \\ \alpha &= 1, \ldots, q, \end{aligned}$$

where δ^i_j is the Kronecker delta. On the other hand, $Z^{(1)} \in \mathcal{D}^{(1)}_{0^{(1)}}$ if and only if $X = U = \widehat{U}^\alpha_{X^j} = \mathrm{D}_{X^j}(U^\alpha) = 0$, with $j = 1, \ldots, p$, and $\alpha = 1, \ldots, q$. Since the matrix (W^i_j) in the definition of the total derivative operators (6.1) is invertible, the constraints for $Z^{(1)}$ to be in $\mathcal{D}^{(1)}_{0^{(1)}}$ are

$$X = U = 0, \qquad 0 = u^\alpha_j = \overline{U}^\alpha_j = U^\alpha_{x^j} + \sum_{\beta=1}^q u^\beta_j U^\alpha_{u^\beta} = U^\alpha_{x^j}, \qquad \begin{array}{c} j = 1, \ldots, p, \\ \alpha = 1, \ldots, q. \end{array}$$

By similar computations, $Z^{(n)} \in \mathcal{T}^{(n)}_{0^{(n)}}$ if and only if

$$X = U = 0, \quad X^i_{x^j} = \delta^i_j, \qquad X^i_J = 0, \qquad |J| \geq 2, \qquad U^\alpha_J = 0, \qquad |J| \geq 1,$$

while $Z^{(n)} \in \mathcal{D}^{(n)}_{0^{(n)}}$ if and only if

$$X = U = 0, \qquad U^\alpha_J = 0, \qquad |J| \geq 1.$$

In other words, at $x = 0$, $\mathcal{D}^{(n)}_{0^{(n)}}$ consists of n jets of diffeomorphisms of the form

$$X = f(x,u), \qquad U = u\, g(x,u), \qquad f(0,0) = 0, \qquad \det(f^i_{x^j})|_{(0,0)} \neq 0, \qquad \prod_{\alpha=1}^q g^\alpha(0,0) \neq 0,$$

while $\mathcal{T}^{(n)}_{0^{(n)}}$ consists of n jets of diffeomorphisms of the form

$$X = x + u\, h(x,u), \qquad U = u\, g(x,u), \qquad \prod_{\alpha=1}^q g^\alpha(0,0) \neq 0.$$

In particular, on the zero section, we have $X = x$ and hence $\mathcal{T}^{(n)}_{0^{(n)}}$ consists of n jets of diffeomorphisms which fix every single point of s, i.e., the jets of the global isotropy subgroup of s. On the other hand, the quotient space $\mathcal{Q}^{(n)}_{0^{(n)}} = \mathcal{D}^{(n)}_{0^{(n)}} / \mathcal{T}^{(n)}_{0^{(n)}}$ can be identified with the space of local diffeomorphisms of the form

$$X = a(x), \qquad U = u, \qquad a(0) = 0, \qquad \det(a^i_j)(0) \neq 0.$$

These are just the reparametrizations of the zero section, which are extended to be diffeomorphisms with identical reparametrizations of the parallel sections, although the method of extension is unimportant and just selects a particular representative of the quotient space.

Thus, pseudo-groups whose reduced action is free differ from freely acting pseudo-groups only by the inclusion of some additional transformations that either belong to the global isotropy subgroup of the section and/or perform reparametrizations of sections. These all preserve the section, and thus do not affect the moving frame calculation nor the computations of differential invariants.

Example 6.7. Suppose $p = q = 1$, and consider the Lie pseudo-group action

$$X = x + a, \qquad U = f(x,u), \tag{6.6}$$

where $f_u \neq 0$. Since the reduced parametric pseudo-group jets are $\overline{X}$, $\overline{U}_{x^n}$, $n \geq 0$, and the prolonged action is $\widehat{U}_{X^n} = \overline{U}_{x^n}$, this pseudo-group admits a free reduced action. On the other hand, the pseudo-group (6.6) does not act freely anywhere on the jet space J^∞. When $p = 1$, $q = 2$, the extended pseudo-group

$$X = x + a, \qquad U = f(x, u), \qquad V = v + b, \tag{6.7}$$

is of the same form, and furthermore is intransitive and so has nontrivial differential invariants, namely v_{x^n} for all $n \geq 1$, despite the fact that it does not act freely. On the other hand, when $p = 2$, $q = 1$, the same pseudo-group

$$X = x + a, \qquad Y = y + b, \qquad U = f(x, u), \tag{6.8}$$

acts freely and transitively on the subset of jet space where $u_y \neq 0$ at all orders ≥ 1. We note that the pseudo-groups (6.6) and (6.7) are not reducible, while (6.8) is reducible by virtue of Theorem 6.5.

6.2 The Reduced Moving Frame Construction.

Coming back to the construction of a reduced moving frame, this is accomplished by selecting a cross-section $\mathcal{K}^{(n)} \subset \mathrm{J}^n$ that is transversal to the orbits of the prolonged group action (6.2). As in most applications, we will always assume that $\mathcal{K}^{(n)}$ is a coordinate cross-section defined by fixing $\mathsf{d}^{(n)}$ values of the individual jet coordinates $z^{(n)} = (x, u^{(n)})$ to suitable constants. Let

$$\mathcal{I}_{\mathcal{K}}^{(n)} \subset \{\, i,\ (\alpha, J) \mid\ i = 1, \ldots, p,\ \alpha = 1, \ldots, q,\ |J| \leq n \,\} \tag{6.9}$$

denote the set of indices of jet coordinates of order $\leq n$ that determine the cross-section, which is thus prescribed by $\mathsf{d}^{(n)} = \# \mathcal{I}_{\mathcal{K}}^{(n)}$ equations, of the form

$$\mathcal{K}^{(n)} = \{x^i = c^i,\ u^\alpha_J = c^\alpha_J \mid i,\, (\alpha, J) \in \mathcal{I}_{\mathcal{K}}^{(n)}\}, \tag{6.10}$$

for suitable constants c^i, c^α_J.

Given a cross-section (6.10), the *reduced right moving frame*[5] $\overline{g}^{(n)} = \overline{\rho}^{(n)}(X, \widehat{U}^{(n)})$ gives the reduced pseudo-group element that maps a submanifold jet $(X, \widehat{U}^{(n)})$ (that belongs to a suitable neighborhood of the cross-section) to the cross-section (normal form) jet $(x, u^{(n)})$ that lies in the same pseudo-group orbit. Freeness guarantees that the reduced pseudo-group element is uniquely determined. Conversely, the pseudo-group inverse $(\overline{g}^{(n)})^{-1} = (\overline{\rho}^{(n)}(X, \widehat{U}^{(n)}))^{-1}$ defines the reduced left moving frame that sends the cross-section jet $(x, u^{(n)})$ to the submanifold jet $(X, \widehat{U}^{(n)})$.

To explicitly determine the moving frame, we begin by switching $(X, \widehat{U}^{(n)})$ and $(x, u^{(n)})$ in the formulas (6.3) for the prolonged action of the (reduced) pseudo-group:

$$(x, u^{(n)}) = P^{(n)}(X, \widehat{U}^{(n)}, \overline{g}^{(n)}), \tag{6.11}$$

[5] By an abuse of notation, we use the same symbol to denote the pseudo-group normalization function and the corresponding moving frame section in Definition 6.2.

bearing in mind that here $\overline{g}^{(n)}$ parametrizes the inverse of the pseudo-group element that appears in (6.3). The reduced right moving frame is constructed by solving the *normalization equations*, which are obtained by equating the components of the preceding map corresponding to the choice of cross-section (6.10) to the corresponding normalization constants,

$$P^i(X,\widehat{U}^{(n)},\overline{g}^{(n)}) = c^i, \qquad P^\alpha_J(X,\widehat{U}^{(n)},\overline{g}^{(n)}) = c^\alpha_J, \qquad \text{with} \qquad i,\,(\alpha,J) \in \mathcal{I}^{(n)}_{\mathcal{K}}, \tag{6.12}$$

for the reduced parametric pseudo-group jets $\overline{g}^{(n)}$. Transversality of the cross-section and freeness of the reduced action guarantee, via the Implicit Function Theorem, that the normalization equations can be locally solved, nearby the cross-section, which produces the right moving frame:

$$\overline{g}^{(n)} = \overline{\rho}^{(n)}(X,\widehat{U}^{(n)}). \tag{6.13}$$

Substituting the moving frame expressions (6.13) into (6.11) produces the *normalized differential invariants*. Those corresponding to the cross-section coordinates, namely

$$c^i = P^i(X,\widehat{U}^{(n)},\overline{\rho}^{(n)}(X,\widehat{U}^{(n)})), \qquad c^\alpha_J = P^\alpha_J(X,\widehat{U}^{(n)},\overline{\rho}^{(n)}(X,\widehat{U}^{(n)})), \qquad i,\ (\alpha,J) \in \mathcal{I}^{(n)}_{\mathcal{K}},$$

reduce, by construction, to the normalization constants, and are known as the *phantom differential invariants*, whereas the remaining functions

$$\begin{aligned} H^j(X,\widehat{U}^{(n)}) &= P^j(X,\widehat{U}^{(n)},\overline{\rho}^{(n)}(X,\widehat{U}^{(n)})), \\ I^\beta_K(X,\widehat{U}^{(n)}) &= P^\alpha_J(X,\widehat{U}^{(n)},\overline{\rho}^{(n)}(X,\widehat{U}^{(n)})), \end{aligned} \qquad j,\ (\beta,K) \notin \mathcal{I}^{(n)}_{\mathcal{K}}, \tag{6.14}$$

form a complete system of functionally independent differential invariants of order $\leq n$, known as the *basic normalized differential invariants*, although in what follows "basic" will often be dropped from the terminology.

Example 6.8. Returning to our running example, under the assumption that $u_{yy} > 0$, a possible cross-section to the prolonged action (6.4) is given by

$$\mathcal{K}^{(2)} = \{x=0,\ y=0,\ u=c_0,\ u_x=c_1,\ u_y=d_0,\ u_{xx}=c_2,\ u_{xy}=d_1,\ u_{yy}=1\}, \tag{6.15}$$

where c_0, c_1, c_2, d_0, d_1 are arbitrary constants. More generally,

$$\mathcal{K}^{(\infty)} = \{x=0,\ y=0,\ u_{yy}=1,\ u_{x^k}=c_k,\ u_{x^ky}=d_k, \quad \text{for all} \quad k\geq 0\}. \tag{6.16}$$

Following the original papers [54, 55], and to simplify the computations, we set the arbitrary constants to zero, i.e., $c_k = d_k = 0$. Interchanging $\widehat{U}^{(3)} \longleftrightarrow u^{(3)}$ in the prolonged action (6.4), the normalization equations, up to order two, are obtained by substituting the cross-section determining equations (6.15) into the prolonged action:

$$0 = \overline{X}, \qquad 0 = \overline{Y}, \qquad 0 = \overline{U}, \qquad 0 = \frac{\overline{U}_x + (\widehat{U}-\overline{U})\overline{U}_y}{\overline{X}_x},$$

$$0 = \frac{\overline{U}_y}{\overline{X}_x}, \qquad 0 = \frac{\overline{U}_{xy} + (\widehat{U}_Y - \overline{U}_y)\overline{U}_y + (\widehat{U}-\overline{U})\widehat{U}_{YY}}{\overline{X}_x^2}, \qquad 1 = \frac{\widehat{U}_{YY}}{\overline{X}_x^2},$$

$$0 = \frac{\overline{U}_{xx} - (\widehat{U}_Y - \overline{U}_y)\overline{U}_x + (\widehat{U}_X - \overline{U}_x)\overline{U}_y + (\widehat{U}-\overline{U})(2\overline{U}_{xy} + 2(\widehat{U}-\overline{U})\widehat{U}_{YY} + (\widehat{U}_Y - \overline{U}_y)\overline{U}_x)}{\overline{X}_x^2},$$

where the reduced pseudo-group jets $(\overline{X}, \overline{Y}, \overline{U}, \overline{X}_x, \overline{U}_x, \overline{U}_y, \overline{U}_{xx}, \overline{U}_{xy})$ are evaluated at the source $(X, Y, \widehat{U}(X,Y))$. Solving these equations for the reduced pseudo-group parameters, we obtain the right moving frame

$$\begin{gathered} \overline{X} = 0, \qquad \overline{Y} = 0, \qquad \overline{U} = 0, \qquad \overline{X}_x = \sqrt{\widehat{U}_{YY}}, \qquad \overline{U}_x = 0, \qquad \overline{U}_y = 0, \\ \overline{U}_{xx} = 0, \qquad \overline{U}_{xy} = -\widehat{U}\widehat{U}_{YY}. \end{gathered} \tag{6.17}$$

Substituting the pseudo-group normalizations (6.17) into the last two equations of (6.4), with $\widehat{U}^{(3)} \longleftrightarrow u^{(3)}$, we obtain the third order normalized differential invariants

$$I_{1,2} = \frac{\widehat{U}_{XYY} + 2\widehat{U}_Y\widehat{U}_{YY} + \widehat{U}\widehat{U}_{YYY}}{\widehat{U}_{YY}^{3/2}}, \qquad I_{0,3} = \frac{\widehat{U}_{YYY}}{\widehat{U}_{YY}^{3/2}}. \tag{6.18}$$

The left moving frame is obtained by inverting the pseudo-group normalizations (6.17). The result is

$$\begin{gathered} \overline{X}(0) = X, \qquad \overline{Y}(0) = Y, \qquad \overline{U}(0) = \widehat{U}, \qquad \overline{X}_x(0) = \frac{1}{\sqrt{\widehat{U}_{YY}}}, \\ \overline{U}_x(0) = \frac{\widehat{U}_X + \widehat{U}\widehat{U}_Y}{\sqrt{\widehat{U}_{YY}}}, \qquad \overline{U}_y(0) = \frac{\widehat{U}_Y}{\sqrt{\widehat{U}_{YY}}}, \qquad \overline{U}_{xy}(0) = \frac{\widehat{U}_{XY} + \widehat{U}_Y^2 + \widehat{U}\widehat{U}_{YY}}{\widehat{U}_{YY}}, \\ \overline{U}_{xx}(0) = \frac{\widehat{U}_{XX} + 2\widehat{U}\widehat{U}_{XY} + 2\widehat{U}_Y\widehat{U}_X + 4\widehat{U}\widehat{U}_Y^2 - \widehat{U}\widehat{U}_Y\widehat{U}_X - \widehat{U}^2\widehat{U}_Y^2}{\widehat{U}_{YY}}. \end{gathered} \tag{6.19}$$

where the pseudo-group parameters are now evaluated at the origin $0 = (0,0,0)$. One can verify the validity of (6.19) by substituting the cross-section (6.15), with $c_0 = c_1 = d_0 = d_1 = 0$, and the pseudo-group normalizations (6.19) into the prolonged action (6.4) to obtain identities.

The method of (reduced) moving frames can naturally be formulated in terms of power series as shown in [54, Section 8]. As explained in Section 2, we identify the submanifold jet $(X, \widehat{U}^{(\infty)}) \in A^\infty$ of an analytic section with the (locally) convergent power series

$$\widehat{U}^\alpha(Y) = \sum_J \frac{\widehat{U}_J^\alpha}{J!}(Y - X)^J, \qquad \alpha = 1, \ldots, q, \tag{6.20}$$

centered at the point $X \in \mathcal{X}$.

Let $\mathcal{K} = \mathcal{K}^{(\infty)} \subset \mathrm{J}^\infty$ be a coordinate cross-section of infinite order. As in (6.9), we let $\mathcal{I}_\mathcal{K}$ denote the set of indices i, (α, J) of jet coordinates that prescribe the cross-section, as in (6.10). We further set

$$\mathcal{I}_\mathcal{K}^\alpha = \{\, J \mid (\alpha, J) \in \mathcal{I}_\mathcal{K} \,\}. \tag{6.21}$$

Given a left moving frame $\overline{\rho}^{(\infty)}$, a formal power series

$$u^\alpha(y) = \sum_J \frac{u_J^\alpha}{J!}(y - x)^J, \qquad \alpha = 1, \ldots, q, \tag{6.22}$$

is said to be a *normal form power series* of (6.20) if the right moving frame sends the submanifold power series (6.20) to the normal form power series (6.22) locally, or, inversely, the left moving frame sends the normal form power series (6.22) back to the submanifold power series (6.20). The coefficients u^α_J with $(\alpha, J) \in \mathcal{I}_\mathcal{K}$ represent the normalization constants prescribed by the cross-section, i.e., the phantom invariants, which serve to fix the normal form power series. The remaining coefficients u^β_K with $(\beta, K) \notin \mathcal{I}_\mathcal{K}$ represent the complete set of basic normalized differential invariants. Namely, given a submanifold $U = \widehat{U}(X)$ that maps to the prescribed normal form (6.22), the resulting formulas for the coefficients u^β_K in terms of the jet of $\widehat{U}$ are the differential invariants, as specified in (6.14).

From the normal form power series (6.22), we can extract the *cross-section power series*

$$C^\alpha(y) = \sum_{J \in \mathcal{I}^\alpha_\mathcal{K}} \frac{c^\alpha_J}{J!} (y-x)^J, \qquad \alpha = 1, \dots, q, \tag{6.23}$$

whose Taylor coefficients are the phantom invariants. If $\mathcal{I}^\alpha_\mathcal{K}$ is a finite set, then $C^\alpha(y)$ is a polynomial, while if $\mathcal{I}^\alpha_\mathcal{K} = \varnothing$, our convention is that $C^\alpha(y)$ does not exist.

Remark 6.9. Since the prolonged pseudo-group transformations only depend on the reduced pseudo-group jets, the moving frame method applies equally well to non-free actions whose reduced action is eventually free. However, we have, as yet, been unable to come up with any truly interesting examples, beyond the rather trivial ones in Example 6.7, and so, as in almost all other treatments of moving frames, we have restricted our attention to pseudo-groups which act freely on an open subset of jet space of suitably high order.

7 The Normal Form Determining Equations.

We now formulate a system of differential equations that a normal form must satisfy. These equations are obtained by suitably manipulating the reduced determining equations for the pseudo-group.

Consider two sections $s, S \subset M$ of the fibered manifold $\pi\colon M \to \mathcal{X}$. In local coordinates, the "source section" has the form $s = \{(x, u(x))\}$, while the "target section" is given by $S = \{(X, \widehat{U}(X))\}$. As in the previous section, we assume that the source section represents the normal form while the target section is a prescribed analytic section that we seek to normalize via a suitable pseudo-group diffeomorphism. In other words, we seek a diffeomorphism $\varphi \in \mathcal{G}$ such that, locally, $S = \varphi(s)$. In terms of the reduced pseudo-group, this requires

$$\overline{U} = \widehat{U}(\overline{X}) \qquad \text{or, more explicitly,} \qquad U(x, u(x)) = \widehat{U}(X(x, u(x))). \tag{7.1}$$

Now, consider the reduced determining equations

$$\overline{\mathcal{G}}^{(n)} = \left\{ \overline{\Delta}^{(n)}\big(x, u^{(n)}, \overline{X}^{(n)}, \overline{U}^{(n)}\big) = 0 \right\} \tag{7.2}$$

for the reduced pseudo-group diffeomorphism $\overline{\varphi}(x) = (\overline{X}, \overline{U})$. Recall that $u^{(n)}, \overline{X}^{(n)}, \overline{U}^{(n)}$ denote derivatives with respect to the source variables x up to order n. Applying the chain rule to differentiate the first equation in (7.1) yields formulae for the x derivatives of $\overline{U}$ in terms of the x derivatives of $\overline{X}$ and the X derivatives of $\widehat{U}$:

$$\overline{U}^{(n)} = \overline{\mathfrak{U}}^{(n)}(\overline{X}^{(n)}, \widehat{U}^{(n)}), \tag{7.3}$$

where $\widehat{U}^{(n)}$ denotes the derivatives of $\widehat{U}$ with respect to X up to order n. These can be explicitly computed by successively applying the *chain rule total differential operators*

$$\mathrm{D}_{x^i} = \sum_{j=1}^{p} \overline{X}^j_{x^i}\, \mathrm{D}_{X^j}, \qquad i = 1, \ldots, p, \tag{7.4}$$

to $\widehat{U}^\alpha$ for $\alpha = 1, \ldots, q$. For example, when $p = q = 1$, we have $\mathrm{D}_x = \overline{X}_x \,\mathrm{D}_X$, and hence, up to order two,

$$\overline{U}_x = \widehat{U}_X \overline{X}_x, \qquad \overline{U}_{xx} = \widehat{U}_{XX} \overline{X}_x^2 + \widehat{U}_X \overline{X}_{xx}$$

Substituting the expressions (7.3) into the reduced determining equations (7.2) produces the *normal form determining equations*

$$\mathcal{N}^{(n)} = \bigl\{ \widetilde{\Delta}^{(n)}\bigl(x, u^{(n)}, \overline{X}^{(n)}, \widehat{U}^{(n)}\bigr) = 0 \bigr\}. \tag{7.5}$$

Thus, given a prescribed function $\widehat{U} = \widehat{U}(\overline{X})$ defining a submanifold (section), whose derivatives $\widehat{U}^{(n)}$ are known, we can view (7.5) as an n-th order system of differential equations for the unknown functions $\overline{X}(x), u(x)$, the latter prescribing the normal form of the prescribed submanifold.

To investigate involutivity of the normal form determining equations, we linearize at the identity, keeping in mind that $\overline{X}^{(n)}$ and $u^{(n)}$ vary, while $\widehat{U}^{(n)}$ is fixed. The vector field used for linearization is

$$\sum_{0 \le |J| \le n} \left(\sum_{i=1}^{p} \overline{\xi}^i_J \frac{\partial}{\partial \overline{X}^i_J} + \sum_{\alpha=1}^{q} \psi^\alpha_J \frac{\partial}{\partial u^\alpha_J} \right).$$

We begin by linearizing the chain rule formula (7.3), for which we write out its individual components.

Lemma 7.1. *For any $\alpha = 1, \ldots, q$ and multi-index $J = (j_1, \ldots, j_n)$, the linearization of the chain rule equation*

$$\overline{U}^\alpha_J = \overline{\mathfrak{U}}^\alpha_J(\overline{X}^{(n)}, \widehat{U}^{(n)}) \tag{7.6}$$

at the identity is

$$\overline{\phi}^\alpha_J = \mathrm{D}^J_x \left(\sum_{i=1}^{p} u^\alpha_i \,\overline{\xi}^i \right) - \sum_{i=1}^{p} u^\alpha_{J,i}\, \overline{\xi}^i. \tag{7.7}$$

Proof: Linearization at the identity amounts to computing the infinitesimal generator of a one parameter group. In the case of (7.1), the group can be identified with the induced action of the inverse of the change of independent variables prescribed by $X = \overline{X}(x) = X(x, u(x))$ on the dependent variables u; for details, see the discussion on pages 105–106 of [47]. Because we are dealing with the inverse, the infinitesimal generator is

$$- \sum_{i=1}^{p} \overline{\xi}^i \frac{\partial}{\partial x^i},$$

which only acts on the independent variables. Linearizing the induced action on the derivatives (7.6) is the same as computing the prolongation of this vector field, which, according to [47, Theorem 2.36] is exactly given by the prolongation formula (7.7), the quantity in parentheses being its characteristic. *Q.E.D.*

Example 7.2. For example, suppose $p = 2$, with independent variables x, y, and $q = 1$, with dependent variable u. Applying

$$\mathrm{D}_x = \overline{X}_x \mathrm{D}_X + \overline{Y}_x \mathrm{D}_Y, \qquad \mathrm{D}_y = \overline{X}_y \mathrm{D}_X + \overline{Y}_y \mathrm{D}_Y,$$

once and twice to $\widehat{U}$ produces the first and second chain rule formulas

$$\begin{aligned}
\overline{U}_x &= \widehat{U}_X \overline{X}_x + \widehat{U}_Y \overline{Y}_x, \qquad \overline{U}_y = \widehat{U}_X \overline{X}_y + \widehat{U}_Y \overline{Y}_y,\\
\overline{U}_{xx} &= \widehat{U}_{XX} \overline{X}_x^2 + 2\,\widehat{U}_{XY} \overline{X}_x \overline{Y}_x + \widehat{U}_{YY} \overline{Y}_x^2 + \widehat{U}_X \overline{X}_{xx} + \widehat{U}_Y \overline{Y}_{xx},\\
\overline{U}_{xy} &= \widehat{U}_{XX} \overline{X}_x \overline{X}_y + \widehat{U}_{XY} (\overline{X}_x \overline{Y}_y + \overline{X}_y \overline{Y}_x) + \widehat{U}_{YY} \overline{Y}_x \overline{Y}_y + \widehat{U}_X \overline{X}_{xy} + \widehat{U}_Y \overline{Y}_{xy},\\
\overline{U}_{yy} &= \widehat{U}_{XX} \overline{X}_y^2 + 2\,\widehat{U}_{XY} \overline{X}_y \overline{Y}_y + \widehat{U}_{YY} \overline{Y}_y^2 + \widehat{U}_X \overline{X}_{yy} + \widehat{U}_Y \overline{Y}_{yy}.
\end{aligned} \tag{7.8}$$

Linearizing at the identity, where (5.9) holds, produces

$$\begin{aligned}
\overline{\phi}_x &= u_x \overline{\xi}_x + u_y \overline{\eta}_x = \mathrm{D}_x(u_x \overline{\xi} + u_y \overline{\eta}) - (u_{xx} \overline{\xi} + u_{xy} \overline{\eta}),\\
\overline{\phi}_y &= u_x \overline{\xi}_y + u_y \overline{\eta}_y = \mathrm{D}_y(u_x \overline{\xi} + u_y \overline{\eta}) - (u_{xy} \overline{\xi} + u_{yy} \overline{\eta}),\\
\overline{\phi}_{xx} &= u_x \overline{\xi}_{xx} + u_y \overline{\eta}_{xx} + 2\,u_{xx} \overline{\xi}_x + 2\,u_{xy} \overline{\eta}_x = \mathrm{D}_x^2(u_x \overline{\xi} + u_y \overline{\eta}) - (u_{xxx} \overline{\xi} + u_{xxy} \overline{\eta}),\\
\overline{\phi}_{xy} &= u_x \overline{\xi}_{xy} + u_y \overline{\eta}_{xy} + u_{xx} \overline{\xi}_x + u_{xy}(\overline{\xi}_y + \overline{\eta}_x) + u_{yy} \overline{\eta}_x = \mathrm{D}_x \mathrm{D}_y(u_x \overline{\xi} + u_y \overline{\eta}) - (u_{xxy} \overline{\xi} + u_{xyy} \overline{\eta}),\\
\overline{\phi}_{yy} &= u_x \overline{\xi}_{yy} + u_y \overline{\eta}_{yy} + 2\,u_{xy} \overline{\xi}_y + 2\,u_{yy} \overline{\eta}_y = \mathrm{D}_y^2(u_x \overline{\xi} + u_y \overline{\eta}) - (u_{xyy} \overline{\xi} + u_{yyy} \overline{\eta}),
\end{aligned} \tag{7.9}$$

in accordance with the general formula (7.7).

Theorem 7.3. *The linearization of the normal form determining equations* (7.5) *at the identity* $(\overline{X}, u^{(n)}) = (x, \widehat{U}^{(n)})$ *coincides with the linearization of the reduced determining equations* (7.2) *at the identity* $(\overline{X}, \overline{U}^{(n)}) = (x, u^{(n)})$ *after the substitutions*

$$\overline{\phi}^\alpha_J \ \longmapsto \ \mathrm{D}_x^J \left(\sum_{i=1}^p u^\alpha_i \overline{\xi}^i \right) - \sum_{i=1}^p u^\alpha_{J,i} \overline{\xi}^i - \psi^\alpha_J, \qquad \alpha = 1, \ldots, q, \qquad J = (j_1, \ldots, j_n). \tag{7.10}$$

Remark 7.4. The linearization of the normal form determining equations (7.5) in Theorem 7.3 occurs at the point $(x, \widehat{U}^{(n)})$. But since at the identity $u^{(n)} = \widehat{U}^{(n)}$, we may substitute $u^{(n)}$ for $\widehat{U}^{(n)}$ in the linearization, which is implicitly done in Theorem 7.3.

Proof: In view of (5.10), the linearized reduced determining equations have the form

$$\overline{L}_\nu = \sum_{i=1}^p \sum_{0 \le |J| \le n} A^{i,J}_{\nu,\mathbb{1}} \, \overline{\xi}^i_J + \sum_{\alpha=1}^q \sum_{0 \le |K| \le n} B^{\alpha,K}_{\nu,\mathbb{1}} \, \overline{\phi}^\alpha_K, \tag{7.11}$$

where the additional $\mathbb{1}$ subscript means that we evaluate the coefficients at the identity. On the other hand, substituting (7.6) into (5.10), we deduce that the normal form determining equations take the form

$$\widetilde{\Delta}_\nu = \sum_{i=1}^{p} \left[\widetilde{A}^i_\nu (\overline{X}^i - x^i) + \widetilde{A}^{i,i}_\nu (\overline{X}^i_i - 1) + \sum_{\substack{J \neq i \\ 1 \le |J| \le n}} \widetilde{A}^{i,J}_\nu \overline{X}^i_J \right] + \sum_{\alpha=1}^{q} \sum_{0 \le |K| \le n} \widetilde{B}^{\alpha,K}_\nu \left[\overline{\mathfrak{U}}^\alpha_K (\overline{X}^{(k)}, \widehat{U}^{(k)}) - u^\alpha_K \right],$$

whose coefficients are obtained from those of (5.10) by using the chain rule substitution (7.3). Linearizing the latter expressions at the identity, using (7.7), and noting that at the identity (7.3) reduces to $\overline{U}^{(n)} = \widehat{U}^{(n)} = u^{(n)}$, produces

$$\widetilde{L}_\nu = \sum_{i=1}^{p} \sum_{0 \le |J| \le n} A^{i,J}_{\nu,\mathbb{1}} \,\overline{\xi}^i_J + \sum_{\alpha=1}^{q} \sum_{0 \le |K| \le n} B^{\alpha,K}_{\nu,\mathbb{1}} \left[\mathrm{D}^K_x \left(\sum_{i=1}^{p} u^\alpha_i \overline{\xi}^i \right) - \sum_{i=1}^{p} u^\alpha_{K,i} \overline{\xi}^i - \psi^\alpha_K \right]. \tag{7.12}$$

Comparing (7.11) and (7.12) completes the proof. *Q.E.D.*

Remark 7.5. Inverting the substitutions (7.10) for ψ^α_J, we recover the usual formula for the prolongation of the vector field

$$-\overline{\mathbf{v}} = -\left(\sum_{i=1}^{p} \overline{\xi}^i(x) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \overline{\phi}^\alpha(x) \frac{\partial}{\partial u^\alpha} \right) \tag{7.13}$$

to jet space. More explicitly, recall from [47] that the n-th order prolongation of $\overline{\mathbf{v}}$ is the vector field

$$\overline{\mathbf{v}}^{(n)} = \sum_{i=1}^{p} \overline{\xi}^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{0 \le |J| \le n} \widehat{\phi}^\alpha_J \frac{\partial}{\partial u^\alpha_J}, \tag{7.14}$$

where the prolonged vector field coefficients are given by the formula

$$\widehat{\phi}^\alpha_J = \overline{\phi}^\alpha_J - \mathrm{D}^J_x \left(\sum_{i=1}^{p} \overline{\xi}^i u^\alpha_i \right) + \sum_{i=1}^{p} \overline{\xi}^i u^\alpha_{J,i}. \tag{7.15}$$

Then, under the substitution (7.10), the prolonged vector field $-\overline{\mathbf{v}}^{(n)}$ is mapped to the vector field

$$\widetilde{\mathbf{v}}^{(n)} = -\sum_{i=1}^{p} \overline{\xi}^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{0 \le |J| \le n} \psi^\alpha_J \frac{\partial}{\partial u^\alpha_J}. \tag{7.16}$$

As an immediate corollary, we are able to characterize the involutivity of the normal form determining equations.

Theorem 7.6. *For any order n, if the reduced determining equations are involutive, then so are the normal form determining equations.*

Proof: The substitution (7.10) is an invertible linear map, which does not alter the algebraic properties of the symbol, and hence does not affect its involutivity. *Q.E.D.*

Example 7.7. In this example we compute the normal form determining equations for the Lie pseudo-group (4.4). We substitute the chain rule formulas (7.8) into the reduced determining equations (5.7). The resulting equations, once simplified, are

$$\begin{aligned} &\overline{X}_y = 0, && \overline{X}_{xx} = \widehat{U}_Y \overline{X}_x^2 - u_y \overline{X}_x, \qquad \overline{X}_{xy} = \overline{X}_{yy} = 0, \\ &\overline{Y}_x = (\widehat{U} - u)\overline{X}_x, && \overline{Y}_{xx} = \big(\widehat{U}_X + 2(\widehat{U} - u)\widehat{U}_Y\big)\overline{X}_x^2 - \big(u_x + (\widehat{U} - u)u_y\big)\overline{X}_x, \\ &\overline{Y}_y = \overline{X}_x, && \overline{Y}_{xy} = \widehat{U}_Y \overline{X}_x^2 - u_y \overline{X}_x, \qquad \overline{Y}_{yy} = 0, \qquad u_{yy} = \widehat{U}_{YY}\overline{X}_x^2, \end{aligned} \tag{7.17}$$

and the parametric jets are $\overline{X}, \overline{Y}, u, \overline{X}_x, u_x, u_y, u_{xx}, u_{xy}$. We now linearize the normal form determining equations (7.17) at the identity transformation. To do so, we apply the vector field

$$\mathrm{j}_\infty \widetilde{\mathbf{v}} = \sum_{i,j=0}^{\infty} \left(\overline{\xi}_{ij} \frac{\partial}{\partial \overline{X}_{ij}} + \overline{\eta}_{ij} \frac{\partial}{\partial \overline{Y}_{ij}} + \psi_{ij} \frac{\partial}{\partial u_{ij}} \right),$$

to all the equations and then set $\overline{X} = x$, $\overline{Y} = y$, $\widehat{U}_{X^iY^j} = u_{x^iy^j}$ for all $i, j \geq 0$. This yields the linear system

$$\begin{aligned} &\overline{\xi}_y = 0, \qquad \overline{\eta}_x = -\psi, \qquad \overline{\eta}_y = \overline{\xi}_x, \qquad \overline{\xi}_{xx} = u_y \overline{\xi}_x - \psi_y, \qquad \overline{\xi}_{xy} = \overline{\xi}_{yy} = 0, \\ &\overline{\eta}_{xx} = u_x \overline{\xi}_x - u_y \psi - \psi_x, \qquad \overline{\eta}_{xy} = u_y \overline{\xi}_x - \psi_y, \qquad \overline{\eta}_{yy} = 0, \qquad \psi_{yy} = 2\,u_{yy}\overline{\xi}_x. \end{aligned} \tag{7.18}$$

Observe that, in accordance with Theorem 7.3, these equations can also be obtained from the linearized reduced determining equations (5.20) by applying the linear substitutions

$$\begin{aligned} \overline{\phi} \longmapsto -\psi, \qquad \overline{\phi}_x \longmapsto u_x \overline{\xi}_x + u_y \overline{\eta}_x - \psi_x, \qquad \overline{\phi}_y \longmapsto u_x \overline{\xi}_y + u_y \overline{\eta}_y - \psi_y, \\ \overline{\phi}_{yy} \longmapsto 2\,u_{xy}\overline{\xi}_y + 2\,u_{yy}\overline{\eta}_y + u_x \overline{\xi}_{yy} + u_y \overline{\eta}_{yy} - \psi_{yy}, \end{aligned} \tag{7.19}$$

coming from (7.9), and then simplifying the resulting equations. Since the linear (algebraic) map defined by (7.19) is invertible, the systems have the same indices and Cartan characters, and involutivity follows immediately.

For later use in Example 8.4, we compute the normal form determining equations of order three

$$\begin{aligned} &\overline{X}_{xxx} = (u_y^2 - u_{xy})\overline{X}_x - 3\,u_y \widehat{U}_Y \overline{X}_x^2 + (\widehat{U}_{XY} + 2\,\widehat{U}_Y^2 + (\widehat{U} - u)\widehat{U}_{YY})\,\overline{X}_x^3, \\ &\overline{X}_{xxy} = \overline{X}_{xyy} = \overline{X}_{yyy} = \overline{Y}_{xyy} = \overline{Y}_{yyy} = 0, \\ &\overline{Y}_{xxx} = (2u_x u_y - u_{xx} + (\widehat{U} - u)(u_y^2 - u_{xy}))\overline{X}_x - 3\,(u_x \widehat{U}_Y + u_y \widehat{U}_X + 2\,(\widehat{U} - u)u_y \widehat{U}_Y)\overline{X}_x^2 \\ &\qquad\qquad + (\widehat{U}_{XX} + 4\,\widehat{U}_X \widehat{U}_Y + 3\,(\widehat{U} - u)(\widehat{U}_{XY} + 2\,\widehat{U}_Y^2) + 2(\widehat{U} - u)^2 \widehat{U}_{YY})\overline{X}_x^3, \\ &\overline{Y}_{xxy} = (u_y^2 - u_{xy})\overline{X}_x - 3\,u_y \widehat{U}_Y \overline{X}_x^2 + (\widehat{U}_{XY} + 2\,\widehat{U}_Y^2 + (\widehat{U} - u)\widehat{U}_{YY})\overline{X}_x^3, \\ &u_{xyy} = -2\,u_y \widehat{U}_{YY} \overline{X}_x^2 + (\widehat{U}_{XYY} + 2\,\widehat{U}_Y \widehat{U}_{YY} - u\,\widehat{U}_{YYY} + \widehat{U}\widehat{U}_{YYY})\overline{X}_x^3, \\ &u_{yyy} = \widehat{U}_{YYY}\overline{X}_x^3. \end{aligned} \tag{7.20}$$

Linearizing the equations (7.20) at the identity jet yields

$$\begin{aligned} &\overline{\xi}_{xxx} = (2\,u_{xy} + u_y^2)\overline{\xi}_x - \psi_{xy} - u_y \psi_y - u_{yy}\psi, && \overline{\xi}_{xxy} = \overline{\xi}_{xyy} = \overline{\xi}_{yyy} = 0, \\ &\overline{\eta}_{xxx} = (2\,u_{xy} + 2\,u_x u_y)\,\overline{\xi}_x - \psi_{xx} - u_y \psi_x - u_x \psi_y - (2\,u_{xy} + u_y^2)\,\psi, && \\ &\overline{\eta}_{xxy} = (2\,u_{xy} + u_y^2)\overline{\xi}_x - \psi_{xy} - u_y \psi_y - u_{yy}\psi, && \overline{\eta}_{xyy} = \overline{\eta}_{yyy} = 0, \\ &\psi_{xyy} = (3\,u_{xyy} + 2\,u_y u_{yy})\,\overline{\xi}_x - 2\,u_{yy}\psi_y - u_{yyy}\psi, && \psi_{yyy} = 3\,u_{yyy}\overline{\xi}_x. \end{aligned} \tag{7.21}$$

8 Involutivity and Reduced Moving Frames.

We now reach the heart of the paper, in which we complete the proof of our general convergence result for normal forms of submanifolds. The key remaining step is to establish the compatibility of the cross-section normalizations producing the moving frame with the involutivity of the normal form determining system. The main complication is they are not necessarily compatible at low order. However, as we will demonstrate, beyond the order of freeness of the prolonged pseudo-group action, this identification can be made. Indeed, this is to be suspected, since this is also where the algebraic moving frame constructions used in [55] apply. As noted in [55], the finite number of normalizations imposed at or below the order of freeness are not, in general, compatible with the algebraic framework used to establish generating sets of differential invariants and syzygies, and so must be appended to the former to obtain a complete system of differential invariants. Here we will see a similar behavior within the involutivity framework. Establishing this connection is a bit technical, as we now explain.

The involutivity of the normal form determining equations (7.5) relies on the class-based ordering of multi-indices, which imposes some restrictions on which jets are parametric and principal. For example, in the normal form determining equations (7.17), the equation

$$u_{yy} = \widehat{U}_{YY}\overline{X}_x^2$$

is solved for u_{yy} since it is a principal jet according to the involutivity framework. On the other hand, recalling the moving frame computations in Example 6.8, we have the normalization equation

$$\widehat{U}_{YY} = \frac{u_{yy}}{\overline{X}_x^2} = \frac{1}{\overline{X}_x^2}$$

which, for the left moving frame, was solved for

$$\overline{X}_x = \sqrt{\frac{u_{yy}}{\widehat{U}_{YY}}} = \frac{1}{\sqrt{\widehat{U}_{YY}}}.$$

Thus, the same equation may be solved for different jets depending on whether we implement the involutivity formalism or the moving frame construction. At the level of the linearized equations, the equation in question is

$$2\,u_{yy}\,\overline{\xi}_x - \psi_{yy} = 0.$$

Since the symbol of the equation is $\psi_{yy} = 0$, involutivity involves solving for ψ_{yy}. However, the moving frame construction requires solving for $\overline{\xi}_x$.

The aim of this section is to show that, while they may differ at low order, if the normal form determining equations are prolonged to a sufficiently high order, then the determination of the parametric jets via the symbol of the normal form determining equations is consistent with the moving frame construction.

Let $\mathcal{L}_{\mathcal{N}}^{(n)}$ denote the linearization of the n-th order normal form determining equations (7.5) at the identity, and let

$$\Sigma_{\mathcal{N}}^n = \mathbf{H}(\mathcal{L}_{\mathcal{N}}^{(n)})$$

be the corresponding symbol with symbol matrix $M^n_{\mathcal{N}}$. We first fix some of the freedom that exists when ordering the columns of $M^n_{\mathcal{N}}$ within a fixed class. To be compatible with the moving frame construction, we require the columns associated to the reduced pseudo-group jets $\overline{X}^i_K$ to appear to the left of the columns corresponding to the normal form jets u^α_J when $\operatorname{cls} K = \operatorname{cls} J$. This ordering stems from the fact that, in the moving frame method, we prioritize solving for the reduced pseudo-group jets $\overline{X}^i_J$ over the normal form jets u^α_K within a fixed class.

For $n \geq 0$, we introduce the n-th order *vertical symbol*

$$\Psi^n = (\operatorname{span}\{\psi^{(n)}\}) \cap \Sigma^n_{\mathcal{N}}, \tag{8.1}$$

consisting of all the equations in the n-th order symbol that only depend on the coefficients ψ^α_J of order $|J| = n$. Combining these spaces, we define the *vertical symbol* to be

$$\Psi = \bigcup_{n=0}^{\infty} \Psi^n. \tag{8.2}$$

Example 8.1. In our running example, keeping only the highest order terms in the linearized equations (7.18), we obtain the symbol equations

$$\begin{gathered} \overline{\xi}_y = 0, \qquad \overline{\eta}_x = 0, \qquad \overline{\eta}_y = \overline{\xi}_x, \\ \overline{\xi}_{xx} = \overline{\xi}_{xy} = \overline{\xi}_{yy} = 0, \qquad \overline{\eta}_{xx} = \overline{\eta}_{xy} = \overline{\eta}_{yy} = 0, \qquad \psi_{yy} = 0. \end{gathered}$$

Therefore, the vertical symbols of order ≤ 2 are

$$\Psi^0 = \Psi^1 = \varnothing, \qquad \text{and} \qquad \Psi^2 = \{\psi_{yy} = 0\}.$$

Similarly, from the order three linearized equations (7.21), we find that the order three vertical symbol is $\Psi^3 = \{\psi_{xyy} = \psi_{yyy} = 0\}$, and more generally,

$$\Psi^k = \{\psi_{x^j y^{k-j}} = 0 \mid 0 \leq j \leq k-2\} \qquad \text{for} \qquad k \geq 2.$$

Upon row reducing the vertical symbol Ψ, the pivots of Ψ_{REF} identify principal normal form jets. Now the question is whether this identification of the principal normal form jets is consistent with the moving frame construction. To answer this question, we introduce the n-th order *linearized differential invariant annihilator subbundle*

$$\mathcal{Z}^{(n)} = (\operatorname{span}\{\overline{\xi}, \psi^{(n)}\}) \cap \mathcal{L}^{(n)}_{\mathcal{N}} \tag{8.3}$$

containing all the linearized normal form equations that only depend on $\overline{\xi}$ and $\psi^{(n)}$. To understand the origin of (8.3), we recall that a function $I(x, u^{(n)})$ is a differential invariant of $\overline{\mathcal{G}}$ if and only if it is annihilated by all prolonged infinitesimal generators (7.14) of the reduced pseudo-group action. In view of (7.16), this is equivalent to the infinitesimal constraint

$$\widetilde{\mathbf{v}}^{(n)}(I) = -\sum_{i=1}^{p} \overline{\xi}^i \frac{\partial I}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{0 \leq |J| \leq n} \psi^\alpha_J \frac{\partial I}{\partial u^\alpha_J} = 0. \tag{8.4}$$

Theorem 8.2. *If $I(x, u^{(n)})$ is a differential invariant, then the infinitesimal invariance equation $\widetilde{\mathbf{v}}^{(n)}(I) = 0$ belongs to $\mathcal{Z}^{(n)}$.*

Proof. By definition, $\mathcal{Z}^{(n)}$ contains all the linear combinations of $\overline{\xi}$ and $\psi^{(n)}$ that vanish. Since the infinitesimal invariance criterion (8.4) is of this form, it must belong to $\mathcal{Z}^{(n)}$. *Q.E.D.*

Applying Theorem 8.2 to the basic normalized differential invariants (6.14), evaluated at the source variables $(x, u^{(n)})$ rather than the target variables $(X, \widehat{U}^{(n)})$, we conclude that

$$\widetilde{\mathbf{v}}^{(n)}(H^j) = \widetilde{\mathbf{v}}^{(n)}(I^\beta_K) = 0 \qquad \text{with} \qquad j,\ (\beta, K) \notin \mathcal{I}^{(n)}_{\mathcal{K}} \tag{8.5}$$

are equations in $\mathcal{Z}^{(n)}$. Since the basic normalized differential invariants form a complete set of functionally independent differential invariants of order $\leq n$, it follows that, at each regular jet,

$$\mathcal{Z}^{(n)}|_{z^{(n)}} = \left\{ \widetilde{\mathbf{v}}^{(n)}(H^j)|_{z^{(n)}} = \widetilde{\mathbf{v}}^{(n)}(I^\beta_K)|_{z^{(n)}} = 0 \mid j,\ (\beta, K) \notin \mathcal{I}^{(n)}_{\mathcal{K}} \right\}. \tag{8.6}$$

Remark 8.3. One needs to be a little careful here. Not every equation defining $\mathcal{Z}^{(n)}$ is necessarily of the form (8.5), because its coefficients need not be partial derivatives of some function. On the other hand, (8.6) says that, at a fixed regular jet, the linear subvariety defined by the differential invariant conditions (8.5) coincides with the n-th order linearized differential invariant annihilator subbundle $\mathcal{Z}^{(n)}$.

Keeping only the highest order terms in (8.3), we introduce the n-th order *linearized differential invariant annihilator symbol*

$$\Upsilon^n = \mathbf{H}(\mathcal{Z}^{(n)}).$$

Since $\overline{\xi}$ has order zero, it follows that for $n \geq 1$, the n-th order linearized differential invariant annihilator symbol Υ^n only involves linear equations in ψ^α_J of order $|J| = n$.

Example 8.4. Recalling the linearized normal form determining equations (7.18) and (7.21), we conclude that when $u_{yy} \neq 0$,

$$\mathcal{Z}^{(3)} = \left\{ \psi_{yyy} = \frac{3\,u_{yyy}}{2\,u_{yy}}\psi_{yy}, \quad \psi_{xyy} = \left(\frac{3}{2}\frac{u_{xyy}}{u_{yy}} + u_y \right)\psi_{yy} - 2\,u_{yy}\psi_y - u_{yyy}\psi \right\}. \tag{8.7}$$

We observe that, in accordance with the preceding remarks, the equations in (8.7) can also found by setting the linearization of the normalized invariants (6.18), re-expressed in terms of the source variables $(x, u^{(n)})$, to zero. Keeping only the highest order terms,

$$\Upsilon^0 = \Upsilon^1 = \Upsilon^2 = \varnothing, \qquad \text{while} \qquad \Upsilon^3 = \{\psi_{xyy} = \psi_{yyy} = 0\} = \Psi^3. \tag{8.8}$$

More generally, $\Upsilon^k = \Psi^k$ for all $k \geq 3$.

On the other hand, when $u_{yy} = 0$, we have

$$\mathcal{Z}^{(2)} = \{\psi_{yy} = 0\} \qquad \text{and} \qquad \mathcal{Z}^{(3)} = \{\psi_{yy} = \psi_{xyy} = \psi_{yyy} = 0\},$$

so that

$$\Upsilon^2 = \{\psi_{yy} = 0\} \qquad \text{and} \qquad \Upsilon^3 = \{\psi_{xyy} = \psi_{yyy} = 0\}.$$

In this case, the equality $\Upsilon^k = \Psi^k$ holds for all $k \geq 2$.

Remark 8.5. It is worth reiterating that all the symbol computations are done at a fixed jet, whose dependence has been omitted throughout the paper. The last example reminds us that we need to pay attention to the base jet when performing computations. This is important when, for example, analyzing singular normal forms, [57].

Example 8.6. As a second example, consider the Lie pseudo-group

$$X = f(x), \qquad Y = \lambda\, y, \qquad U = u + b, \qquad V = v + c, \tag{8.9}$$

where $f \in \mathcal{D}(\mathbb{R})$, $\lambda > 0$, and $b, c \in \mathbb{R}$. Here we assume that $p = q = 2$ with $u = u(x,y)$, $v = v(x,y)$. Working under the assumption that $y \neq 0$, the normal form determining equations $\mathcal{N}^{(2)}$ of order two are

$$\begin{gathered}
\overline{X}_x = \frac{u_x}{\widehat{U}_X}, \quad \overline{X}_y = 0, \quad \overline{Y}_x = 0, \quad \overline{Y}_y = \frac{\overline{Y}}{y}, \quad u_y = \frac{\overline{Y}\widehat{U}_Y}{y}, \quad v_x = \frac{u_x \widehat{V}_X}{\widehat{U}_X}, \quad v_y = \frac{\overline{Y}\widehat{V}_Y}{y}, \\
\overline{X}_{xx} = \frac{u_{xx}}{\widehat{U}_X} - \frac{u_x^2 \widehat{U}_{XX}}{\widehat{U}_X^3}, \quad \overline{X}_{xy} = \overline{X}_{yy} = \overline{Y}_{xx} = \overline{Y}_{xy} = \overline{Y}_{yy} = 0, \quad u_{xy} = \frac{u_x \overline{Y}\widehat{U}_{XY}}{y\widehat{U}_X}, \\
u_{yy} = \frac{\overline{Y}^2 \widehat{U}_{YY}}{y^2}, \quad v_{xx} = \frac{u_{xx}\widehat{V}_X}{\widehat{U}_X} + u_x^2\left(\frac{\widehat{V}_{XX}\widehat{U}_X - \widehat{U}_{XX}\widehat{V}_X}{\widehat{U}_X^3}\right), \quad v_{xy} = \frac{u_x \overline{Y}\widehat{V}_{XY}}{y\widehat{U}_X}, \quad v_{yy} = \frac{\overline{Y}^2\widehat{V}_{YY}}{y^2}.
\end{gathered}$$

We remark that the equations for u_y, v_x, v_y, ..., can be obtained by successively applying the chain rule operators

$$\mathrm{D}_x = \overline{X}_x \mathrm{D}_X = \frac{u_x}{\widehat{U}_X}\,\mathrm{D}_X, \qquad \mathrm{D}_y = \overline{Y}_y \mathrm{D}_Y = \frac{\overline{Y}}{y}\,\mathrm{D}_Y,$$

to the last two transformations in (8.9). Linearization at the identity yields the system of linear equations $\mathcal{L}_{\mathcal{N}}^{(2)}$ given by

$$\begin{gathered}
\overline{\xi}_x = \frac{\psi_x}{u_x}, \quad \overline{\xi}_y = 0, \quad \overline{\eta}_x = 0, \quad \overline{\eta}_y = \frac{\overline{\eta}}{y}, \quad \psi_y = \frac{u_y}{y}, \quad \gamma_x = \frac{v_x}{u_x}\,\psi_x, \quad \gamma_y = \frac{v_y}{y}\,\overline{\eta}, \\
\overline{\xi}_{xx} = \frac{\psi_{xx}}{u_x} - 2\frac{u_{xx}\psi_x}{u_x^2}, \quad \overline{\xi}_{xy} = \overline{\xi}_{yy} = \overline{\eta}_{xx} = \overline{\eta}_{xy} = \overline{\eta}_{yy} = 0, \quad \psi_{xy} = u_{xy}\left(\frac{\psi_x}{u_x} + \frac{\overline{\eta}}{y}\right), \\
\psi_{yy} = 2\frac{u_{yy}}{y}\,\overline{\eta}, \quad \gamma_{xx} = \frac{v_x}{u_x}\,\psi_{xx} + 2\left(\frac{v_{xx}u_x - u_{xx}v_x}{u_x^2}\right)\psi_x, \quad \gamma_{xy} = v_{xy}\left(\frac{\psi_x}{u_x} + \frac{\overline{\eta}}{y}\right), \quad \gamma_{yy} = 2\frac{v_{yy}}{y}\,\overline{\eta},
\end{gathered}$$

where $\overline{\xi}$, $\overline{\eta}$, ψ, γ denotes the linearization of $\overline{X}$, $\overline{Y}$, u, v, respectively. Up to order two, the symbols are

$$\begin{aligned}
&\Sigma_{\mathcal{N}}^0 = \varnothing, \qquad \Sigma_{\mathcal{N}}^1 = \left\{\overline{\xi}_x = \frac{\psi_x}{u_x}, \quad \overline{\xi}_y = \overline{\eta}_x = \overline{\eta}_y = 0, \quad \psi_y = 0, \quad \gamma_x = \frac{v_x}{u_x}\,\psi_x, \quad \gamma_y = 0\right\}, \\
&\Sigma_{\mathcal{N}}^2 = \left\{\overline{\xi}_{xx} = \frac{\psi_{xx}}{u_x}, \quad \overline{\xi}_{xy} = \overline{\xi}_{yy} = \overline{\eta}_{xx} = \overline{\eta}_{xy} = \overline{\eta}_{yy} = \psi_{xy} = \psi_{yy} = 0, \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. \gamma_{xx} = \frac{v_x}{u_x}\,\psi_{xx}, \quad \gamma_{xy} = \gamma_{yy} = 0\right\}.
\end{aligned}$$

In this example,

$$\mathcal{Z}^{(2)} = \left\{\psi_y = \frac{u_y}{y}\,\overline{\eta}, \quad \gamma_x = \frac{v_x}{u_x}\,\psi_x, \quad \gamma_y = \frac{v_y}{y}\,\overline{\eta}, \quad \psi_{xy} = u_{xy}\left(\frac{\psi_x}{u_x} + \frac{\overline{\eta}}{y}\right), \quad \psi_{yy} = 2\frac{u_{yy}}{y}\,\overline{\eta},\right.$$

$$\gamma_{xx} = \frac{v_x}{u_x}\,\psi_{xx} + 2\left(\frac{v_{xx}u_x - u_{xx}v_x}{u_x^2}\right)\psi_x, \quad \gamma_{xy} = v_{xy}\left(\frac{\psi_x}{u_x} + \frac{\overline{\eta}}{y}\right), \quad \gamma_{yy} = 2\frac{v_{yy}}{y}\,\overline{\eta}\Bigg\},$$

which can also be found by applying the vector field

$$\widetilde{\mathbf{v}}^{(\infty)} = -\overline{\xi}\frac{\partial}{\partial x} - \overline{\eta}\frac{\partial}{\partial y} + \sum_J \left(\psi_J \frac{\partial}{\partial u_J} + \gamma_J \frac{\partial}{\partial v_J}\right)$$

to the differential invariants

$$I_{0,1} = y\,u_y, \qquad I_{1,1} = \frac{y\,u_{xy}}{u_x}, \qquad I_{0,2} = y^2 u_{yy},$$

$$J_{1,0} = \frac{v_x}{u_x}, \qquad J_{0,1} = y\,v_y, \qquad J_{2,0} = \frac{v_{xx}u_x - v_x u_{xx}}{u_x^3}, \qquad J_{1,2} = \frac{y\,v_{xy}}{u_x}, \qquad J_{0,2} = y^2 v_{yy}$$

and setting the result to zero. Finally, we note that

$$\Upsilon^0 = \Psi^0 = \varnothing, \qquad \Upsilon^1 = \Psi^1 = \left\{\psi_y = 0, \quad \gamma_x = \frac{v_x}{u_x}\,\psi_x, \quad \gamma_y = 0\right\},$$

$$\Upsilon^2 = \Psi^2 = \left\{\psi_{xy} = \psi_{yy} = 0, \quad \gamma_{xx} = \frac{v_x}{u_x}\,\psi_{xx}, \quad \gamma_{xy} = \gamma_{yy} = 0\right\}.$$

Remark 8.7. The linear spaces defined above are related to the algebraic constructions introduced in [55]. First, the vertical symbol (8.2) is related to the prolonged symbol submodule defined in [55, Definition 4.2]. On the other hand, the linearized differential invariant annihilator subbundle $\mathcal{Z}^{(n)}$ is equivalent to the prolonged annihilator bundle introduced in [55, eq. (4.26)].

Since the equations in $\mathcal{Z}^{(n)}$ only depend on $\overline{\xi}$ and $\psi^{(n)}$, all the pivots of the row reduced symbol Υ^n_{REF} identify principal normal form jets in the normal form determining equations (7.1) that are compatible with the reduced moving frame construction. Indeed, the normal form equations whose linearization at the identity are in $\mathcal{Z}^{(n)}$ must be independent of the reduced pseudo-group jets $\overline{X}^i_J$ of order $|J| \geq 1$. Thus, these equations are of the form

$$F(x, \widehat{U}^{(n)}, \overline{X}, u^{(n)}) = 0. \tag{8.10}$$

The equations that depend on $u^{(n)}$ can then be solved for certain principal normal form jets u^α_J, in a manner that is compatible with the moving frame implementation, as we now explain in detail. We note that these equations are not solved for any of the base coordinates $\overline{X}^i$, since, in the moving frame implementation, $\overline{X}$ is set equal to the point

$$X_0 = (X_0^1, \ldots X_0^p) \tag{8.11}$$

at which the jet of the source section $S = \{(X, \widehat{U}(X))\}$ is to be evaluated.

As Example 8.4 exemplifies, in general $\Psi^n \subseteq \Upsilon^n$, where the containment means that the solution space of Ψ^n is contained in the solution space of Υ^n. But if there exists an order n such that

$$\Psi^k = \Upsilon^k \qquad \text{for all} \quad k > n, \tag{8.12}$$

then we can conclude that the principal normal form jets determined by the pivots of Ψ^k_{REF} are compatible with the moving frame construction. We now show that (8.12) holds with $n = n_f$, the order at which the prolonged reduced pseudo-group becomes free.

According to Definition 6.4, the reduced Lie pseudo-group $\overline{\mathcal{G}}$ acts freely at $z^{(n)} \in \mathrm{J}^n$ if and only if

$$\overline{\mathcal{G}}^{(n)}_{z^{(n)}} = \left\{ \, (\overline{X}^{(n)}, \overline{U}^{(n)}) \in \overline{\mathcal{G}}^{(n)} \mid P^{(n)}(z^{(n)}, \overline{X}^{(n)}, \overline{U}^{(n)}) = z^{(n)} \, \right\} = \left\{ \, \overline{\mathbb{1}}^{(n)}_z \, \right\}, \tag{8.13}$$

where $P^{(n)}(z^{(n)}, \overline{X}^{(n)}, \overline{U}^{(n)})$ is the function that prescribes the prolonged action (6.2) at order n. At the infinitesimal level, the Lie pseudo-group acts locally freely if and only if

$$\overline{\mathfrak{g}}^{(n)}_{z^{(n)}} = \left\{ \, (\overline{\xi}^{(n)}, \overline{\phi}^{(n)}) \in \mathcal{L}^{(n)}_{\overline{\mathcal{G}}}(z^{(n)}, \overline{\xi}^{(n)}, \overline{\phi}^{(n)}) \mid \overline{\mathbf{v}}^{(n)} = 0 \, \right\} = \{0\}, \tag{8.14}$$

where the prolonged vector field $\overline{\mathbf{v}}^{(n)}$ is defined in (7.14).

The next result shows that *persistence of freeness* also holds for reduced Lie pseudo-group actions.

Theorem 8.8. *If the reduced pseudo-group $\overline{\mathcal{G}}$ acts (locally) freely at $z^{(n)} \in \mathrm{J}^n$, then for all $k > 0$ it acts (locally) freely at $z^{(n+k)} \in \mathrm{J}^{n+k}$ where $\pi^{n+k}_n(z^{(n+k)}) = z^{(n)}$.*

Proof. The linearized equations (8.14) imply that the symbol of the system of equations (8.13) is trivial. Therefore the system (8.13) is involutive with vanishing Cartan characters $\overline{\mathsf{c}}^{(i)}_n = 0$ for $i = 1, \ldots, p$. Since $\overline{\mathcal{G}}^{(n+k)}_{z^{(n+k)}}$ can be obtained by prolonging $\overline{\mathcal{G}}^{(n)}_{z^{(n)}}$, and involutivity is preserved under prolongation, we conclude, recalling (5.19), that the Cartan characters of $\overline{\mathcal{G}}^{(n+k)}_{z^{(n+k)}}$ also vanish, which means that all the jets of order $n+k$ are uniquely determined. Since $\overline{\mathbb{1}}^{(n+k)}_{z^{(n+k)}} \in \overline{\mathcal{G}}^{(n+k)}_{z^{(n+k)}}$, this is the only solution and the reduced pseudo-group remains free at order $n+k$. *Q.E.D.*

We now make the substitutions (7.3) in (8.13) to obtain

$$\left\{ \, (\overline{X}^{(n)}, u^{(n)}) \in \mathcal{N}^{(n)} \mid P^{(n)}(z^{(n)}, \overline{X}^{(n)}, \overline{\mathfrak{U}}^{(n)}(\overline{X}^{(n)}, \widehat{U}^{(n)})) = z^{(n)} \, \right\} = \left\{ \, (\mathbb{1}^{(n)}_x, \widehat{U}^{(n)}) \, \right\}, \tag{8.15}$$

which holds whenever the reduced pseudo-group acts freely. At the infinitesimal level, we use the equality (7.16) to conclude that $0 = \overline{\mathbf{v}}^{(n)} = -\widetilde{\mathbf{v}}^{(n)}$, the latter being equivalent to $\{\overline{\xi} = 0, \psi^{(n)} = 0\}$. Thus, the linearization of (8.15), at the identity transformation, yields

$$\left\{ \, (\overline{\xi}^{(n)}, \psi^{(n)}) \in \mathcal{L}^{(n)}_{\mathcal{N}}(z^{(n)}, \overline{\xi}^{(n)}, \psi^{(n)}) \mid \overline{\xi} = 0, \psi^{(n)} = 0 \, \right\} = \{0\}. \tag{8.16}$$

Remark 8.9. The local freeness condition (8.16) implies that the system of equations $\mathcal{L}^{(n)}_{\mathcal{N}}(z^{(n)}, \overline{\xi}^{(n)}, \psi^{(n)}) \cap \{\overline{\xi} = 0, \psi^{(n)} = 0\}$ is equivalent to $\{\overline{\xi}^{(n)} = 0, \psi^{(n)} = 0\}$. Therefore any linear combination $Y \in \operatorname{span}\{\overline{\xi}^{(n)}, \psi^{(n)}\}$ can be written in the form $Y = U + V$, with $U \in \operatorname{span}\{\overline{\xi}, \psi^{(n)}\}$ and the equation $V = 0$ belonging to $\mathcal{L}^{(n)}_{\mathcal{N}}$.

We now establish the key moving frame/involutivity compatibility result.

Theorem 8.10. *If $\overline{\mathcal{G}}$ acts (locally) freely at $z^{(n)} \in \mathrm{J}^n$, then the equality $\Psi^k|_{z^{(k)}} = \Upsilon^k|_{z^{(k)}}$ holds for all $k > n$ and all $z^{(k)} \in (\pi^k_n)^{-1}\{z^{(n)}\}$.*

Proof. By an inductive argument that relies on the persistence of freeness, it suffices to prove the equality for $k = n+1$. Since $\Psi^n \subseteq \Upsilon^n$ for any $n \in \mathbb{N}$, it suffices to show the reverse inclusion. In other words, if $Q = 0$ is in Ψ^{n+1}, by which we mean that $Q = 0$ is one of the equations defining Ψ^{n+1}, we must show that there exists $U \in \operatorname{span}\{\overline{\xi}, \psi^{(n)}\}$ such that

$$Q + U = 0 \ \in \ \mathcal{Z}^{(n+1)}.$$

If this is the case, then $Q = \mathbf{H}(Q+U) = 0$ is in Υ^{n+1}.

Now, since $Q = 0$ is an equation in the symbol $\Sigma^{n+1}_{\mathcal{N}}$, there exists $Y \in \operatorname{span}\{\overline{\xi}^{(n)}, \psi^{(n)}\}$ such that

$$Q + Y = 0 \ \in \ \mathcal{L}^{(n+1)}_{\mathcal{N}}.$$

Using Remark 8.9, we have that

$$Y = U + V,$$

with $U \in \operatorname{span}\{\overline{\xi}, \psi^{(n)}\}$ and $V = 0$ in $\mathcal{L}^{(n)}_{\mathcal{N}}$. Thus, the equation

$$Q + U = (Q + Y) - V = 0 \ \in \ \mathcal{L}^{(n+1)}_{\mathcal{N}}.$$

Since $Q = 0$ is in Ψ^{n+1} and $U \in \operatorname{span}\{\overline{\xi}, \psi^{(n)}\}$, we conclude that $0 = Q + U \in \mathcal{Z}^{(n+1)}$. *Q.E.D.*

In light of Theorem 8.10, once freeness is attained at order n_f, we can use $\Psi^{n_f+1}_{\text{REF}}$ to identify the principal normal form jets of order n_f+1. Given a multi-index I of order n_f+1 we explicitly identify its class by rewriting it as

$$I = (I \setminus i), i \qquad\qquad \text{where} \qquad \operatorname{cls}(I) = i.$$

With this notation, the principal normal form jets u^α_I of order n_f+1 satisfy the equations

$$u^\alpha_{(I\setminus i),i} = \Delta^\alpha_I(x, \widehat{U}^{(n_f+1)}, \overline{X}, \ldots, u^\beta_J, \ldots, u^\gamma_{(K\setminus k),k}, \ldots), \tag{8.17}$$

where the normal form jets occurring on the right hand side of (8.17) are parametric with $|J| \leq n_f$, $|K| = n_f + 1$, and $k = \operatorname{cls}(K) \leq \operatorname{cls}(I) = i$.

Freeness implies that, at order n_f, all the reduced horizontal pseudo-group jets of orders $1 \leq k \leq n_f$ can be normalized so that

$$\overline{X}^i_L = \Xi^i_L(x, \widehat{U}^{(n_f)}, \overline{X}, \ldots, u^\beta_J, \ldots), \qquad i = 1, \ldots, p, \quad 1 \leq |L| \leq n_f, \tag{8.18}$$

where the normal form jets u^β_J appearing on the right hand side of (8.18) are parametric with $|J| \leq n_f$. Similarly, the principal normal form jets of order $\leq n_f$ satisfy the equations

$$u^\alpha_I = \Delta^\alpha_I(x, \widehat{U}^{(n_f)}, \overline{X}, \ldots, u^\beta_J, \ldots), \qquad \alpha = 1, \ldots, q, \qquad |I| \leq n_f, \tag{8.19}$$

where the u^β_J are also parametric with $|J| \leq n_f$. Since the equations (8.18) and (8.19) are obtained by implementing the reduced moving construction, these are not necessarily class respecting. This means that the class of the parametric normal form jets on the right hand side of an equation may be greater than the class of the jet occurring on the left hand side of the same equation. To obtain class respecting equations for the reduced horizontal pseudo-group jets, we differentiate the equations in (8.18) for the reduced pseudo-group jets $\overline{X}^i_L$ of order $|L| = n_f$ with respect to the multiplicative variables $\ell \leq \text{cls}(L)$, thereby obtaining the equations

$$\overline{X}^i_{L,\ell} = \widetilde{\Xi}^i_{L,\ell}(x, \widehat{U}^{(n_f+1)}, \overline{X} \ldots, u^\beta_J, \ldots, u^\beta_{J,\ell}, \ldots), \qquad i = 1, \ldots, p, \quad |L| = n_f, \tag{8.20}$$

where we used (8.18) to remove the first order jets $\overline{X}^i_j$ from (8.20). We note that the class of $\overline{X}^i_{L,\ell}$ is now

$$\text{cls}(L, \ell) = \ell.$$

We also observe that all the normal form jets $u^\beta_{J,\ell}$ on the right hand side of (8.20) satisfy the class requirement

$$\text{cls}(J, \ell) = \min\{\text{cls}(J), \ell\} \leq \ell.$$

Using (8.17) and (8.19), we can remove any principal normal form jet of order $\leq n_f + 1$ appearing in (8.20) to obtain

$$\overline{X}^i_{L,\ell} = \Xi^i_{L,\ell}(x, \widehat{U}^{(n_f+1)}, \overline{X} \ldots, u^\beta_J, \ldots, u^\kappa_{(N\setminus n),n}, \ldots), \qquad i = 1, \ldots, p, \quad |L| = n_f, \tag{8.21}$$

where all the normal form jets on the right hand side of the equations are parametric with $|J| \leq n_f$, $|N| = n_f + 1$, and $n = \text{cls}(N) \leq \text{cls}(L, \ell) = \ell$.

The equations (8.17) and (8.21) account for all the equations of order $n_f + 1$ contained in the normal form determining equations $\mathcal{N}^{(n_f+1)}$. Since the normal form equations $\mathcal{N}^{(n_f)}$ of order n_f are involutive, and involutivity is preserved under prolongation, the msystem of differential equations given by (8.17) and (8.21) combined with $\mathcal{N}^{(n_f)}$ is involutive.

The preceding discussion proves the following result.

Theorem 8.11. *Assume the normal form determining equations* (7.5) *become involutive at order $\overline{n}_\star$ and that the prolonged action of a Lie pseudo-group is free at order $n_f \geq \overline{n}_\star$. Then the normal form determining equations of order $n_f + 1$ are involutive and the determination of the principal jets of order $n_f + 1$ is consistent with the moving frame construction.*

Remark 8.12. Example 8.4 shows that freeness is not necessary to obtain (8.12). Non-free actions will arise, in particular, in equivalence problems where there are non-trivial isotropy groups. By appropriately dealing with the isotropy group, a modified version of Theorem 8.11 should still hold. The details are, however, deferred to future study.

Remark 8.13. The normal form equations $\mathcal{N}^{(n_f)}$ are given by the equations (8.18) and (8.19). However, we note that the jets occurring on the left hand sides of the equations are not necessarily in accordance with the pivots of the symbol matrix $M^{(n_f)}_{\mathcal{N}}$.

Example 8.14. Continuing Example 7.7, we saw that the order two normal form determining equations are not compatible with the moving frame construction. But since the prolonged action becomes free at order two, based on Theorem 8.11, those of order three given in (7.20) will be compatible. The normal form determining equation of order three remain involutive with Cartan characters $\overline{\mathsf{c}}^{(1)}_3 = 2$, $\overline{\mathsf{c}}^{(2)}_3 = 0$.

Assuming the prolonged action becomes free at order $n_f \geq \overline{n}_\star$, we follow the discussion on page 10 to rewrite the normal form determining equations $\mathcal{N}^{(n_f+1)}$ as an equivalent system of first order differential equations

$$\widetilde{\mathcal{N}}^{(1)} = \left\{ \begin{array}{lll} \widetilde{\Delta}^{(1)}(x, \widehat{U}^{(n_f+1)}, (\overline{X}^{(n_f)})^{(1)}, (u^{(n_f)})^{(1)}) = 0, & & \\ \partial_i \overline{X}^j_J = X^j_{J,i}, \quad \partial_i u^\alpha_J = u^\alpha_{J,i}, & & |J| \leq n_f,\ 1 \leq i \leq p, \\ \partial_i \overline{X}^j_J = \partial_k \overline{X}^j_{J,i\setminus k}, \quad \partial_i u^\alpha_J = \partial_k u^\alpha_{J,i\setminus k}, & & |J| = n_f,\ k = \text{cls}\, J < i \leq p. \end{array} \right\}. \tag{8.22}$$

According to Proposition 3.9, this first order system remains involutive with the same Cartan characters as the original normal form determining system $\mathcal{N}^{(n_f+1)}$. Furthermore, we write (8.22) in reduced Cartan normal form. Since the second and third lines of (8.22) are already in Cartan normal form, we focus on the equations $\widetilde{\Delta}^{(1)} = 0$. When expressing the order $n_f + 1$ jets as first order derivatives, we use the substitutions (3.6) and make the blanket assumption that when writing $\partial_\ell u^\gamma_{L\setminus\ell}$, the multi-index L is of order $n_f + 1$ and class ℓ. Doing so, we obtain the first order system of differential equations

$$\begin{aligned} \partial_i u^\alpha_{I\setminus i} &= \Delta^\alpha_I\big(x, \widehat{U}^{(n_f+1)}, \overline{X}, \ldots, u^\beta_J, \ldots, \partial_k u^\gamma_{K\setminus k}, \ldots\big), \\ \partial_\ell \overline{X}^i_L &= \Xi^i_{L,\ell}\big(x, \widehat{U}^{(n_f+1)}, \overline{X}, \ldots, u^\beta_J, \ldots, \partial_n u^\kappa_{N\setminus n}, \ldots\big), \end{aligned} \tag{8.23}$$

where the normal form jets u^β_J, $\partial_k u^\gamma_{K\setminus k}$, $\partial_n u^\kappa_{N\setminus n}$ appearing on the right hand side of the equations are parametric with $|J| \leq n_f$, $|L| = n_f$, $|I| = |K| = |N| = n_f + 1$ and $k \leq i$, $n \leq \ell \leq \text{cls}(L)$. We note that the equations in (8.23) are just the equations (8.17) and (8.21) written as first order differential equations. These equations are supplemented with the algebraic equations

$$u^\alpha_I = \Delta^\alpha_I(x, \widehat{U}^{(n_f)}, \overline{X}, \ldots, u^\beta_J, \ldots), \qquad \overline{X}^i_L = \Xi^i_L(x, \widehat{U}^{(n_f)}, \overline{X}, \ldots, u^\beta_J, \ldots), \tag{8.24}$$

given by (8.18) and (8.19). According to Theorem 3.11, provided all the functions Δ^α_I, Ξ^i_L, and $\Xi^i_{L,\ell}$ in (8.23), (8.24) are real-analytic at the origin, the formally well-posed initial conditions

$$\begin{aligned} u^\gamma_{K\setminus p}(x^1, \ldots, x^p) &= f^\gamma_{K\setminus p}(x^1, \ldots, x^p), \\ u^\gamma_{K\setminus p-1}(x^1, \ldots, x^{p-1}, 0) &= f^\gamma_{K\setminus p-1}(x^1, \ldots, x^{p-1}), \\ &\ \vdots \\ u^\gamma_{K\setminus 1}(x^1, 0, \ldots, 0) &= f^\gamma_{K\setminus 1}(x^1), \\ u^\beta_J(0, \ldots, 0) &= f^\beta_J, \end{aligned} \tag{8.25}$$

specifying the parametric jets occurring on the right hand side of the equations (8.24) are analytic at the origin, and the algebraic equations

$$\begin{aligned} u^\alpha_I(0, \ldots, 0) &= \Delta^\alpha_I(x, \widehat{U}^{(n_f)}, \overline{X}, \ldots, u^\beta_J, \ldots)|_{(0,\ldots,0)}, \\ \overline{X}^i_L(0, \ldots, 0) &= \Xi^i_L(x, \widehat{U}^{(n_f)}, \overline{X}, \ldots, u^\beta_J, \ldots)|_{(0,\ldots,0)}, \\ \overline{X}^i(0, \ldots, 0) &= X^i_0, \end{aligned} \tag{8.26}$$

are satisfied, then the normal form determining system admits one and only one solution that is analytic at the origin. In particular, the normal form $u(x)$, which forms part of the solution is analytic. In (8.26), the right hand side of the third equation are the components of the point $X_0 = (X_0^1, \ldots, X_0^p) \in \mathcal{X}$ at which the submanifold is being considered.

Remark 8.15. The initial conditions (8.25) and (8.26) are stated under the assumption that the pseudo-group $\mathcal{G}$ can map the origin $0 \in \mathcal{X}$ to the point X_0. In applications, the origin can be replaced by any convenient point $\boldsymbol{p} \in \mathcal{X}$. For example, the points where $y = 0$ are singular for the pseudo-group (8.9). Here, the origin can be replaced by the point $\boldsymbol{p} = (0, 1)$, and any point (X_0, Y_0) with $Y_0 > 0$ is on its group orbit. In general, given $\boldsymbol{p} \in \mathcal{X}$, the initial conditions (8.25) can be modified by considering hyperplanes passing through $\boldsymbol{p}$. Of course, it is also possible to make a local change of coordinates preserving δ-regularity so that $\boldsymbol{p}$ is mapped to 0 and the initial conditions are given by (8.25) and (8.26).

Example 8.16. For our running example, based on the cross-section (6.16), the standard moving frame implementation yields the general normal form

$$u(x, y) = c(x) + y\, d(x) + \frac{y^2}{2}\, w(x, y), \tag{8.27}$$

where $w(0, 0) = 1$. To use our result to show that the resulting formal power series converges, since the prolonged action becomes free at order $n_f = 2$, we must consider the order three normal form determining equations given in (7.20). We note that the last two equations of (7.20) are solved for the principal normal form jets u_{xyy} and u_{yyy}, in accordance with the order three vertical symbol (8.8). As first order partial differential equations, these determine $\partial_x(u_{yy})$ and $\partial_y(u_{yy})$. On the other hand, the order three normal form jets $u_{xxy} = \partial_x(u_{xy})$ and $u_{xxx} = \partial_x(u_{xx})$ are parametric of class one. In accordance with the general initial conditions (8.25), u_{xx} and u_{xy} are fixed by initial conditions along the x-axis. Differentiating (8.27), those are given by

$$u_{xx}(x, 0) = c_{xx}(x), \qquad u_{xy}(x, 0) = d_x(x), \tag{8.28}$$

These initial conditions are supplemented with the algebraic initial conditions

$$\begin{aligned} &X(0,0) = X_0, \qquad Y(0,0) = Y_0, \\ u(0,0) = c_0, \qquad &u_x(0,0) = c_1, \qquad u_y(0,0) = d_0, \qquad u_{yy}(0,0) = 1. \end{aligned} \tag{8.29}$$

We note that the initial conditions (8.28), (8.29) can be simplified to

$$X(0,0) = X_0, \quad Y(0,0) = Y_0, \quad u(x,0) = c(x), \quad u_y(x,0) = d(x), \quad u_{yy}(0,0) = 1.$$

Provided the functions $c(x)$ and $d(x)$ are analytic, part of the solution to the normal form determining equations (7.20) is given by (8.27), thereby establishing its convergence.

We now incorporate the moving frame construction within the algebraic construction introduced above. As seen in Section 6.2, a moving frame is obtained by selecting a (coordinate) cross-section (6.10) transversal to the prolonged pseudo-group orbits. Let

$$\mathcal{L}_{\mathcal{K}}^{(n)} = \{\overline{\xi}^i = 0,\, \psi_J^\alpha = 0 \mid i,\ (\alpha, J) \in \mathcal{I}_{\mathcal{K}}^{(n)}\}$$

denote the linearization of the n-th order cross-section determining equations $\mathcal{K}^{(n)}$. Let $\Sigma_{\mathcal{K}}^n = \mathbf{H}(\mathcal{L}_{\mathcal{K}}^{(n)})$ its symbol, which must be disjoint from the symbol of the linearized differential invariant annihilator subbundle, in order that the only solution to all the equations in the disjoint union

$$\Sigma^n_{\mathcal{K}} \sqcup \Upsilon^n = \{0\}$$

is the trivial solution. Thus, the pivots of $\Sigma^n_{\mathcal{K},\mathrm{REF}}$ must be complementary to the pivots of $\Upsilon^n_{\mathrm{REF}}$. We call such a cross-section a *well-posed cross-section*. This terminology stems from the fact that it provides the formally well-posed initial conditions (8.25), for the normal form determining equations (8.23) and the algebraic equations (8.26). A well-posed cross-section is a refinement on the notion of algebraic cross-section introduced in [55], which is prescribed by a Gröbner basis of the submodule Ψ. On the other hand, implicit in our implementation of the theory of involutive differentiation equations is the fact that the determination of a well-posed cross-section is prescribed by a Pommaret basis, [63], and, in general, Pommaret and Gröbner bases are not necessarily the same. Only when the ideal is stable can one guarantee that its reduced Pommaret basis equals its reduced Gröbner basis, [40].

Remark 8.17. A well-posed or algebraic cross-section is the Lie pseudo-group analogue of a minimal order cross-section introduced in [50] for finite-dimensional Lie group actions. In both cases, it has the property that pseudo-group jets are normalized as soon as possible. More precisely, a cross-section $\mathcal{K} \subset \mathrm{J}^\infty$ is of *minimal order* if for all $n \geq 0$ its projection $\mathcal{K}^{(n)} = \pi^\infty_n(\mathcal{K}) \subset \mathrm{J}^n$ forms a cross-section to the orbits of $\overline{\mathcal{G}}^{(n)}$ on J^n.

In light of Theorem 8.10, for all $n > n_f$, the pivots of $\Sigma^n_{\mathcal{K},\mathrm{REF}}$ are complementary to the pivots of Ψ^n_{REF}. Therefore, the cross-section equations $\mathcal{K}^{>n_f} = \{u^\alpha_J = c^\alpha_j \mid (\alpha, J) \in \mathcal{I}_{\mathcal{K}} > n_f\}$ specify the parametric jets of order $> n_f$ in the normal form determining equations. On the other hand, the cross-section $\mathcal{K}^{(n_f)} = \{x^i = c^i, u^\alpha_J = c^\alpha_J \mid i,\ (\alpha, J) \in \mathcal{I}^{(n_f)}_{\mathcal{K}}\}$ of order $\leq n_f$ determines the parametric derivatives in the normal form equations (8.18), (8.19) of order $\leq n_f$ resulting from the moving frame implementation. When combined, the whole cross-section $\mathcal{K}$ therefore determines the Taylor coefficients of the functions occurring on the right hand side of the initial conditions (8.25). Indeed, $\mathcal{K}^{>n_f}$ determines the Taylor coefficients of order ≥ 1, while $\mathcal{K}^{\leq n_f}$ specifies the order zero Taylor coefficients. This brings us to the main theorem of the paper.

Theorem 8.18. *Let $\mathcal{G}$ be an analytic Lie pseudo-group acting transitively on $\mathcal{X}$ with its prolonged action acting eventually freely. If the cross-section*

$$\mathcal{K} = \{x^i = 0, u^\alpha_J = c^\alpha_J \mid i,\ (\alpha, J) \in \mathcal{I}_{\mathcal{K}}\} \tag{8.30}$$

is well-posed, with the normalization constants c^α_J defining analytic functions

$$C^\alpha(y) = \sum_{J \in \mathcal{I}^\alpha_{\mathcal{K}}} \frac{c^\alpha_J}{J!}\, y^J, \qquad \alpha = 1, \ldots, q, \tag{8.31}$$

for the cross-section power series, then the corresponding normal form power series

$$u^\alpha(y) = \sum_J \frac{u^\alpha_J}{J!} y^J$$

defines an analytic function in the neighborhood of the origin.

Remark 8.19. Implicit in the statement of Theorem 8.18 is the fact that the coordinates used to express the well-posed cross-section (8.30) are δ-regular. Indeed, by definition $\mathcal{G}$ is a Lie pseudo-group if its elements are the solutions to an involutive system of differential equations, and involutivity, within our framework, requires δ-regularity.

Remark 8.20. Kossovskiy and Zaitsev have realized in [37] the importance of working with well-posed cross-sections when constructing convergent normal forms. As mentioned in the first paragraph of section two of their work, they resolve the problem of divergence of Kolář's normal form for degenerate hypersurfaces in $\mathbb{C}^2$, [35], by selecting a well-posed/minimal order cross-section.

Since the determining equations $\mathcal{K}^{>n_f}$ of a well-posed cross-section of order $> n_f$ are specified by the well-posed initial conditions (8.25), the set of indices $\mathcal{I}_{\mathcal{K}}^{>n_f}$ determining the parametric normal form jets of order $> n_f$ admits the *Rees decomposition*

$$\mathcal{I}_{\mathcal{K}}^{>n_f} = \biguplus_{(\alpha,J)\in\mathcal{I}_{\mathcal{K}}^{n_f+1}} \mathcal{C}^\alpha(J) \tag{8.32}$$

consisting of the disjoint union of involutive cones

$$\mathcal{C}^\alpha(J) = \{(\alpha, J, k^1, \ldots, k^n) \mid 1 \le k^j \le \mathrm{cls}(J) \text{ and } n \ge 0\}$$

with (α, J) ranging over all the tuples in $\mathcal{I}_{\mathcal{K}}^{n_f+1}$. According to [63, Proposition 5.1.6], the Rees decomposition (8.32) is sufficient to guarantee the existence of a Pommaret basis for the ideal $\Upsilon^{>n_f} = \Psi^{>n_f}$. This allows us to introduce a simple criterion to determine that a cross-section is well-posed without having to compute the normal form determining equations (8.23), (8.24).

Theorem 8.21. *Let $\mathcal{G}$ be a Lie pseudo-group whose prolonged action becomes free at order n_f. A cross-section $\mathcal{K}$ is well-posed if it is of minimal order and its set of defining indices $\mathcal{I}_{\mathcal{K}}^{>n_f}$ admits a Rees decomposition* (8.32)*.*

Proof. We need to show that there exists a system of normal form determining equations that is involutive at order $n_f + 1$ with $\mathcal{K}$ providing well-posed initial conditions.

Since the prolonged action becomes free, by Theorem 6.5 the pseudo-group $\mathcal{G}$ is reducible with reduced determining equations (7.2). The normal form determining equations are then obtained by substituting the chain rule formulas (7.3) into (7.2) to obtain (7.5), which are subsequently solved for the principal jet pseudo-group jets $\overline{X}^i_J$ and the principal normal form jets u^β_K, with $(\beta, K) \notin \mathcal{I}_{\mathcal{K}}^{(n)}$. In order that $\mathcal{I}_{\mathcal{K}}^{(n)}$ be as large as possible, we must require that as many as possible reduced pseudo-group jets $\overline{X}^i_J$ be principal. This, in other words, is equivalent to requiring that the cross-section $\mathcal{K}$ be of minimal order.

The order $n_f + 1$ normal form determining equations are given by equations of the form (8.23). The equations for the reduced pseudo-group jets do not provide any obstruction to involutivity, and therefore it suffices to consider the equations for the normal form jets. By assumption, since $\mathcal{I}_{\mathcal{K}}^{>n_f}$ admits a Rees decomposition (8.32), the symbols $\Upsilon^{>n_f} = \Psi^{>n_f}$ admit a Pommaret basis. The existence of the Pommaret basis implies that it is possible to express the differential equations for the normal form jets of order $n_f + 1$ is such a way that (8.23) is involutive with the parametric normal form jets u^γ_K of order $|K| = n_f + 1$ specified by the cross-section. *Q.E.D.*

Example 8.22. For our running example, consisting of the Lie pseudo-group (4.4), a well-posed cross-section is given by (6.16), which we can verify satisfies the hypotheses of Theorem 8.21. First, for all $n \geq 0$, $\mathcal{K}^{(n)}$ is transversed to the prolonged pseudo-group action and thus is of minimal order. Next, since the prolonged action becomes free at order $n_f = 2$, consider the cross-section determining equations of order > 2 given by

$$\mathcal{K}^{>2} = \{u_{x^{k+1}} = c_{k+1},\, u_{x^k y} = d_k \mid k \geq 0\}.$$

The corresponding set of determining indices has the Rees decomposition

$$\begin{aligned} \mathcal{I}_{\mathcal{K}}^{>2} &= \{(k+1,0),\, (k,1) \mid k \geq 2\} \\ &= \{(k+1,0) \mid k \geq 2\} \biguplus \{(k,1) \mid k \geq 2\} = \mathcal{C}(3,0) \biguplus \mathcal{C}(2,1). \end{aligned}$$

9 Chains.

In [12], Chern and Moser introduced the concept of a chain as a tool for proving the convergence of their normal form power series for CR hypersurfaces $S \subset \mathbb{C}^{n+1}$. A regular curve $\boldsymbol{\mathcal{C}} \subset S$ in the hypersurface S is said to be a *chain* if its projection $\pi(\boldsymbol{\mathcal{C}}) \subset \mathcal{X}$ onto the space of independent variables can be rectified by a biholomorphic transformation that also normalizes the Taylor coefficients of the hypersurface S appearing in the Chern–Moser normal form. In their paper, Chern and Moser employ a finite sequence of transformations to successively place the Taylor expansion of the transformed surface in normal form. Since each transformation is analytic as it either satisfies an algebraic constraint or is the solution to an analytic system of ordinary differential equations, the final transformed hypersurface is analytic, and hence its Taylor series, which is now in normal form, converges.

To make the discussion more precise, let us review the convergence argument in [12] when $n = 1$ so that $S \subset \mathbb{C}^2$ is a three-dimensional hypersurface locally parametrized by

$$S = \{(Z, \overline{Z}, U, \widehat{V}(Z, \overline{Z}, U))\},$$

where $(Z, \overline{Z}, U) \in \mathcal{X}$ are the independent variables with $Z \in \mathbb{C}$ and $U \in \mathbb{R}$. We refer the reader to [42] for a detailed account of this particular case. As in [12], let $W = U + \mathrm{i}\,V$. After translation, we can work at the origin and consider the Taylor expansion

$$\widehat{V}(Z, \overline{Z}, U) = \sum_{j,k=0}^{\infty} Z^j \overline{Z}^k F_{j,k}(U), \tag{9.1}$$

where the Taylor coefficients and powers of U are contained in the functions $F_{j,k}$. By assumption $F_{0,0}(0) = 0$, since the hypersurface has been translated to the origin. One then seeks to find a chain, meaning a curve

$$\boldsymbol{\mathcal{C}} = \{(\psi(u), \varphi(u))\} \subset S, \qquad \text{with} \qquad \varphi_u(0) \neq 0 \tag{9.2}$$

whose projection $\pi(\boldsymbol{\mathcal{C}})$ is holomorphically rectified to the line $\ell = \{(0,0,u)\}$ and sends the hypersurface Taylor series (9.1) to the Chern–Moser normal form. This is accomplished by the following sequence of analytic transformations, each of which serves to normalize some of the Taylor coefficients in the normal form.

Step 1: The transformation

$$Z = z + \psi(w), \qquad W = \varphi(w)$$

is holomorphic, takes $\pi(\boldsymbol{\mathcal{C}})$ into ℓ and sends (9.1) to[6]

$$v = \sum_{j+k\geq 1} z^j \overline{z}^k F_{j,k}(u).$$

We observe that such a transformation does not impose any constraint on the chain.

Step 2: Cancel the harmonic terms $z^j F_{j,0}(u)$ and $\overline{z}^k F_{0,k}(u)$ using a transformation of the form

$$z^* = z, \qquad w^* = w + g(z, w) \qquad \text{with} \qquad g(0, w) = 0, \tag{9.3}$$

so that the new power series is

$$v = \sum_{j\geq 1 \text{ or } k\geq 1} z^j \overline{z}^k F_{j,k}(u).$$

The function $g(z, w)$ is derived in the proof of [12, Lemma 3.2] and is found by solving an algebraic equation. We note that (9.3) does not affect the line $\ell = \{(0, 0, u)\}$, which is also the case for all the upcoming transformations.

Step 3: Under the assumption that the hypersurface is Levi nondegenerate, which means that $\partial^2 \widehat{V}/\partial Z\, \partial \overline{Z} \neq 0$, normalize $z\overline{z}^k F_{1,k}(u) = 0$ and $z^j \overline{z} F_{j,1}(u) = 0$ using

$$z^* = z + f(z, w), \qquad w^* = w,$$

with $f(0, w) = 0$, $f_z(0, w) = 0$ so that

$$v = z\overline{z} F_{1,1}(u) + \sum_{j,k\geq 2} z^j \overline{z}^k F_{j,k}(u),$$

where $F_{1,1}(0) \neq 0$. The function $f(z, w)$ satisfies an algebraic equation given in the proof of [12, Lemma 3.3].

Step 4: Normalize $F_{1,1}(u) = 1$ using

$$z^* = C(w)\, z, \qquad w^* = w \tag{9.4}$$

so that the transformed power series is

$$v = z\overline{z} + \sum_{j,k\geq 2} z^j \overline{z}^k F_{j,k}(u).$$

To do so, it suffices to take

$$C(u) = \sqrt{F_{1,1}(u)} \tag{9.5}$$

and then replace u by w to obtain the transformation (9.4).

[6]During the course of the procedure, the expressions for the Taylor coefficient functions F_{jk} will change. We avoid introducing new notation for each version.

Step 5: Normalize $F_{2,2}(u) = 0$, $F_{3,2}(u) = \overline{F_{3,2}(u)} = 0$, and $F_{3,3}(u) = 0$, so that the Chern–Moser normal form is

$$v = z\overline{z} + z^4\overline{z}^2 F_{4,2}(u) + z^2\overline{z}^4 F_{2,4}(u) + \sum_{\substack{j+k\geq 7\\ j,k\geq 2}} z^j\overline{z}^k F_{j,k}(u).$$

The normalization $F_{3,2}(u) = 0$ imposes a differential constraint on first component of the chain (9.2) given by a second order ordinary differential equation

$$\psi_{uu}(u) = Q(\psi(u), \overline{\psi}(u), \psi_u(u), \overline{\psi}_u(u), u).$$

The explicit formula for Q is not provided in [12]. For three-dimensional hypersurfaces, a Lie theoretic description of this equation is given in [42].

The normalization $F_{22}(u) = 0$ is achieved using the transformation

$$z^* = \lambda(w)\, z, \qquad w^* = w, \tag{9.6}$$

satisfying $\lambda(u)\,\overline{\lambda(u)} = 1$, $\lambda(0) = 1$, and solving the first order ordinary differential equation

$$\lambda_u(u) = -\frac{\mathrm{i}}{2} F_{2,2}(u)\,\lambda(u).$$

The transformations (9.4) and (9.6) are slightly different. In light of (9.5), the function $C(w)$ in (9.4) is real-valued or purely imaginary depending on whether $F_{1,1}(u) > 0$ or $F_{1,1}(u) < 0$, while the function $\lambda(w)$ in (9.6) is complex-valued.

Finally, the normalization $F_{3,3}(u) = 0$ is achieved via the transformation

$$z^* = z\sqrt{\varphi_w(w)}, \qquad w^* = \varphi(w),$$

with $\varphi(\mathbb{R}) \subset \mathbb{R}$, $\varphi(0) = 0$, $\varphi_w(0) \in (0,\infty)$ and satisfying the third order ordinary differential equation

$$\varphi_{uuu}(u) = \frac{3}{2}\frac{\varphi_{uu}^2(u)}{\varphi_u(u)} - 3F_{3,3}(u)\varphi_u(u).$$

This provides constraints on the second component of the chain (9.2).

We now see how this particular Chern–Moser construction can be formulated within our general framework. To make the connection evident, let us assume for now that the class one Cartan character of the involutive normal form determining equations is the only nonzero character, so

$$\overline{\mathsf{c}}_n^{(1)} \neq 0, \qquad \overline{\mathsf{c}}_n^{(2)} = \cdots = \overline{\mathsf{c}}_n^{(p)} = 0. \tag{9.7}$$

In this particular setting, the general solution depends only on functions of one variable, and the initial conditions (8.25) reduce to

$$u^\gamma_{K\setminus 1}(x^1, 0, \ldots, 0) = f^\gamma_{K\setminus 1}(x^1), \qquad u^\beta_J(0,\ldots,0) = f^\beta_J. \tag{9.8}$$

Since the Taylor coefficients of the initial conditions (9.8) determine the cross-section $\mathcal{K}$, the left hand side of the equations (9.8) can be replaced by the cross-section functions (8.31) so that

$$C^{\gamma}_{K\backslash 1}(x^1, 0\ldots,0) = f^{\gamma}_{K\backslash 1}(x^1) \qquad \text{and} \qquad C^{\beta}_{J}(0,\ldots,0) = f^{\beta}_{J}. \tag{9.9}$$

We observe that the equations (9.9) are defined on the line $\ell = \{(x^1, 0, \ldots, 0)\} \subset \mathcal{X}$. Then, a one-dimensional chain $\boldsymbol{\mathcal{C}}$ is a regular curve in the section S with the property that there exists a pseudo-group transformation $\varphi \in \mathcal{G}$ mapping $\boldsymbol{\mathcal{C}}$ to the curve $\varphi^{-1}(\boldsymbol{\mathcal{C}}) = \boldsymbol{\ell} = (\ell, C(\ell))$ contained in the normal form s, where $C(y)$ is the cross-section function (8.31). In particular, we note that the projection of the chain onto the space of independent variables $\pi(\boldsymbol{\mathcal{C}}) \subset \mathcal{X}$ is rectified to the line $\varphi^{-1}|_{\mathcal{X}}(\pi(\boldsymbol{\mathcal{C}})) = \ell$. In other words, $\pi(\boldsymbol{\mathcal{C}}) = \varphi|_{\mathcal{X}}(\ell)$.

Thus, to find the chain $\boldsymbol{\mathcal{C}} = \varphi(\ell, C(\ell))$ passing through $(X_0, \widehat{U}(X_0))$, it suffices to find $\varphi \in \mathcal{G}$ such that

$$(\varphi|_{\mathcal{X}}(\ell), \widehat{U}(\varphi|_{\mathcal{X}}(\ell))) = \varphi(\ell, C(\ell)). \tag{9.10}$$

Setting $(X(x,u), U(x,u)) = \varphi(x,u)$ and $(\overline{X}|_{\boldsymbol{\ell}}, \overline{U}|_{\boldsymbol{\ell}}) = \varphi(\ell, C(\ell)) = (X(\ell, C(\ell)), U(\ell, C(\ell)),$ equation (9.10) reduces to solving

$$\widehat{U}(\overline{X}|_{\boldsymbol{\ell}}) = \overline{U}|_{\boldsymbol{\ell}}. \tag{9.11}$$

We note that (9.11) is the same equation as (7.1) but restricted to the curve $\boldsymbol{\ell} = (\ell, C(\ell))$. With $\boldsymbol{\ell}$ being one-dimensional, the equations (9.11) form a system of ordinary differential equations for the parametric reduced pseudo-group jets with initial value $(\overline{X}|_{\boldsymbol{\ell}}(0), \overline{U}|_{\boldsymbol{\ell}}(0)) = (X_0, \widehat{U}(X_0))$. We now show how this works with two examples.

Example 9.1. Consider the Lie pseudo-group

$$X = f(x), \qquad Y = y + b, \qquad U = \frac{u}{f_x(x)}, \tag{9.12}$$

acting on surfaces $u(x,y)$, where $f \in \mathcal{D}(\mathbb{R})$ and $b \in \mathbb{R}$. We assume $u \neq 0$ in what follows. The normal form determining equations of order one are

$$\overline{X}_y = \overline{Y}_x = 0, \qquad \overline{Y}_y = 1, \qquad \widehat{U}\,\overline{X}_x = u, \qquad \widehat{U}_Y \overline{X}_x = u_y. \tag{9.13}$$

These equations are involutive with indices and Cartan characters

$$\overline{\mathsf{b}}^{(1)}_1 = 2, \qquad \overline{\mathsf{b}}^{(2)}_1 = 3, \qquad \overline{\mathsf{c}}^{(1)}_1 = 1, \qquad \overline{\mathsf{c}}^{(2)}_1 = 0.$$

As it can be seen from the pseudo-group (9.12), the parametric reduced pseudo-group jets are $\overline{X}_{x^k}$, $k \geq 0$. A moving frame for the pseudo-group (9.12) was constructed in [54] using the cross-section

$$\mathcal{K} = \{x = y = 0,\ u = 1,\ u_{x^k} = 0,\ k \geq 1\}.$$

This induces the initial conditions

$$X(0,0) = X_0, \qquad Y(0,0) = Y_0, \qquad u(x,0) = 1$$

for the system of equations (9.13). The corresponding cross-section function is $u(x,0) = C(x) = 1$, and defines the line

$$\boldsymbol{\ell} = \{(x,0,1)\} \subset s \tag{9.14}$$

in the normal form.

A chain $\boldsymbol{\mathcal{C}} = \{(\overline{X}(x), Y_0, \widehat{U}(\overline{X}(x), Y_0))\} \subset S$ is a regular curve that is rectified to the line (9.14) by a pseudo-group transformation (9.12). First, for the y-coordinate of $\boldsymbol{\ell}$ to be sent to Y_0 in the chain, a translation with $b = Y_0$ is performed. On the other hand, the function $\overline{X}(x)$ satisfies the chain determining equation (9.11), which yields the differential equation

$$\widehat{U}(\overline{X}(x), Y_0) = \overline{U} = \frac{1}{\overline{X}_x(x)}.$$

In other words,

$$\overline{X}_x(x) = \frac{1}{\widehat{U}(\overline{X}(x), Y_0)} \qquad \text{with the initial condition} \qquad \overline{X}(0) = X_0.$$

This is an ordinary differential equation for $\overline{X}(x)$, whose right hand side is analytic when the surface $\widehat{U}(X, Y)$ is analytic, and hence defines an analytic normalizing transformation.

Example 9.2. As a second example, we consider our running example, which consists of the Lie pseudo-group (4.4). The cross-section function corresponding to the normal form (8.27) is

$$C(x, y) = c(x) + y\, d(x) + \frac{y^2}{2}, \tag{9.15}$$

where $c(x)$ and $d(x)$ are specified functions, which in the simplest version can be set to zero $c(x) = d(x) = 0$, such as in Example 6.8. We now find the chain corresponding to the two initial conditions

$$u(x, 0) = c(x) \qquad \text{and} \qquad u_y(x, 0) = d(x).$$

The first initial condition gives the ordinary differential equation

$$\widehat{U}(\overline{X}(x), \overline{Y}(x, 0)) = c(x) + \frac{\overline{Y}_x(x, 0)}{\overline{X}_x(0)}.$$

The second initial condition gives

$$\widehat{U}_Y(\overline{X}(x), \overline{Y}(x, 0))\overline{X}_x(x) = d(x) + \frac{\overline{X}_{xx}(x)}{\overline{X}_x(x)}.$$

Thus, the chain $\boldsymbol{\mathcal{C}} = \{(\overline{X}(x), \overline{Y}(x, 0), \widehat{U}(\overline{X}(x), \overline{Y}(x, 0))\}$ is obtained by solving a system of two ordinary differential equations

$$\begin{aligned} \overline{X}_{xx}(x) &= \widehat{U}_Y(\overline{X}(x), \overline{Y}(x, 0))\overline{X}_x^2(x) - d(x)\overline{X}_x(x), \\ \overline{Y}_x(x, 0) &= (\widehat{U}(\overline{X}(x), \overline{Y}(x, 0)) - c(x))\overline{X}_x(x), \end{aligned} \tag{9.16}$$

subject to the initial conditions

$$\overline{Y}(0, 0) = Y_0, \qquad \overline{X}(0, 0) = X_0, \qquad \overline{X}_x(0) = \overline{X}_x^0.$$

Again, analyticity of the surface $\widehat{U}(X, Y)$ and the cross-section function (9.15) implies analyticity of the right hand sides of the differential equations (9.16), and thus analyticity of the normalizing transformation. To obtain the quadratic term in y in the normal form series (9.15), we need to impose the algebraic constraint $\overline{X}_x^0 = \sqrt{\widehat{U}_{YY}(X_0, Y_0)}$ on the initial conditions.

The above discussion focused on one-dimensional chains when the constraint on the Cartan characters (9.7) holds. In the more general situation, when there are one or more nonzero higher order Cartan characters, the appropriate analog of chains will include submanifolds of dimension ≥ 2. For example, if the largest nonzero Cartan character is $\bar{\mathsf{c}}_n^{(k)}$, then a k-dimensional chain $\boldsymbol{\mathcal{C}}_k$ is a submanifold in S that can be mapped to

$$\boldsymbol{\mathcal{P}}_k = (\mathcal{P}_k, C(\mathcal{P}_k)),$$

where $C(y)$ is the cross-section function (6.23), and such that the projection $\pi(\boldsymbol{\mathcal{C}}_k) \subset \mathcal{X}$ is rectified to the k-dimensional plane $\mathcal{P}_k = \{(x^1, \ldots, x^k, 0, \ldots, 0)\}$. The pseudo-group transformation rectifying the chain will satisfy a system of partial differential equations for the parametric reduced pseudo-group jets given by

$$\widehat{U}(\overline{X}|_{\boldsymbol{\mathcal{P}}_k}) = \overline{U}|_{\boldsymbol{\mathcal{P}}_k}.$$

Inside the k-dimensional chain $\boldsymbol{\mathcal{C}}_k$ there may be a sequence of lower dimensional chains $\boldsymbol{\mathcal{C}}_1 \subset \boldsymbol{\mathcal{C}}_2 \subset \cdots \subset \boldsymbol{\mathcal{C}}_{k-1} \subset \boldsymbol{\mathcal{C}}_k$, with $\pi(\boldsymbol{\mathcal{C}}_j)$ mapped to the j-dimensional coordinate $\mathcal{P}_j = \{(x^1, \ldots, x^j, 0, \ldots, 0)\}$ under a suitable pseudo-group transformation. The existence of these subchains will depend on the form of the initial values (8.25), which is ultimately determined by the Cartan characters, [63, Proposition 8.2.10]. These higher dimensional chains appear in [16], where Ežov and Schmalz introduce two-dimensional chains when studying normal forms for elliptic CR submanifolds in $\mathbb{C}^4$.

10 Additional Examples.

In this final section we provide four more basic examples illustrating the results of the paper. We conclude by showing how the convergence theorem of Chern and Moser, [12], can be deduced from our general theorem; this requires finding suitable coordinates that assure involutivity of the determining equations. In these examples, we will omit the bar notation over X and Y and the hat notation on U and its derivatives, which will unclutter the equations while hopefully not leading to any confusion now that the procedures and meanings are clear.

Example 10.1. In our previous examples, the pseudo-group considered only involved functions depending on one independent variable, namely x. In this example we consider the pseudo-group

$$X = f(x), \qquad Y = g(y), \qquad U = u + c,$$

where $f, g \in \mathcal{D}(\mathbb{R})$ and $c \in \mathbb{R}$. The first order reduced determining equations are

$$X_y = Y_x = 0, \qquad U_x = u_x, \qquad U_y = u_y, \tag{10.1}$$

while the order two equations are

$$X_{xy} = X_{yy} = Y_{xx} = Y_{xy} = 0, \quad U_{xx} = u_{xx}, \quad U_{xy} = u_{xy}, \quad U_{yy} = u_{yy}.$$

Using the ordering $x \prec y$, the indices for the order one equations (10.1) are $\mathsf{b}_1^{(1)} = 2$ and $\mathsf{b}_1^{(2)} = 2$ so that

$$\mathsf{b}_1^{(1)} + 2\,\mathsf{b}_1^{(2)} = 6 \neq 7 = \mathsf{r}_2.$$

Alternatively, the Cartan characters are $\mathsf{c}_1^{(1)} = 1$, $\mathsf{c}_1^{(2)} = 1$ and

$$\mathsf{c}_1^{(1)} + 2\,\mathsf{c}_1^{(2)} = 3 \neq 2 = \mathsf{d}_2.$$

Thus the equations (10.1) are not involutive. In fact at any order n the reduced determining equations are not involutive. To see this, we observe that the order n determining equations for Y are

$$Y_{x^n} = Y_{x^{n-1}y} = \cdots = Y_{xy^{n-1}} = 0,$$

which are all of class one with respect to our chosen ordering. From those equations it is not possible to obtain the equation $Y_{xy^n} = 0$ at order $n+1$ since y is not a multiplicative variable. Changing the ordering to $y \prec x$ would not resolve the issue as the same problem would now appear among the determining equations for X. The conclusion is that the current coordinates are not δ-regular.

As emphasized in mkregularity-r, we must thus introduce new δ-regular coordinates. This is done, for example, by setting

$$x = t + s \qquad \text{and} \qquad y = t - s.$$

The pseudo-group then becomes

$$T + S = f(t+s), \qquad T - S = g(t-s), \qquad U = u + c$$

or

$$T = \frac{f(t+s) + g(t-s)}{2}, \qquad S = \frac{f(t+s) - g(t-s)}{2}, \qquad U = u + c.$$

Relabeling the variables and functions, we now consider the Lie pseudo-group

$$X = f(x+y) + g(x-y), \qquad Y = f(x+y) - g(x-y), \qquad U = u + c. \tag{10.2}$$

The normal form determining equations can be obtained by recursively applying the total differential operators

$$\mathrm{D}_x = X_x\,\mathrm{D}_X + Y_x\,\mathrm{D}_Y, \qquad \mathrm{D}_y = X_y\,\mathrm{D}_X + Y_y\,\mathrm{D}_Y, \tag{10.3}$$

to the pseudo-group transformations (10.2) and eliminating the derivatives of the functions f, g from the resulting equations. At first order, this results in

$$\begin{gathered} X_x = f_t + g_t, \qquad X_y = f_t - g_t, \qquad Y_x = f_t - g_t, \qquad Y_y = f_t + g_t, \\ X_x U_X + Y_x U_Y = u_x, \qquad X_y U_X + Y_y U_Y = u_y, \end{gathered} \tag{10.4}$$

where f_t, g_t represent the first order derivatives of f, g. Provided $U_X^2 - U_Y^2 \neq 0$, we can eliminate f_t, g_t to produce the first order normal form determining equations:

$$X_x = Y_y = \frac{u_x U_X - u_y U_Y}{U_X^2 - U_Y^2}, \qquad X_y = Y_x = \frac{u_y U_X - u_x U_Y}{U_X^2 - U_Y^2}, \tag{10.5}$$

where we take u_x, u_y to be the parametric derivatives. This is consistent with the moving frame construction, but not the theory of involutivity, which would require solving for u_y, assuming the ordering $x \prec y$. In accordance with the discussion in Section 8, this is a second example illustrating the discrepancy between the two theories at low order.

The normal form determining equations of order two can be obtained by differentiating (10.5) using (10.3) — or, alternatively applying them to (10.4) and eliminating the first and second derivatives of f, g — which produces

$$X_{xy} = Y_{xx}, \qquad X_{yy} = X_{xx}, \qquad Y_{xy} = X_{xx}, \qquad Y_{yy} = Y_{xx}, \tag{10.6}$$

along with

$$\begin{aligned} X_{xx} &= \frac{1}{U_X^2 - U_Y^2}\big[(u_{xx} - U_{XX}X_x^2 - 2U_{XY}X_xY_x - U_{YY}Y_x^2)U_X \\ &\qquad\qquad - (u_{xy} - U_{XX}X_xY_x - U_{XY}(X_x^2 + Y_x^2) - U_{YY}X_xY_x)U_Y\big], \\ Y_{xx} &= \frac{1}{U_X^2 - U_Y^2}\big[(u_{xy} - U_{XX}X_xY_x - U_{XY}(X_x^2 + Y_x^2) - U_{YY}X_xY_x)U_X \\ &\qquad\qquad - (u_{xx} - U_{XX}X_x^2 - 2U_{XY}X_xY_x - U_{YY}Y_x^2)U_Y\big], \end{aligned} \tag{10.7}$$

and

$$u_{yy} = u_{xx} - \frac{(u_x^2 - u_y^2)(U_{XX} - U_{YY})}{U_X^2 - U_Y^2}. \tag{10.8}$$

Note that to place (10.7) in the proper Cartan normal form, we should replace X_x, X_y, Y_x, Y_y by their formulas from (10.5), although the resulting expressions are a bit unwieldy. The additional second order parametric derivatives are u_{xx}, u_{xy}.

We can easily verify that the order two normal form determining equations are involutive. Indeed, the indices and Cartan characters[7] are

$$\mathsf{b}_2^{(1)} = 4, \quad \mathsf{b}_2^{(2)} = 3, \qquad \mathsf{c}_2^{(1)} = 2, \quad \mathsf{c}_2^{(2)} = 0,$$

and they satisfy the algebraic involutivity tests

$$\mathsf{b}_2^{(1)} + 2\,\mathsf{b}_2^{(2)} = 10 = \mathsf{r}_3 \qquad \text{or} \qquad \mathsf{c}_2^{(1)} + 2\,\mathsf{c}_2^{(2)} = 2 = \mathsf{d}_3.$$

Since there is no integrability condition, the equations are involutive.

On the space of regular jets $V^{(\infty)} = \{U_X^2 \neq U_Y^2\} \subset \mathrm{J}^\infty$, the prolonged action becomes free at order one[8] and a cross-section is given by

$$\mathcal{K} = \{x = y = 0, \quad u_{x^k} = c_k, \quad u_{yx^k} = d_k \mid k \geq 0 \text{ and } c_1^2 - d_0^2 \neq 0\}. \tag{10.9}$$

The corresponding cross-section function is

$$C(x,y) = c(x) + y\,d(x) \qquad \text{with} \qquad c_x^2(0) - d^2(0) \neq 0,$$

and the normal form is

$$u(x,y) = c(x) + y\,d(x) + y^2\,w(x,y). \tag{10.10}$$

[7] As above, we only need to compute one of these sets to verify involutivity.

[8] Every Lie pseudo-group is free at order $n = 0$. Freeness is only of interest when $n \geq 1$, [54].

In the simplest case, we can take $c(x) = x$ and $d(x) = 0$. Observe that the initial conditions depend on functions of the same variable x, which is not the case if we were to write the system in the original δ irregular coordinates.

According to the general theory, since the action becomes free at order one, the involutivity of the order two normal form determining equations (10.6), (10.7), (10.8) guarantees the convergence of the normal form (10.10) provided well-posed analytic initial conditions are provided and the target function $U(X, Y)$ is analytic. On the other hand, the equations (10.5) provide algebraic constraints among the order one jets at the origin. The desired initial conditions are given by

$$\begin{gathered} X(0,0) = X_0, \qquad Y(0,0) = Y_0, \qquad u(0,0) = C(0,0) = c_0, \\ u_x(x,0) = C(x,0) = c_x(x), \qquad u_y(x,0) = C_y(x,0) = d(x), \end{gathered}$$

where $c_x^2(0) - d^2(0) \neq 0$ and $c(x)$, $d(x)$ are two analytic functions. This shows that (10.9) is a well-posed cross-section satisfying the hypotheses of Theorem 8.21. Indeed the cross-section is of minimal order with the set of defining indices of order > 1 admitting the Rees decomposition

$$\begin{aligned} \mathcal{I}_{\mathcal{K}}^{>1} &= \{(k+1, 0),\ (k, 1) \mid k \geq 1\} \\ &= \{(k+1, 0) \mid k \geq 1\} \biguplus \{(k, 1) \mid k \geq 1\} = \mathcal{C}(2, 0) \biguplus \mathcal{C}(1, 1). \end{aligned}$$

Example 10.2. In the examples considered thus far, the Lie pseudo-group actions were all quasi-horizontal, in the chosen system of coordinates, in the terminology of [2]. This property is not necessary for the results of this paper to be valid, and we illustrate this fact by considering the Lie pseudo-group

$$X = x + a, \qquad Y = y + b, \qquad U = f(u), \tag{10.11}$$

where $a, b \in \mathbb{R}$ and $f \in \mathcal{D}(\mathbb{R})$. Of course, the pseudo-group (10.11) can be transformed into a quasi-horizontal action via the hodograph transformation $(x, y, u) \to (u, y, x)$, but we will not make this transformation here.

Provided $U_X \neq 0$, the normal form determining equations of order ≤ 2 are

$$X_x = Y_y = 1, \qquad X_y = Y_x = 0, \qquad u_y = \frac{u_x U_Y}{U_X}, \tag{10.12}$$

$$\begin{gathered} X_{xx} = X_{xy} = X_{yy} = Y_{xx} = Y_{xy} = Y_{yy} = 0, \\ u_{xy} = \frac{u_x U_{XY} + u_{xx} U_Y}{U_X} - \frac{u_x U_Y U_{XX}}{U_X^2}, \qquad u_{yy} = \frac{u_x U_{YY}}{U_X} + \frac{u_{xx} U_Y^2}{U_X^2} - \frac{u_x U_Y^2 U_{XX}}{U_X^3}. \end{gathered} \tag{10.13}$$

The indices and Cartan characters for the order one determining equations (10.12) are

$$\mathsf{b}_1^{(1)} = 2, \quad \mathsf{b}_1^{(2)} = 3, \qquad \mathsf{c}_1^{(1)} = 1, \quad \mathsf{c}_1^{(2)} = 0$$

so that the involutivity condition

$$\mathsf{b}_1^{(1)} + 2\,\mathsf{b}_1^{(2)} = 8 = \mathsf{r}_2, \qquad \text{or, equivalently,} \qquad \mathsf{c}_1^{(1)} + 2\,\mathsf{c}_1^{(2)} = 1 = \mathsf{d}_2$$

is satisfied. Since there are no integrability conditions, the order one determining equations (10.12) are involutive.

The pseudo-group action becomes free at order one. A well-posed cross-section is given by

$$\mathcal{K} = \{x = y = 0,\ u_{x^k} = c_k \mid k \geq 0 \text{ and } c_1 \neq 0\},$$

with the set of defining indices of order > 1 admitting the Rees decomposition

$$\mathcal{I}_{\mathcal{K}}^{>1} = \{(k, 0) \mid k \geq 2\} = \mathcal{C}(2, 0).$$

The corresponding cross-section function is

$$C(x) = c(x) \qquad \text{with} \qquad c_x(0) \neq 0,$$

and the normal form is given by

$$u(x, y) = c(x) + y\, w(x, y). \tag{10.14}$$

For simplicity, we could take $c(x) = x$. The action being free at order one, the general theory dictates that, assuming analyticity of the function $U(X, Y)$, analyticity of the normal form (10.14) will follow from the involutivity of the order two normal form determining equations (10.13) along with the equations (10.12) providing algebraic constraints among the first order jets at the origin. Formally well-posed initial conditions are given by

$$X(0, 0) = X_0, \quad Y(0, 0) = Y_0, \quad u(0, 0) = c(0), \qquad u_x(x, 0) = c_x(x),$$

with $c_x(0) \neq 0$.

Example 10.3. Up to this point, all the Lie pseudo-group actions considered only involved local diffeomorphisms of the real line. We now consider the pseudo-group

$$X = x + a, \qquad Y = g(x, y), \qquad Z = z + b, \qquad U = u, \tag{10.15}$$

where $g(x, y)$ depends on two variables with $g_y(x, y) \neq 0$ and $a, b \in \mathbb{R}$. In this example, we assume that $u = u(x, y, z)$ is a function of three variables.

Again, omitting bars and hats, the first order normal form determining equations can be obtained by applying the total differential operators

$$\mathrm{D}_x = \mathrm{D}_X + Y_x \mathrm{D}_Y, \qquad \mathrm{D}_y = Y_y \mathrm{D}_Y, \qquad \mathrm{D}_z = \mathrm{D}_Z, \tag{10.16}$$

to (10.15). Assuming that $U_Y \neq 0$, we can rewrite them in the form

$$X_x = 1,\ X_y = X_z = 0,\ Y_x = \frac{u_x - U_X}{U_Y},\ Y_y = \frac{u_y}{U_Y},\ Y_z = Z_x = Z_y = 0,\ Z_z = 1,\ u_z = U_Z, \tag{10.17}$$

where the parametric derivatives are u_x, u_y. We note that this is compatible, with both the theory of moving frames and involutivity. The second order normal form determining equations can be obtained by applying the differential operators (10.16) to the first order equations giving

$$\begin{gathered}X_{xx}=X_{xy}=X_{yy}=X_{xz}=X_{yz}=X_{zz}=0,\qquad Y_{xz}=Y_{yz}=Y_{zz}=0,\\ Z_{xx}=Z_{xy}=Z_{yy}=Z_{xz}=Z_{yz}=Z_{zz}=0,\\ Y_{xx}=\frac{u_{xx}-U_{XX}}{U_Y}-\frac{2U_{XY}(u_x-U_X)}{U_Y^2}-\frac{U_{YY}(u_x-U_X)^2}{U_Y^3},\\ Y_{xy}=\frac{u_{xy}}{U_Y}-\frac{U_{XY}u_y}{U_Y^2}-\frac{U_{YY}u_y(u_x-U_X)}{U_Y^3},\qquad Y_{yy}=\frac{u_{yy}}{U_Y}-\frac{U_{YY}u_y^2}{U_Y^3},\\ u_{xz}=U_{XZ}+\frac{U_{YZ}(u_x-U_X)}{U_Y},\qquad u_{yz}=\frac{U_{YZ}u_y}{U_Y},\qquad u_{zz}=U_{ZZ},\end{gathered}\tag{10.18}$$

and similarly for the higher order versions. The indices and Cartan characters for the order one normal form determining equations (10.17) are

$$\mathsf{b}_1^{(1)}=3,\quad \mathsf{b}_1^{(2)}=3,\quad \mathsf{b}_1^{(3)}=4,\qquad \mathsf{c}_1^{(1)}=1,\quad \mathsf{c}_1^{(2)}=1,\quad \mathsf{c}_1^{(3)}=0,$$

which satisfy involutivity condition

$$\mathsf{b}_1^{(1)}+2\,\mathsf{b}_1^{(2)}+3\,\mathsf{b}_1^{(3)}=21=\mathsf{r}_2\qquad\text{or, equivalently,}\qquad \mathsf{c}_1^{(1)}+2\,\mathsf{c}_1^{(2)}+3\,\mathsf{c}_1^{(3)}=3=\mathsf{d}_2.$$

Since there are no integrability constraints, the order one normal form determining equations are involutive.

The pseudo-group action becomes free at order one and a well-posed cross-section is given by

$$\mathcal{K}=\{x=y=0,\ u_{x^{k+1}}=c_k,\ u_{x^jy^{k+1}}=d_{j,k}\mid j,k\geq 0\text{ and }d_{0,0}\neq 0\}$$

with the defining indices of order >1 admitting the Rees decomposition

$$\begin{aligned}\mathcal{I}_{\mathcal{K}}^{>1}&=\{(i+2,0),\ (j,k+1)\mid i\geq 0,\ j+k\geq 1\}\\ &=\{(i+2,0)\mid i\geq 0\}\biguplus\{(j+1,1)\mid j\geq 0\}\biguplus\{(j,k+2)\mid j,k\geq 0\}\\ &=\mathcal{C}(2,0)\biguplus\mathcal{C}(1,1)\biguplus\mathcal{C}(0,2).\end{aligned}$$

The corresponding cross-section function $C(x,y)$ satisfies the constraints

$$C(0,0)=0,\qquad C_x(x,0)=c(x),\qquad C_y(x,y)=d(x,y)\quad\text{with}\quad C_y(0,0)=d(0,0)\neq 0.$$

In the simplest case, we could let $C(x,y)=y$. In general, the normal form is given by

$$u(x,y,z)=U_0+C(x,y)+z\,w(x,y,z),\tag{10.19}$$

where $U_0=U(X_0,Y_0,Z_0)$ is a constant. Since the prolonged action becomes free at order one, the convergence of the normal form (10.19) follows from the involutivity of the order two normal form determining equations (10.18) with the equations (10.17) providing algebraic constraints on the order one jets at the origin. Since the pseudo-group action (10.15) is intransitive, we also have the order zero normal form determining equation $u=U$, which also needs to be evaluated at the origin. Well-posed initial conditions are given by

$$\begin{gathered}X(0,0,0)=X_0,\qquad Y(0,0,0)=Y_0,\qquad Z(0,0,0)=Z_0,\\ u(0,0,0)=U_0,\qquad u_x(x,0,0)=C_x(x,0)=c(x),\qquad u_y(x,y,0)=C_y(x,y)=d(x,y).\end{gathered}$$

Example 10.4. As our penultimate example, we consider the Lie pseudo-group

$$X = x + a, \qquad Y = y + b, \qquad Z = z + f(x,y), \qquad U = u + g(x,y), \tag{10.20}$$

where f, g satisfy the Cauchy–Riemann equations

$$f_x = g_y, \qquad f_y = -g_x. \tag{10.21}$$

As in Example 10.3, we obtain the normal form determining equations by recursively applying the total differential operators

$$\mathrm{D}_x = \mathrm{D}_X + Z_x \mathrm{D}_Z, \qquad \mathrm{D}_y = \mathrm{D}_Y + Z_y \mathrm{D}_Z, \qquad \mathrm{D}_z = \mathrm{D}_Z, \tag{10.22}$$

to the pseudo-group transformations (10.20). At first order, we have

$$X_x = 1, \qquad X_y = X_z = 0, \qquad Y_y = 1, \qquad Y_x = Y_z = 0, \qquad Z_z = 1, \tag{10.23}$$

along with

$$Z_x = f_x, \quad Z_y = f_y, \quad U_X + Z_x U_Z = u_x + g_x, \quad U_Y + Z_y U_Z = u_y + g_y, \quad U_Z = u_z. \tag{10.24}$$

Eliminating the derivatives of f, g from the latter equations using (10.21) produces

$$Z_x = \frac{U_Z(u_x - U_X) - (u_y - U_Y)}{1 + U_Z^2}, \qquad Z_y = \frac{U_Z(u_y - U_Y) + u_x - U_X}{1 + U_Z^2}, \qquad u_z = U_Z, \tag{10.25}$$

where the parametric derivatives are u_x, u_y. As in Example 10.1, this is compatible with the moving frame construction but not involutivity, which would require solving for u_y in the first equation of (10.25), assuming the ordering $x \prec y \prec z$. The second order normal form determining equations can be obtained by using (10.16) to differentiate (10.23), (10.24), and then solving the resulting equations, or differentiating (10.23), (10.25) directly. We find

$$\begin{aligned}
&X_{xx} = X_{xy} = X_{yy} = X_{xz} = X_{yz} = X_{zz} = 0,\\
&Y_{xx} = Y_{xy} = Y_{yy} = Y_{xz} = Y_{yz} = Y_{zz} = 0, \qquad Z_{xz} = Z_{yz} = Z_{zz} = 0, \quad Z_{yy} = -Z_{xx},\\
&Z_{xx} = [-u_{xy} + U_{XY} + (u_{xx} - U_{XX})U_Z + (U_{YZ} - 2U_{XZ}U_Z)Z_x\\
&\qquad\qquad\qquad - U_Z U_{ZZ} Z_x^2 + U_{XZ} Z_y + U_{ZZ} Z_x Z_y]/[1 + U_Z^2],\\
&Z_{xy} = [u_{xx} - U_{XX} + (u_{xy} - U_{XY})U_Z - (2U_{XZ} + U_{YZ}U_Z)Z_x\\
&\qquad\qquad\qquad - U_{ZZ} Z_x^2 - U_{XZ} U_Z Z_y - U_Z U_{ZZ} Z_x Z_y]/[1 + U_Z^2],\\
&u_{xz} = U_{XZ} + U_{ZZ} Z_x, \qquad u_{yz} = U_{YZ} + U_{ZZ} Z_y, \qquad u_{zz} = U_{ZZ}.
\end{aligned} \tag{10.26}$$

One should replace Z_x, Z_y by their expressions in (10.25) to express the right hand sides in terms of only the parametric derivatives u_x, u_y, u_{xx}, u_{xy}; however, the resulting formulas are too unwieldy to display.

The indices and Cartan characters for the order two normal form determining equations (10.26) are

$$\mathsf{b}_2^{(1)} = 10, \quad \mathsf{b}_2^{(2)} = 7, \quad \mathsf{b}_2^{(3)} = 4, \qquad \mathsf{c}_2^{(1)} = 2, \quad \mathsf{c}_2^{(2)} = 1, \quad \mathsf{c}_2^{(3)} = 0.$$

Since

$$\mathsf{b}_2^{(1)} + 2\,\mathsf{b}_2^{(2)} + 3\,\mathsf{b}_2^{(3)} = 36 = \mathsf{r}_3 \qquad \text{or, equivalently,} \qquad \mathsf{c}_2^{(1)} + 2\,\mathsf{c}_2^{(2)} + 3\,\mathsf{c}_2^{(3)} = 4 = \mathsf{d}_3,$$

and there are no integrability conditions, the order two normal form equations are involutive.

The prolonged pseudo-group action becomes free at order one and a well-posed cross-section is given by

$$\mathcal{K} = \{x = y = z = 0, \quad u_{x^k} = c_k, \quad u_{x^j y^{k+1}} = d_{j,k} \mid j, k \geq 0\}.$$

Similar to the previous example, the set of defining indices of order > 1 admits the Rees decomposition

$$\mathcal{I}_{\mathcal{K}}^{>1} = \mathcal{C}(2,0,0) \biguplus \mathcal{C}(1,1,0) \biguplus \mathcal{C}(0,2,0).$$

The corresponding cross-section function $C(x,y)$ satisfies

$$C(x,0) = c(x) \qquad \text{and} \qquad C_y(x,y) = d(x,y).$$

and the normal form is

$$u(x,y,z) = C(x,y) + z\,w(x,y,z).$$

The convergence of the normal form follows from the involutivity of the order two normal form determining equations (10.26), together with the algebraic constraints obtained by evaluating the order one equations (10.23), (10.25) at the origin. Formally well-posed initial conditions are given by

$$\begin{gathered} X(0) = X_0, \qquad Y(0) = Y_0, \qquad Z(0) = Z_0, \qquad u(0,0,0) = C(0,0) = c_0, \\ u_x(x,0,0) = C_x(x,0) = c_x(x), \qquad u_y(x,y,0) = C_y(x,y) = d(x,y). \end{gathered}$$

Example 10.5. In [57] we revisited the Chern–Moser normal form problem for nondegenerate real hypersurfaces in $\mathbb{C}^2$ under the action of the pseudo-group of holomorphic transformations, [12], obtaining five inequivalent classes of normal forms termed locally umbilic, non-umbilic, generic, circular, and semi-circular. The convergence of these normal forms relied on results from [12]. We now use Theorems 8.18 and 8.21 to give an alternative argument.

Let $z, w = u + \mathrm{i}\,v$ be local coordinates on $\mathbb{C}^2$. Accordingly, the pseudo-group of holomorphic transformations $(z,w) \mapsto (Z(z,w), W(z,w))$ of $\mathbb{C}^2$, with $W = U + \mathrm{i}\,V$, is determined by the differential equations

$$Z_{\overline{z}} = 0, \qquad Z_v = \mathrm{i}\,Z_u, \qquad V_{\overline{z}} = \mathrm{i}\,U_{\overline{z}}, \qquad V_u = -U_v, \qquad V_v = U_u. \tag{10.27}$$

In [57], a real hypersurface $M \subset \mathbb{C}^2$ was locally parametrized as the graph of the real function

$$v = v(z, \overline{z}, u). \tag{10.28}$$

A partial cross-section to the prolonged action was found in [57, eq. (3.14)] and given by

$$\begin{aligned} \{v_{z\overline{z}} = 1,\ z = \overline{z} = u = v = v_{z^k u^\ell} = v_{\overline{z}^k u^\ell} = v_{z\overline{z}u^{\ell+1}} = v_{z^{k+2}\overline{z}u^\ell} \\ = v_{z\overline{z}^{k+2}u^\ell} = v_{z^2\overline{z}^2u^\ell} = v_{z^3\overline{z}^2u^\ell} = v_{z^2\overline{z}^3u^\ell} = v_{z^3\overline{z}^3u^\ell} = 0 \mid k, \ell \geq 0\}. \end{aligned} \tag{10.29}$$

Depending on class of the normal form, only a finite number of normalizations must be added to (10.29) to obtain a complete cross-section. These normalizations do not affect the convergence argument, and we therefore work with the partial cross-section (10.29). The normal form for locally umbilic hypersurfaces is given by the Heisenberg sphere $v = z\overline{z}$, which is obviously analytic. We thus focus on the remaining four classes of normal forms. Since the equations

$$v_{\overline{z}^k u^\ell} = v_{z\overline{z}^{k+2} u^\ell} = v_{z^2\overline{z}^3 u^\ell} = 0$$

can be obtained by conjugating $v_{z^k u^\ell} = v_{z^{k+2}\overline{z} u^\ell} = v_{z^3\overline{z}^2 u^\ell} = 0$, those can be omitted from (10.29). No information is lost as, for example, the pseudo-group normalization originating from the normalization $v_{\overline{z}^k u^\ell} = 0$ is recovered by taking the conjugate of the pseudo-group normalization obtained by solving $v_{z^k u^\ell} = 0$. Also, each Taylor coefficient normalization of the real-valued function (10.28) induces a normalization of its conjugated Taylor coefficient. We thus focus on the reduced partial cross-section

$$\begin{aligned}\mathcal{K} = \big\{ v_{z\overline{z}} = 1,\ z = u = v_{z^k u^\ell} = v_{z\overline{z} u^{\ell+1}} = v_{z^{k+2}\overline{z} u^\ell} \\ = v_{z^2\overline{z}^2 u^\ell} = v_{z^3\overline{z}^2 u^\ell} = v_{z^3\overline{z}^3 u^\ell} = 0 \,|\, k,\ell \geq 0 \big\}.\end{aligned} \tag{10.30}$$

Similar to Example 10.1, we need to make a change of variables for the pseudo-group determining equations (10.27) to become involutive. Reverting back to complex variables, let

$$u = \frac{w + \overline{w}}{2}, \qquad v = \frac{w - \overline{w}}{2\,\mathrm{i}}. \tag{10.31}$$

The determining equations of the pseudo-group then become

$$Z_{\overline{z}} = Z_{\overline{w}} = W_{\overline{z}} = W_{\overline{w}} = 0. \tag{10.32}$$

Introducing the ordering $w \prec z \prec \overline{z} \prec \overline{w}$, the indices and Cartan characters of (10.32) are

$$\mathsf{b}_1^{(1)} = \mathsf{b}_1^{(2)} = 0, \quad \mathsf{b}_1^{(3)} = \mathsf{b}_1^{(4)} = 2 \qquad \text{and} \qquad \mathsf{c}_1^{(1)} = \mathsf{c}_1^{(2)} = 2, \quad \mathsf{c}_1^{(3)} = \mathsf{c}_1^{(4)} = 0.$$

Since the second order determining equations are

$$\begin{gathered} Z_{z\overline{z}} = Z_{w\overline{z}} = Z_{\overline{z}\overline{z}} = Z_{\overline{z}\overline{w}} = Z_{z\overline{w}} = Z_{w\overline{w}} = Z_{\overline{w}\overline{w}} = 0, \\ W_{z\overline{z}} = W_{w\overline{z}} = W_{\overline{z}\overline{z}} = W_{\overline{z}\overline{w}} = Z_{z\overline{w}} = W_{w\overline{w}} = W_{\overline{w}\overline{w}} = 0, \end{gathered}$$

the involutivity test $\mathsf{b}_1^{(1)} + 2\mathsf{b}_1^{(2)} + 3\mathsf{b}_1^{(3)} + 4\mathsf{b}_1^{(4)} = 14 = \mathsf{r}_2$ is satisfied.

Substituting the change of variables (10.31) into the hypersurface defining equation (10.28), and solving for $\overline{w}$ using the Implicit Function Theorem, we obtain the complex defining equation[9]

$$\overline{w} = \overline{w}(z, \overline{z}, w) \tag{10.33}$$

of the hypersurface M. Thus, in the new coordinates, the jet variables are $\overline{w}_{z^j\overline{z}^k w^\ell}$ with $j, k, \ell \geq 0$. To find the cross-section in these new jet variables, we substitute the real and

[9]Kossovskiy and Zaitsev also used the complex defining equation (10.33) in their convergence argument; see the acknowledgements in their paper [37].

complex defining equations (10.28), (10.33) into the second equation of (10.31) to obtain the relationship

$$\overline{w}(z,\overline{z},w) = w - 2\,\mathrm{i}\,v\left(z,\overline{z},\frac{w+\overline{w}(z,\overline{z},w)}{2}\right). \tag{10.34}$$

Differentiating (10.34) produces the expressions for the new jet coordinates $\overline{w}_J$ in terms of the original ones v_K. For example, at order one, we have

$$\overline{w}_z = -2\,\mathrm{i}\,v_z - \mathrm{i}\,v_u\overline{w}_z, \qquad \overline{w}_{\overline{z}} = -2\,\mathrm{i}\,v_{\overline{z}} - \mathrm{i}\,v_u\overline{w}_{\overline{z}}, \qquad \overline{w}_w = 1 - \mathrm{i}\,v_u(1+w_{\overline{w}}).$$

For orders ≥ 2, one finds using induction that

$$\overline{w}_{z^j\overline{z}^k w^\ell} = -2\,\mathrm{i}\,v_{z^j\overline{z}^k u^\ell} + \mathcal{S}_{j,k,\ell}(\overline{w}_J, v_K), \tag{10.35}$$

where $\mathcal{S}_{j,k,\ell}$ is a polynomial involving $\overline{w}_J$, with $|J| \leq j+k+\ell$, and $v_K = v_{z^\alpha\overline{z}^\beta u^\gamma}$, with $\alpha \leq j$, $\beta \leq k$, $\gamma \geq 1$ and $|K| \leq j+k+\ell$. Moreover,

$$\mathcal{S}_{j,k,\ell}(w_J, 0) = 0.$$

Using (10.35) and induction, the partial cross-section in the new complex jet coordinates is

$$\begin{aligned}\widetilde{\mathcal{K}} = \big\{\overline{w}_w = 1,\ \overline{w}_{z\overline{z}} = -2\,\mathrm{i}\,,\ z = w = \overline{w}_{z^k w^\ell} = \overline{w}_{z\overline{z}w^{\ell+1}} = \overline{w}_{z^{k+2}\overline{z}w^\ell} \\ = \overline{w}_{z^2\overline{z}^2 w^\ell} = \overline{w}_{z^3\overline{z}^2 w^\ell} = \overline{w}_{z^3\overline{z}^3 w^\ell} = 0 \,|\, k,\ell \geq 0\big\}.\end{aligned}$$

As shown in [57, Section 4], the prolonged action of the holomorphic pseudo-group becomes free at some order $n_0 \geq 7$ for generic, non-umbilic, and semi-circular hypersurfaces. Circular hypersurfaces retain a one-dimensional isotropy group, but the convergence argument remains valid at some order $n_0 \geq 8$. In all cases, it is possible to construct a minimal cross-section and at the appropriate order n_0, one observes that $\mathcal{I}_{\widetilde{\mathcal{K}}}^{\geq n_0+1}$ admits the following Rees decomposition with respect to the ordering $w \prec z \prec \overline{z}$

$$\biguplus_{j=0}^{n_0+1} \mathcal{C}(n_0+1-j,j,0) \biguplus_{j=1}^{n_0} \mathcal{C}(n_0-j,j,1) \biguplus \mathcal{C}(n_0-3,2,2) \biguplus \mathcal{C}(n_0-4,3,2) \biguplus \mathcal{C}(n_0-5,3,3).$$

By Theorem 8.21, the cross-section is well-posed and by Theorem 8.18 we conclude that the normal form of a nondegenerated hypersurface converges, reproducing Chern and Moser's celebrated convergence result.

11 Final Comments.

In this paper, we have proven a fundamental result establishing the convergence of normal form power series for suitably regular submanifolds under a large class of Lie pseudo-group actions, which includes, in particular, all those for which the equivariant moving frame methods developed in [54, 55] can be applied. To do so, we introduced the normal form determining equations (7.5), whose solution includes the normal form. In Section 8, we showed that, beyond the order of freeness, the involutivity of the normal form determining equations is compatible with the moving frame construction, and that a well-posed cross-section provides suitable analytic initial conditions. The convergence of the normal form is then guaranteed by an application of the Cartan–Kähler Theorem.

The results of the paper have been obtained under the assumption that the pseudo-group eventually acts eventually freely, which is a necessary requirement for the construction of a moving frame. That said, there are many circumstances where the prolonged pseudo-group action never becomes free, in which case the geometric problem admits a non-trivial isotropy groups. In these situations one can construct a partial moving frame, [51,65]. Extending the results of the current paper to Lie pseudo-groups that do not eventually act freely, and to singular submanifolds that admit nontrivial isotropy, will be the subject of future research.

We anticipate that our general convergence result will find a wide range of applications in the construction of normal forms. This include, for example, the investigation of Bishop surfaces, [27], in CR geometry, the construction of Poincaré–Dulac normal forms, [21,33], as well as normal forms in control theory, dynamical systems, partial differential equations, and so on.

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