Equivariant Moving Frames for Euclidean Surfaces

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The purpose of this note is to explain how to use the equivariant method of moving frames, [**1**, **10**], to derive the differential invariants, invariant differential operators and invariant differential forms for surfaces in three-dimensional Euclidean space. This is, of course, a very classical problem and the results are not new; for the classical moving frame derivation, see, for instance, [**2**, **3**]. But there are several reasons for performing this calculation. First, in contrast to the classical treatment, the equivariant moving frame approach does not require any a priori insight into surface geometry, relying only on the explicit formulas for the Euclidean group transformations and their infinitesimal generators; the resulting curvature and higher order differential invariants, invariant differential operators, invariant differential forms, etc., all follow by direct and algorithmic calculations. Further systematic calculations will produce the invariant contact forms, and associated invariant variational bicomplex, [**4**], leading to the explicit formulas governing Euclidean signatures, used to solve the equivalence problem for surfaces under Euclidean motions, [**6**, **7**], Euclidean-invariant variational problems, [**4**], and Euclidean-invariant geometric surface flows, [**8**]. Furthermore, some of these formulae — the invariant differential operators and invariant differential forms — do not, as far as I know, appear in the existing literature, making this a useful exercise for further developing such applications.

We assume the reader is familiar with the basics of Lie transformation groups, jet bundles, [**6**], and the equivariant approach to moving frames, [**1**, **10**]. In this computation, we will concentrate on the right-equivariant moving frame map. As we will see, the classical Darboux moving frame for Euclidean surfaces, [**3**], can be interpreted as a left-equivariant moving frame map, which is the group-theoretic inverse of our map: the orthonormal frame

March 30, 2020

vectors at a point on the surface form its orthogonal components, while the point on the surface (which Cartan is always careful to include in what he calls the "repére mobile") is the translation component.

Our starting point is the six-dimensional Euclidean group $\mathrm{E}(3) = \mathrm{O}(3) \ltimes \mathbb{R}^3$, the semidirect product of the three-dimensional orthogonal group and the three-dimensional abelian translation group. It acts on $\mathbb{R}^3$ by orientation-preserving rigid motions that map a point $\mathbf{z} \in \mathbb{R}^3$ to the transformed point[†]

$$\begin{pmatrix} X \\ Y \\ U \end{pmatrix} = \mathbf{Z} = R\mathbf{z} + \mathbf{a}, \qquad R \in \mathrm{O}(3), \qquad \mathbf{a} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3, \qquad \mathbf{z} = \begin{pmatrix} x \\ y \\ u \end{pmatrix} \in \mathbb{R}^3. \tag{1}$$

If we let $\mathbf{r}^1, \mathbf{r}^2, \mathbf{r}^3$ denote the orthonormal *row vectors* of the orthogonal matrix[‡] $R = (r^i_j)$, then

$$X = \mathbf{r}^1 \cdot \mathbf{z} + a, \qquad Y = \mathbf{r}^2 \cdot \mathbf{z} + b, \qquad U = \mathbf{r}^3 \cdot \mathbf{z} + c.$$

We are interested in the induced action of $\mathrm{E}(3)$ on surfaces $S \subset \mathbb{R}^3$. For simplicity, we assume that the surface is (locally) given as the graph of a function[§] $u = f(x,y)$, sometimes referred to as "a Monge patch", [**2**]. The *normal* to the surface at a point is given by

$$\mathbf{N} = \begin{pmatrix} -u_x \\ -u_y \\ 1 \end{pmatrix}, \qquad \text{with Euclidean norm} \qquad n = \|\,\mathbf{N}\,\| = \sqrt{1 + u_x^2 + u_y^2}\,, \tag{2}$$

so that $\mathbf{n} = \mathbf{N}/n$ is the (upwards) unit surface normal. At the same point, the tangent plane to S is thus spanned by the particular (non-orthogonal) tangent vectors

$$\mathbf{t}_1 = \begin{pmatrix} 1 \\ 0 \\ u_x \end{pmatrix}, \qquad \mathbf{t}_2 = \begin{pmatrix} 0 \\ 1 \\ u_y \end{pmatrix}, \tag{3}$$

and we denote their norms by

$$t_1 = \|\,\mathbf{t}_1\,\| = \sqrt{1 + u_x^2}\,, \qquad t_2 = \|\,\mathbf{t}_2\,\| = \sqrt{1 + u_y^2}\,. \tag{4}$$

[†] We will employ Cartan's convention of using lower case letters for the source (domain) variables, and upper case letters for the corresponding target (range) variables of a map.

[‡] We use the tensorial convention that superscripts are row indices and subscripts column indices.

[§] This requires that the surface intersects the vertical lines $\{\,u = \text{constant}\,\}$ transversally. Non-transversal surfaces can be treated by interchanging the roles of independent and dependent variables. Moreover, it is not difficult to adapt the moving frame computations to arbitrarily parametrized surfaces, although one does then need to properly account for the infinite-dimensional reparametrization pseudo-group.

The horizontal differentials† of the transformed independent variables are

$$\begin{pmatrix} d_H X \\ d_H Y \end{pmatrix} = M \begin{pmatrix} dx \\ dy \end{pmatrix}, \qquad \text{where} \qquad M = \begin{pmatrix} D_x X & D_y X \\ D_x Y & D_y Y \end{pmatrix} = \begin{pmatrix} \mathbf{r}^1 \cdot \mathbf{t}_1 & \mathbf{r}^1 \cdot \mathbf{t}_2 \\ \mathbf{r}^2 \cdot \mathbf{t}_1 & \mathbf{r}^2 \cdot \mathbf{t}_2 \end{pmatrix}, \tag{5}$$

where D_x, D_y denote total derivative operators for the independent variables, [**6**]. The dual *implicit differentiation operators* D_X, D_Y are defined by the standard formula

$$d_H F = D_X F \, d_H X + D_Y F \, d_H Y = D_x F \, dx + D_y F \, dy, \tag{6}$$

valid for any differential function F, that is, function of x, y, u and a finite number of derivatives of u. Explicitly,

$$\begin{pmatrix} D_X \\ D_Y \end{pmatrix} = N \begin{pmatrix} D_x \\ D_y \end{pmatrix} \tag{7}$$

uses the inverse transpose matrix

$$N = M^{-T} = \frac{1}{\Delta} \begin{pmatrix} \mathbf{r}^2 \cdot \mathbf{t}_2 & -\mathbf{r}^2 \cdot \mathbf{t}_1 \\ -\mathbf{r}^1 \cdot \mathbf{t}_2 & \mathbf{r}^1 \cdot \mathbf{t}_1 \end{pmatrix} \qquad \text{where} \qquad \Delta = \det M = \mathbf{r}^3 \cdot \mathbf{n}, \tag{8}$$

the latter formula relying on the fact that R is an orthogonal matrix.

The action of the Euclidean group on surfaces induces an action on their jets (derivatives) known as the *prolonged action*, [**6**]. We let $\mathrm{J}^n = \mathrm{J}^n(\mathbb{R}^3, 2)$ denote the n^{th} order surface jet bundle, with local coordinates consisting of the independent variables, x, y, the dependent variable, u, and "higher order" coordinates representing its partial derivatives $u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots$, up to order n. The prolonged Euclidean group action on J^n is obtained by applying the implicit differentiation operators (7) to the transformed dependent variable U, so the transformed jet coordinates are

$$U_X = D_X U, \quad U_Y = D_Y U, \quad U_{XX} = D_X^2 U, \quad U_{XY} = D_X D_Y U = D_Y D_X U, \quad U_{YY} = D_Y^2 U,$$

and so on. The explicit formulas rapidly get very complicated, and so will not be written down here. However, they are eminently computable in symbolic software such as MATHEMATICA, MAPLE, etc. Indeed, MATHEMATICA was used to facilitate and verify the present calculations‡.

Remark: Observe that the matrix (5) is the upper 2×2 block of the matrix whose columns are the rotated tangent vectors: $(R\mathbf{t}_1, R\mathbf{t}_2)$, whose final row contains the coefficients of dx and dy in

$$d_H U = D_x U \, dx + D_y U \, dy = (\mathbf{r}^3 \cdot \mathbf{t}_1) \, dx + (\mathbf{r}^3 \cdot \mathbf{t}_2) \, dy. \tag{9}$$

† This means that, for simplicity, we are ignoring contact forms, [**4**, **6**], in this calculation, or, equivalently, regard u as a function of x, y when differentiating. These will be dealt with in the second part of the paper, although we could include them from the beginning.

‡ However, the later computations are not straightforward, owing to MATHEMATICA's poor handling and simplification of rational algebraic functions! I suspect that MAPLE is not any better in this regards.

To compute the equivariant moving frame, we must normalize the $6 = \dim \mathrm{E}(2)$ independent group parameters by setting 6 of the transformed jet variables equal to conveniently chosen constants — this corresponds to the choice of a *cross-section* to the prolonged group orbits, or, equivalently, to placing the surface in normal form, [**12**]. The *standard cross-section* that produces the classical moving frame, [**3**, **9**], is given by

$$x = y = u = u_x = u_y = u_{xy} = 0. \tag{10}$$

The determination of the moving frame associated with the cross-section (10) relies on solving the corresponding *normalization equations*

$$X = Y = U = U_X = U_Y = U_{XY} = 0 \tag{11}$$

for the group parameters $g = (R, \mathbf{a}) \in \mathrm{E}(3)$, using the explicit formulas for the prolonged Euclidean transformations that were obtained by implicit differentiation. The result will be a right-equivariant moving frame map $\rho\colon \mathcal{V}^{(2)} \to \mathrm{E}(3)$ defined on a certain open subset $\mathcal{V}^{(2)} \subset \mathrm{J}^2$ consisting of "regular" surface 2–jets; see (53) below for details.

The preceding moving frame construction based on the cross-section (10) can, alternatively, be viewed as "moving" the surface so that it is placed into a certain *normal form*

$$S_0 = g \cdot S = \{\, u = f_0(x, y) \,\}, \tag{12}$$

at the point $z_0 = g \cdot z \in S_0$. First, we apply a suitably chosen translation to make S_0 go through the origin, so $z_0 = 0$. Then a subsequent rotation is applied that makes its tangent plane horizontal at z_0, and, further, such that the x and y axis align with the principal axes or Darboux frame, [**3**]. (Again, the algorithm does not require knowing what the latter geometric terms mean.) The required group transformation is precisely the one given by the moving frame map:

$$g = \rho(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}). \tag{13}$$

As a result, the *Euclidean normal form* surface S_0 has the following Taylor expansion at $z_0 = 0$:

$$u = \tfrac{1}{2}\kappa_1 x^2 + \tfrac{1}{2}\kappa_2 y^2 + \tfrac{1}{6}\kappa_{1,1} x^3 + \tfrac{1}{2}\kappa_{1,2} x^2 y + \tfrac{1}{2}\kappa_{2,1} x y^2 + \tfrac{1}{6}\kappa_{2,2} y^3 + \ \cdots\ , \tag{14}$$

whose coefficients, when written in terms of the surface jet coordinates at the original point $z = g^{-1} \cdot z_0$, form a complete system of independent differential invariants, [**12**]. In particular, κ_1, κ_2 are the *principal curvatures*.

Remark: The *regular subset* $\mathcal{V}^{(2)} \subset \mathrm{J}^2$, where the moving frame map is well-defined, consists of those surface 2–jets belonging to the six-dimensional orbits where the prolonged six-dimensional Euclidean group acts locally freely, meaning the isotropy subgroup of any jet in $\mathcal{V}^{(2)}$ is discrete. Geometrically, as we will see below, the 2–jet of a surface belongs to the regular subset if and only if it is *non-umbilic*, meaning that the principal curvatures are unequal: $\kappa_1 \neq \kappa_2$. It is worth emphasizing that the ensuing moving frame computation does not require any a priori knowledge of what an "umbilic point" means.

Let us now present the details of this calculation. The order zero normalizations $X = Y = U = 0$ prescribe the translation parameters $\mathbf{a} = -R\mathbf{z}$ of the group element $g \in \mathrm{E}(3)$. Since they play no further role in the prolonged action or other moving frame formulae, we can effectively ignore them from here on.

The first order normalizations require that

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} U_X \\ U_Y \end{pmatrix} = N \begin{pmatrix} D_x U \\ D_y U \end{pmatrix} = N \begin{pmatrix} \mathbf{r}^3 \cdot \mathbf{t}_1 \\ \mathbf{r}^3 \cdot \mathbf{t}_2 \end{pmatrix}, \tag{15}$$

cf. (7–9). Thus $\mathbf{r}^3 \cdot \mathbf{t}_1 = \mathbf{r}^3 \cdot \mathbf{t}_2 = 0$, which implies that $\mathbf{r}^3$ is a unit vector orthogonal to the tangent plane, and hence, up to sign† $\mathbf{r}^3 = \mathbf{n}^T$ (keeping in mind that the $\mathbf{r}^j$ are row vectors). This requires that the first two rows $\mathbf{r}^1, \mathbf{r}^2$ of the orthogonal matrix R, which must be orthogonal to $\mathbf{r}^3$, form an orthonormal basis of the tangent space to S at z, and hence

$$\mathbf{r}^1 = \widehat{\mathbf{r}}^1 \cos\phi - \widehat{\mathbf{r}}^2 \sin\phi, \qquad \mathbf{r}^2 = \widehat{\mathbf{r}}^1 \sin\phi + \widehat{\mathbf{r}}^2 \cos\phi, \qquad \mathbf{r}^3 = \mathbf{n}^T, \tag{16}$$

for some as yet undetermined rotation angle ϕ. Here

$$\widehat{\mathbf{r}}^1 = \frac{\mathbf{t}_1^T}{t_1}, \qquad \widehat{\mathbf{r}}^2 = \frac{\widetilde{\mathbf{t}}_2^T}{t_1 n}, \tag{17}$$

form an orthonormal basis of the tangent space, whereby

$$\widetilde{\mathbf{t}}_2 = \begin{pmatrix} -u_x u_y \\ 1 + u_x^2 \\ u_y \end{pmatrix}, \qquad \text{with norm} \qquad \|\, \widetilde{\mathbf{t}}_2 \,\| = t_1 n = \sqrt{1 + u_x^2}\, \sqrt{1 + u_x^2 + u_y^2}\,, \tag{18}$$

is a vector orthogonal to both $\mathbf{t}_1$ and $\mathbf{n}$, which can be easily constructed either via the Gram–Schmidt process or directly by inspection.

Note: The above normalization means that we are, effectively, parametrizing the orthogonal group in factored form:

$$R = R_\phi \cdot \widehat{R} \in \mathrm{O}(3), \tag{19}$$

where

$$R_\phi = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \widehat{R} = \begin{pmatrix} \widehat{\mathbf{r}}^1 \\ \widehat{\mathbf{r}}^2 \\ \mathbf{r}^3 \end{pmatrix}. \tag{20}$$

So we are, in essence, employing the recursive moving frame algorithm developed in [**11**]. However, because in this example the required prolonged actions can be computed relatively painlessly, the full recursive machinery will not be used here.

† In this simplified presentation, which is in accordance with the classical treatment in all texts I know, we will ignore this and later discrete (sign) ambiguities in our specification of the moving frame. It would be worth taking these properly into account, as in the treatment of Euclidean space curves in [**7**].

Substituting the normalized values (16) for rows of the orthogonal matrix R into the second order prolonged Euclidean action yields

$$\begin{aligned} U_{XX} &= V_1 \cos^2\phi - 2\,V_2 \cos\phi \sin\phi + V_3 \sin^2\phi, \\ U_{XY} &= V_1 \cos\phi \sin\phi + V_2(\cos^2\phi - \sin^2\phi) - V_3 \cos\phi \sin\phi, \\ U_{YY} &= V_1 \sin^2\phi + 2\,V_2 \cos\phi \sin\phi + V_3 \cos^2\phi, \end{aligned} \tag{21}$$

where

$$\begin{aligned} V_1 &= \frac{u_{xx}}{t_1^2\, n} = \frac{u_{xx}}{(1+u_x^2)\sqrt{1+u_x^2+u_y^2}}\,, \\ V_2 &= \frac{-\,u_x u_y u_{xx} + t_1^2\, u_{xy}}{t_1^2\, n^2} = \frac{-\,u_x u_y u_{xx} + (1+u_x^2) u_{xy}}{(1+u_x^2)(1+u_x^2+u_y^2)}\,, \\ V_3 &= \frac{u_x^2 u_y^2 u_{xx} - 2\,u_x u_y\, t_1^2\, u_{xy} + t_1^4\, u_{yy}}{t_1^2\, n^3} = \frac{u_x^2 u_y^2 u_{xx} - 2\,u_x u_y (1+u_x^2) u_{xy} + (1+u_x^2)^2 u_{yy}}{(1+u_x^2)(1+u_x^2+u_y^2)^{3/2}}\,. \end{aligned} \tag{22}$$

In view of (21), we can thus normalize $U_{XY} = 0$ by setting

$$\tan 2\phi = \widetilde{J}, \qquad \text{where} \qquad \widetilde{J} = \frac{2\,V_2}{V_3 - V_1}\,. \tag{23}$$

Note that, modulo discrete ambiguities stemming from the fact that the Euclidean action is only locally free on $\mathcal{V}^{(2)} \subset \mathrm{J}^2$,

$$\cos\phi = \sqrt{\frac{1+\widetilde{I}}{2}}\,, \qquad \sin\phi = -\sqrt{\frac{1-\widetilde{I}}{2}}\,, \qquad \text{where} \qquad \widetilde{I} = \frac{1}{\sqrt{1+\widetilde{J}^2}}\,. \tag{24}$$

The denominator in (23) is nonzero at non-umbilic points — see (53) below — and the resulting orthogonal matrix $R(u_x, u_y, u_{xx}, u_{xy}, u_{yy})$ obtained from (19, 24) serves to define the O(3) component of the right-equivariant moving frame map $\rho\colon \mathcal{V}^{(2)} \longrightarrow \mathrm{E}(3)$, whose translation component $\mathbf{a} = -\,R\mathbf{z}$ was derived at the beginning of our calculation.

Remark: The corresponding left-equivariant moving frame is obtained by applying the group inversion to the right moving frame map, and hence has corresponding parameter values $(R^{-1}, -R^{-1}\mathbf{a}) = (R^T, \mathbf{z})$. Thus, as noted above, the columns of the orthogonal matrix R^T defined by the left moving frame are the orthonormal Darboux frame vectors and the translation component $\mathbf{z}$ is the point on the surface at which they are based.

With the moving frame map in hand, we are now able to compute invariant objects — differential invariants, invariant differential operators, invariant differential forms, etc. — through the process of *invariantization*. This is performed by first transforming the object in question according to a general Euclidean transformation, and then substituting the moving frame formulas for the groups parameters that appear in the transformed object. Invariantization preserves all algebraic operations, but does not respect differentiation.

In particular, substituting (23) back into (21) produces the two independent second order differential invariants:

$$
\begin{aligned}
U_{XX} \;\longmapsto\; \kappa_1 &= \frac{V_1 + V_3 + \sqrt{(V_1 - V_3)^2 + 4\,V_2^2}}{2} = H + \sqrt{H^2 - K}\,, \\
U_{YY} \;\longmapsto\; \kappa_2 &= \frac{V_1 + V_3 - \sqrt{(V_1 - V_3)^2 + 4\,V_2^2}}{2} = H - \sqrt{H^2 - K}\,,
\end{aligned} \tag{25}
$$

where κ_1, κ_2 are the *principal curvatures*, while

$$
\begin{aligned}
H &= \tfrac{1}{2}(\kappa_1 + \kappa_2) = \tfrac{1}{2}(V_1 + V_3) = \frac{(1 + u_y^2)u_{xx} - 2\,u_x u_y u_{xy} + (1 + u_x^2)u_{yy}}{2\,(1 + u_x^2 + u_y^2)^{3/2}}\,, \\
K &= \kappa_1 \kappa_2 = V_1 V_3 - V_2^2 = \frac{u_{xx}u_{yy} - u_{xy}^2}{(1 + u_x^2 + u_y^2)^2}\,,
\end{aligned} \tag{26}
$$

are, respectively, the *mean* and *Gauss curvatures*, thereby reproducing their classical formulas in a Monge patch, [**2**; p. 409]. (As noted above, their formulas for a general parametrized surface can also be obtained by a similar but more intricate computation.)

The invariant differential operators are obtained by substituting the moving frame normalizations (16, 24) into the implicit differentiation operators (7), leading to

$$
\begin{pmatrix} \mathcal{D}_1 \\ \mathcal{D}_2 \end{pmatrix} = N^* \begin{pmatrix} D_x \\ D_y \end{pmatrix}, \tag{27}
$$

where, in view of (19),

$$
N^* = Q_\phi \, \widehat{N} \tag{28}
$$

is the invariantized version of the coefficient matrix (8), whereby

$$
Q_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1 + \widetilde{I}} & \sqrt{1 - \widetilde{I}} \\ -\sqrt{1 - \widetilde{I}} & \sqrt{1 + \widetilde{I}} \end{pmatrix}, \tag{29}
$$

cf. (23, 24), while, using (8),

$$
\begin{aligned}
\widehat{N} = \frac{1}{n} \begin{pmatrix} \widehat{\mathbf{r}}^2 \cdot \mathbf{t}_2 & -\widehat{\mathbf{r}}^2 \cdot \mathbf{t}_1 \\ -\widehat{\mathbf{r}}^1 \cdot \mathbf{t}_2 & \widehat{\mathbf{r}}^1 \cdot \mathbf{t}_1 \end{pmatrix} &= \begin{pmatrix} 1/t_1 & 0 \\ -\,u_x u_y/(n\,t_1) & t_1/n \end{pmatrix} \\
&= \frac{1}{\sqrt{1 + u_x^2}\,\sqrt{1 + u_x^2 + u_y^2}} \begin{pmatrix} \sqrt{1 + u_x^2 + u_y^2} & 0 \\ -\,u_x u_y & 1 + u_x^2 \end{pmatrix}.
\end{aligned} \tag{30}
$$

One can apply the same invariantization process to all of the higher order transformed jet coordinates, $U_{XXX}, U_{XXY}, \ldots$, to construct the corresponding higher order differential invariants. However, a better strategy is to note that, as a consequence of the recurrence formulae the higher order differential invariants can all be obtained by invariant differentiation of the Gauss and mean curvatures. Details appear below, cf. (61, 67).

The one remaining issue is that, surprisingly, the formulas (27–30) for the invariant differential operators do not appear to be symmetric in x and y. However, they are — indeed they must be — but the symmetry is by no means evident, and is, in fact, not easy to verify directly, even with the aid of MATHEMATICA. (Again, this is due to its substandard simplification routines for rational algebraic functions.)

To reveal the underlying symmetry, we need to place the initial orthonormal basis vectors (17) for the surface's tangent space in a more symmetric form. This will require subjecting them to an intermediate rotation in order that their projections onto the xy plane, which are also the rows of the matrix $\widehat{N}$ in (30), are symmetric under interchange of the x and y coordinates. To this end, let $\widehat{\psi}$ denote the angle between their projections, which is, in fact, equal π minus the angle between the original tangent vectors (3), and is given by the usual dot product formula:

$$\cos\widehat{\psi} = -\frac{u_x u_y}{t_1 t_2} = -\frac{\mathbf{t}_1 \cdot \mathbf{t}_2}{t_1 t_2}\,. \tag{31}$$

Let us set

$$\psi = \tfrac{1}{2}\,\widehat{\psi} - \tfrac{1}{4}\,\pi, \tag{32}$$

so that

$$\cos\psi = \sqrt{\frac{1+P}{2}}\,, \quad \sin\psi = -\sqrt{\frac{1-P}{2}}\,, \quad \text{where} \quad P = \frac{n}{t_1 t_2} = \sqrt{\frac{1+u_x^2+u_y^2}{(1+u_x^2)(1+u_y^2)}}\,. \tag{33}$$

By construction, applying the corresponding rotation matrix

$$R_\psi = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{34}$$

to the orthonormal basis tangent vectors $\widehat{\mathbf{r}}^1, \widehat{\mathbf{r}}^2$ will make their projections suitably symmetric. Indeed, a short computation using (33) reveals that

$$\widetilde{R} = R_\psi \cdot \widehat{R} = \frac{1}{n}\begin{pmatrix} t_2\cos\psi & t_1\sin\psi & u_x t_2\cos\psi + u_y t_1\sin\psi \\ t_2\sin\psi & t_1\cos\psi & u_x t_2\sin\psi + u_y t_1\cos\psi \\ -u_x & -u_y & 1 \end{pmatrix}. \tag{35}$$

(The nonconstant entries in the third column are most easily determined by the fact that the first two rows must be orthogonal to the normal vector in the last row.) The effect of the combined orthogonal transformation (35) on the second order derivatives is found by replacing ϕ by ψ in (21) and then using the formulas (33) for the trigonometric functions thereof; we write the result as

$$
\begin{aligned}
W_1 &= \frac{t_2^2\,(t_1 t_2 + n) u_{xx} - 2\,u_x u_y t_1 t_2\, u_{xy} + t_1^2\,(t_1 t_2 - n) u_{yy}}{2\,t_1 t_2 n^3} \\
&= \frac{\left[\begin{aligned} &\sqrt{1+u_x^2}\,\sqrt{1+u_y^2}\,\big[\,(1+u_y^2)u_{xx} - 2\,u_x u_y u_{xy} + (1+u_x^2)u_{yy}\,\big] \\ &\qquad\qquad + \sqrt{1+u_x^2+u_y^2}\,\big[\,(1+u_y^2)u_{xx} - (1+u_x^2)u_{yy}\,\big]\end{aligned}\right]}{2\sqrt{1+u_x^2}\,\sqrt{1+u_y^2}\,(1+u_x^2+u_y^2)^{3/2}}\,, \\
W_2 &= \frac{-\,u_x u_y t_2^2\, u_{xx} + 2\,t_1^2\, t_2^2\, u_{xy} - u_x u_y t_1^2\, u_{yy}}{2\,t_1 t_2 n^3} \\
&= \frac{-\,u_x u_y (1+u_y^2) u_{xx} + 2\,(1+u_x^2)(1+u_y^2) u_{xy} - u_x u_y (1+u_x^2) u_{yy}}{2\sqrt{1+u_x^2}\,\sqrt{1+u_y^2}\,(1+u_x^2+u_y^2)^{3/2}}\,, \\
W_3 &= \frac{t_2^2\,(t_1 t_2 - n) u_{xx} - 2\,u_x u_y t_1 t_2\, u_{xy} + t_1^2\,(t_1 t_2 + n) u_{yy}}{2\,t_1 t_2 n^3} \\
&= \frac{\left[\begin{aligned} &\sqrt{1+u_x^2}\,\sqrt{1+u_y^2}\,\big[\,(1+u_y^2)u_{xx} - 2\,u_x u_y u_{xy} + (1+u_x^2)u_{yy}\,\big] \\ &\qquad\qquad - \sqrt{1+u_x^2+u_y^2}\,\big[\,(1+u_y^2)u_{xx} - (1+u_x^2)u_{yy}\,\big]\end{aligned}\right]}{2\sqrt{1+u_x^2}\,\sqrt{1+u_y^2}\,(1+u_x^2+u_y^2)^{3/2}}\,.
\end{aligned}
\tag{36}
$$

With this in hand, let us return to the final moving frame normalization. To effect this, we use (35) to rewrite the orthogonal matrix factorization (19) in the form

$$
R = R_\varphi \cdot \widetilde{R} = R_\varphi \cdot R_\psi \cdot \widehat{R} = R_\phi \cdot \widehat{R} \in \mathrm{O}(3), \tag{37}
$$

where $\phi = \varphi + \psi$. The remaining unnormalized rotation

$$
R_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{38}
$$

acts on the second order quantities (36) in the same fashion as in (21), thus producing symmetric versions of the second order transformed derivative formulae:

$$
\begin{aligned}
U_{XX} &= W_1 \cos^2\varphi - 2\,W_2 \cos\varphi \sin\varphi + W_3 \sin^2\varphi, \\
U_{XY} &= W_1 \cos\varphi \sin\varphi + W_2 (\cos^2\varphi - \sin^2\varphi) - W_3 \cos\varphi \sin\varphi, \\
U_{YY} &= W_1 \sin^2\varphi + 2\,W_2 \cos\varphi \sin\varphi + W_3 \cos^2\varphi.
\end{aligned}
\tag{39}
$$

We can thus normalize $U_{XY} = 0$ by setting

$$
\tan 2\varphi = J, \tag{40}
$$

where

$$J = \frac{2\,W_2}{W_3 - W_1} = \frac{-\,u_x u_y\,t_2^2\,u_{xx} + 2\,t_1^2\,t_2^2\,u_{xy} - u_x u_y\,t_1^2\,u_{yy}}{n\,(-\,t_2^2\,u_{xx} + t_1^2\,u_{yy})} = \frac{u_x u_y (1+u_y^2) u_{xx} - 2\,(1+u_x^2)(1+u_y^2) u_{xy} + u_x u_y (1+u_x^2) u_{yy}}{\sqrt{1+u_x^2+u_y^2}\,\left[\,(1+u_y^2)u_{xx} - (1+u_x^2)u_{yy}\,\right]}\,. \tag{41}$$

Because we are performing the identical moving frame normalizations, just based on the alternative parametrization (37) of the orthogonal group, the resulting differential invariants and invariant differential operators will be exactly the same as above. In particular, substituting (23) back into (21) produces the same principal curvature invariants

$$\begin{aligned} U_{XX} &\longmapsto \kappa_1 = \frac{W_1 + W_3 + \sqrt{(W_1 - W_3)^2 + 4\,W_2^2}}{2} = H + \sqrt{H^2 - K}, \\ U_{YY} &\longmapsto \kappa_2 = \frac{W_1 + W_3 - \sqrt{(W_1 - W_3)^2 + 4\,W_2^2}}{2} = H - \sqrt{H^2 - K}, \end{aligned} \tag{42}$$

where H, K are the *mean* and *Gauss curvatures* (26). Finally, (40) implies that (again modulo discrete ambiguities)

$$\cos\varphi = \sqrt{\frac{1+I}{2}}\,, \qquad \sin\varphi = -\sqrt{\frac{1-I}{2}}\,, \tag{43}$$

where

$$I = \frac{1}{\sqrt{1+J^2}} = \frac{W}{\sqrt{H^2-K}}\,, \qquad W = W_1 - W_3 = \frac{(1+u_y^2)u_{xx} - (1+u_x^2)u_{yy}}{\sqrt{1+u_x^2}\,\sqrt{1+u_y^2}\,(1+u_x^2+u_y^2)}\,. \tag{44}$$

We finally recompute the invariant differential operators (27), where, in view of equations (33, 35, 43), formula (28) becomes

$$\begin{aligned} N^* &= \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix} = Q_\varphi\,Q_\psi\,\widehat{N} \\ &= \frac{1}{2\sqrt{1+u_x^2+u_y^2}} \begin{pmatrix} \sqrt{1+I} & \sqrt{1-I} \\ -\sqrt{1-I} & \sqrt{1+I} \end{pmatrix} \begin{pmatrix} \sqrt{1+u_y^2}\,\sqrt{1+P} & -\sqrt{1+u_x^2}\,\sqrt{1-P} \\ -\sqrt{1+u_y^2}\,\sqrt{1-P} & \sqrt{1+u_x^2}\,\sqrt{1+P} \end{pmatrix}. \end{aligned} \tag{45}$$

that, upon substituting the expressions (33, 44) for P, I, produces the explicit formulas for the invariant differential operators (27) in an evidently symmetric form. Finally note that, in view of (33),

$$\det N^* = n = \sqrt{1+u_x^2+u_y^2}\,.$$

Recurrence Formulae and the Invariant Variational Quasi–Tricomplex

In more advanced applications, including Euclidean-invariant variational problem and invariant flows, one requires the full algebra of invariant differential forms on jet space, or, more generally, the *invariant variational quasi-tricomplex*; see [**4**] for the general results and computational machinery underlying the remainder of this note.

To continue, we will return to our earlier moving frame calculations, but now retain all contact form components. Thus, the transformed horizontal forms (5) are given by†

$$dX = (D_x X)\, dx + (D_y X)\, dy + X_u \theta = (\mathbf{r}^1 \cdot \mathbf{t}_1)\, dx + (\mathbf{r}^1 \cdot \mathbf{t}_2)\, dy + r^1_3\, \theta,$$
$$dY = (D_x Y)\, dx + (D_y Y)\, dy + Y_u \theta = (\mathbf{r}^2 \cdot \mathbf{t}_1)\, dx + (\mathbf{r}^2 \cdot \mathbf{t}_2)\, dy + r^2_3\, \theta,$$

where

$$\theta = du - u_x\, dx - u_y\, dy \tag{46}$$

is the basic order 0 contact form. Thus, the fully invariant horizontal forms are

$$\begin{aligned} \varpi^1 &= \iota(dx) = \frac{N_4\, dx - N_3\, dy + (-u_x N_4 + u_y N_3)\,\theta}{\sqrt{1+u_x^2+u_y^2}}, \\ \varpi^2 &= \iota(dy) = \frac{-N_2\, dx + N_1\, dy + (u_x N_2 - u_y N_1)\,\theta}{\sqrt{1+u_x^2+u_y^2}}, \end{aligned} \tag{47}$$

where N_1, N_2, N_3, N_4 are the entries of the matrix N^* given in (45).

Next, under a general diffeomorphism, the basic order 0 contact form (46) is transformed to

$$\Theta = (U_u - U_X X_u - U_Y Y_u)\, \theta.$$

In particular, for a Euclidean transformation (1), $X_u = r^1_3$, $Y_u = r^2_3$, $U_u = r^3_3$. Under the moving frame normalization, U_X and U_Y both map to 0, cf. (15), while $r^3_3 \longmapsto 1/n$, cf. (35, 37). Hence, the invariantized order 0 contact form is simply

$$\vartheta = \iota(\theta) = \frac{\theta}{n} = \frac{\theta}{\sqrt{1+u_x^2+u_y^2}}\,. \tag{48}$$

The higher order invariant contact forms can be obtained by a similar invariantization process; alternatively, as we now explain, they are obtained by invariant differentiation of the order zero form (48) via the recurrence formulae.

With these in hand, there is a natural bigrading of the invariant differential forms on jet space into the invariant horizontal forms, spanned by (47), and invariant vertical or contact forms, spanned by (48) and its higher order counterparts. Under this bigrading, the differential splits into three components, $d = d_{\mathcal{H}} + d_{\mathcal{V}} + d_{\mathcal{W}}$, in which the *invariant horizontal differential* $d_{\mathcal{H}}$ increases the invariant horizontal degree by 1, the *invariant vertical differential* $d_{\mathcal{V}}$ increases the invariant vertical degree by 1, and the *anomalous invariant differential* $d_{\mathcal{W}}$, both decreases the invariant horizontal degree by -1 and increases the invariant vertical degree by 2, thus endowing the algebra of invariant differential forms with the structure known as the invariant variational quasi-tricomplex, [**4**]. In particular,

† Lower case subscripts on X, Y, U denote partial derivatives. Thus, $D_x X = X_x + u_x X_u$, etc.

the invariant horizontal differential of an (invariant) differential function or form Ω is given by the formula

$$d_{\mathcal{H}}\Omega = \varpi^1 \wedge \mathcal{D}_1\Omega + \varpi^2 \wedge \mathcal{D}_2\Omega, \tag{49}$$

where $\mathcal{D}_1, \mathcal{D}_2$ are the invariant differential operators (27), acting on Ω by Lie differentiation.

The *recurrence formula* for invariantized differential functions and differential forms on jet space is contained in the following theorem. In general, suppose G is an r-dimensional Lie group that acts on locally effectively on a manifold M and hence, by prolongation, on the jet bundles $\mathrm{J}^n = \mathrm{J}^n(M,p)$ for p-dimensional submanifolds $N \subset M$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ denote the infinitesimal generators, which we identify with a basis of its Lie algebra $\mathfrak{g}$. We will employ the same notation $\mathbf{v}_i$ for their prolonged action on differential functions and differential forms (by Lie differentiation) on the submanifold jet spaces J^n. As usual, we identify, and use the same symbols for, differential functions and forms on any jet space J^n with their pull-backs via the standard projections $\mathrm{J}^k \to \mathrm{J}^n$ for any $k \geq n$.

Theorem 1. *Let ι denote the invariantization map associated with a moving frame $\rho\colon \mathrm{J}^n \to G$. If Ω is any differential function or differential form on the submanifold jet space J^k, then*

$$d\,\iota(\Omega) = \iota(d\Omega) + \sum_{\kappa=1}^{r} \nu^\kappa \wedge \iota[\mathbf{v}_\kappa(\Omega)], \tag{50}$$

where $\nu^1, \ldots, \nu^r$ denote the pulled-back Maurer–Cartan forms dual to the infinitesimal generators $\mathbf{v}_1, \ldots, \mathbf{v}_r$ under the moving frame map $\rho\colon \mathcal{V}^{(2)} \to G$.

Suppose that the cross-section defining the moving frame is specified by setting a collection of r differential functions to suitably chosen constants: $F_i(x, u^{(n)}) = c_i$ for $i = 1, \ldots, r$. (Typically, the F_i are individual jet coordinates, meaning that, as here, we choose a coordinate cross-section.) If we take Ω to be one of the F_i, then $\iota(F_i) = c_i$ and hence the left hand side of the corresponding recurrence formula (50) vanishes. The result is a system of r linear equations that can always be uniquely solved for the pulled-back Maurer–Cartan forms $\nu^1, \ldots, \nu^r$ in terms of known invariant differential functions and forms. Moreover, substituting these expressions into (50) produces a complete system of explicit recurrence formulae relating the differentiated and invariantized differential functions and forms, thereby completely specifying the structure of both the differential invariant algebra and the entire invariant variational quasi-tricomplex. Note especially that this calculation *does not require explicit knowledge of the differential invariants, nor the invariant differential forms, nor even the moving frame map*! We only need to know the formulae for the prolonged infinitesimal generators $\mathbf{v}_\kappa$, which can be easily found using the well-known prolongation formula for vector fields on jet space, [**5**, **6**].

For the action of $\mathrm{E}(3)$ under study here, a basis for the prolonged infinitesimal generators is provided by the following six vector fields:

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_y, \qquad \mathbf{v}_3 = \partial_u, \tag{51}$$

representing infinitesimal translations, and

$$\begin{aligned}
\mathbf{v}_4 &= -y\partial_x + x\partial_y - u_y\partial_{u_x} + u_x\partial_{u_y} - 2\,u_{xy}\partial_{u_{xx}} + (u_{xx}-u_{yy})\partial_{u_{xy}} + 2\,u_{xy}\partial_{u_{yy}} + \ \cdots\ , \\
\mathbf{v}_5 &= -u\partial_x + x\partial_u + (1+u_x^2)\partial_{u_x} + u_x u_y\partial_{u_y} \\
&\qquad + 3\,u_x\,u_{xx}\partial_{u_{xx}} + (u_y\,u_{xx} + 2\,u_x\,u_{xy})\partial_{u_{xy}} + (2\,u_y\,u_{xy} + u_x\,u_{yy})\partial_{u_{yy}} + \ \cdots\ , \qquad (52)\\
\mathbf{v}_6 &= -u\partial_y + y\partial_u + u_x u_y\partial_{u_x} + (1+u_y^2)\partial_{u_y} \\
&\qquad + (u_y\,u_{xx} + 2\,u_x\,u_{xy})\partial_{u_{xx}} + (2\,u_y\,u_{xy} + u_x\,u_{yy})\partial_{u_{xy}} + 3\,u_y\,u_{yy}\partial_{u_{yy}} + \ \cdots\ ,
\end{aligned}$$

representing infinitesimal rotations, where we just display the terms up to second order, although it is straightforward to prolong further, to any desired order.

Remark: The regular subset $\mathcal{V}^{(2)} \subset \mathrm{J}^2$ is where the second prolongation of E(3) acts locally freely, and hence where the second order prolonged infinitesimal generators (51, 52) are linearly independent. Its complement, the *singular subset* $\mathcal{S}^{(2)} = \mathrm{J}^2 \setminus \mathcal{V}^{(2)}$, is thus defined by the (generalized) Lie determinant conditions

$$u_x u_y u_{xx} = (1+u_x^2)u_{xy}, \qquad u_x u_y u_{yy} = (1+u_y^2)u_{xy}, \qquad (1+u_y^2)u_{xx} = (1+u_x^2)u_{yy}, \quad (53)$$

which imply that the infinitesimal generators $\mathbf{v}_1, \ldots, \mathbf{v}_6$ are linearly dependent at the given 2-jet. Note that the second condition in (53) is a consequence of the first and the third, or, equivalently, the first is a consequence of the second and the third. Furthermore, (53) are necessary and sufficient for the 2 jet to correspond to an *umbilic* point on the surface, meaning that its principal curvatures (42) are equal: $\kappa_1 = \kappa_2$, or, equivalently, $K = H^2$. In particular, if the tangent plane is horizontal, then the point is umbilic if and only if

$$u_x = u_y = 0, \qquad u_{xx} = u_{yy}, \qquad u_{xy} = 0. \qquad (54)$$

Thus, the regular subset $\mathcal{V}^{(2)}$ consists of all non-umbilic surface 2-jets.

The basic invariantized differential functions and forms that prescribe the invariant variational quasi-tricomplex will be denoted by†

$$I_J = \iota(u_J), \qquad \varpi^1 = \iota(dx), \qquad \varpi^2 = \iota(dy), \qquad \vartheta_J = \iota(\theta_J). \qquad (55)$$

In particular, the cross-section variables (10) have trivial invariantizations

$$\iota(x) = \iota(y) = \iota(u) = \iota(u_x) = \iota(u_y) = \iota(u_{xy}) = 0, \qquad (56)$$

and are known as the *phantom invariants*. The explicit formulas for the lowest order nontrivial ones

$$\kappa_1 = \iota(u_{xx}), \qquad \kappa_2 = \iota(u_{yy}), \qquad \vartheta = \iota(\theta), \qquad (57)$$

are given in (42, 48). All higher order differential invariants I_J for $\#J \geq 3$ and higher order invariant contact forms ϑ_J for $\#J \geq 1$ can be obtained therefrom by invariant

† Here, the subscripts on I_J and ϑ_J are indices, not partial derivatives.

(Lie) differentiation of (57) using the invariant differential operators (27). Fortunately the recurrence formulae allow us to rather easily determine the relations without any need for deriving the complicated explicit formulae for these fundamental quantities.

In our case, letting, in turn, $\Omega = x, y, u, u_x, u_y, u_{xy}$ be the cross-section coordinates produces zero on the left hand side of the recurrence formula (50), while using the prolonged infinitesimal generators (51, 52) and the elementary formulae

$$du = u_x\,dx + u_y\,dy + \theta, \qquad du_x = u_{xx}\,dx + u_{xy}\,dy + \theta_x, \qquad du_y = u_{xy}\,dx + u_{yy}\,dy + \theta_y,$$

and so on, to write out the right hand side produces the *phantom recurrence relations*

$$\begin{aligned}
0 &= \iota(dx) + \iota(1)\,\nu^1 + \iota(-y)\,\nu^4 + \iota(-u)\,\nu^5 = \varpi^1 + \nu^1,\\
0 &= \iota(dy) + \iota(1)\,\nu^2 + \iota(x)\,\nu^4 + \iota(-u)\,\nu^6 = \varpi^2 + \nu^2,\\
0 &= \iota(du) + \iota(1)\,\nu^3 + \iota(x)\,\nu^5 + \iota(y)\,\nu^6 = \vartheta + \nu^3,\\
0 &= \iota(du_x) + \iota(-u_y)\,\nu^4 + \iota(1+u_x^2)\,\nu^5 + \iota(u_x u_y)\,\nu^6 = \kappa_1\varpi^1 + \vartheta_x + \nu^5,\\
0 &= \iota(du_y) + \nu^4\iota(u_x) + \nu^5\iota(u_x u_y) + \nu^6\iota(1+u_y^2) = \kappa_2\varpi^2 + \vartheta_y + \nu^6,\\
0 &= \iota(du_{xy}) + \iota(u_{xx} - u_{yy})\,\nu^4 + \iota(u_y\,u_{xx} + 2\,u_x\,u_{xy})\,\nu^5 + \iota(2\,u_y\,u_{xy} + u_x\,u_{yy})\,\nu^6\\
&= I_{xxy}\varpi^1 + I_{xyy}\varpi^2 + \vartheta_{xy} + (\kappa_1 - \kappa_2)\,\nu^4.
\end{aligned} \tag{58}$$

This linear system can be easily solved for the pulled-back Maurer–Cartan forms:

$$\begin{gathered}
\nu^1 = -\varpi^1, \qquad \nu^2 = -\varpi^2, \qquad \nu^3 = -\vartheta,\\
\nu^4 = \frac{I_{xxy}\varpi^1 + I_{xyy}\varpi^2 + \vartheta_{xy}}{\kappa_2 - \kappa_1}, \qquad \nu^5 = -\kappa_1\varpi^1 - \vartheta_x, \qquad \nu^6 = -\kappa_2\varpi^2 - \vartheta_y.
\end{gathered} \tag{59}$$

As with the translational coordinates, the first three of these — the translational forms — play no role in the subsequent calculations.

Next, taking $\Omega = u_{xx}$ and u_{yy} in (50) and using (59) produces the second order recurrence formulae

$$\begin{aligned}
d\kappa_1 &= d\iota(u_{xx}) = \iota(du_{xx}) + \iota(-2\,u_{xy})\,\nu^4 + \iota(3\,u_x\,u_{xx})\,\nu^5 + \iota(u_y\,u_{xx} + 2\,u_x\,u_{xy})\,\nu^6\\
&= I_{xxx}\varpi^1 + I_{xxy}\varpi^2 + \vartheta_{xx},\\
d\kappa_2 &= d\iota(u_{yy}) = \iota(du_{yy}) + \iota(2\,u_{xy})\,\nu^4 + \iota(u_y\,u_{xx} + 2\,u_x\,u_{xy})\,\nu^5 + \iota(3\,u_y\,u_{yy})\,\nu^6\\
&= I_{xxx}\varpi^1 + I_{xxy}\varpi^2 + \vartheta_{yy}.
\end{aligned} \tag{60}$$

Hence, by (49),

$$I_{xxx} = \kappa_{1,1}, \qquad I_{xxy} = \kappa_{1,2}, \qquad I_{xyy} = \kappa_{2,1}, \qquad I_{yyy} = \kappa_{2,2}, \tag{61}$$

where we abbreviate $\kappa_{i,j} = \mathcal{D}_j\kappa_i$. We can thus re-express the rotational Maurer–Cartan forms (59) in the form

$$\nu^4 = Y_1 \varpi^1 + Y_2 \varpi^2 + \vartheta_{xy}, \qquad \nu^5 = -\kappa_1 \varpi^1 - \vartheta_x, \qquad \nu^6 = -\kappa_2 \varpi^2 - \vartheta_y, \tag{62}$$

where

$$Y_1 = \frac{\kappa_{1,2}}{\kappa_2 - \kappa_1}, \qquad Y_2 = \frac{\kappa_{2,1}}{\kappa_2 - \kappa_1}, \tag{63}$$

are known as the *commutator invariants* for reasons that will soon become apparent[†]

The recurrence formulae for the invariant horizontal forms (47) are obtained by setting $\Omega = dx$ and dy in the general recurrence formula (50) and using (62), producing

$$\begin{aligned} d\varpi^1 = d\iota(dx) &= \iota(d^2x) - \nu^4 \wedge \iota(dy) - \nu^5 \wedge \iota(du) \\ &= -Y_1 \varpi^1 \wedge \varpi^2 + \frac{\varpi^2 \wedge \vartheta_{xy}}{\kappa_1 - \kappa_2} + \kappa_1 \varpi^1 \wedge \vartheta + \vartheta_x \wedge \vartheta, \\ d\varpi^2 = d\iota(dy) &= \iota(d^2y) + \nu^4 \wedge \iota(dx) - \nu^6 \wedge \iota(du) \\ &= -Y_2 \varpi^1 \wedge \varpi^2 + \frac{\varpi^1 \wedge \vartheta_{xy}}{\kappa_2 - \kappa_1} + \kappa_2 \varpi^2 \wedge \vartheta + \vartheta_y \wedge \vartheta. \end{aligned} \tag{64}$$

Each formula splits into three components, according to the bigrading induced by the invariant variational quasi-tricomplex: the first summand is the invariant horizontal differential $d_{\mathcal{H}} \varpi^i$; the middle two summands are the invariant vertical differential $d_{\mathcal{V}} \varpi^i$; the final summand is the anomalous invariant differential $d_{\mathcal{W}} \varpi^i$ that stems from the non-projectability of the Euclidean action on $M = \mathbb{R}^3$. In particular, if F is any differential function, then, in view of (49),

$$\begin{aligned} 0 = d_{\mathcal{H}}^2 F &= d_{\mathcal{H}} \big[(\mathcal{D}_1 F) \varpi^1 + (\mathcal{D}_2 F) \varpi^2 \big] \\ &= \big[\mathcal{D}_1 \mathcal{D}_2 F - \mathcal{D}_2 \mathcal{D}_1 F - Y_1 \mathcal{D}_1 F - Y_2 \mathcal{D}_2 F \big] \varpi^1 \wedge \varpi^2, \end{aligned}$$

which, since F is arbitrary, implies the *commutation formula*

$$\big[\mathcal{D}_1, \mathcal{D}_2 \big] = \mathcal{D}_1 \mathcal{D}_2 - \mathcal{D}_2 \mathcal{D}_1 = Y_1 \mathcal{D}_1 + Y_2 \mathcal{D}_2 \tag{65}$$

for the invariant differential operators, whence our designation of Y_1, Y_2 as commutator invariants.

[†] Comparing with Guggenheimer's treatment, [**3**; eq. (10-53)], we have $\rho_1 = -Y_1$, $\rho_2 = -Y_2$, where there is a misprint in his first formula that gives the wrong sign for ρ_1; this can be seen by substituting the equation before (10-15) and (10-50) back into (10-10), with the ω's replaced by the corresponding π's.

The third order recurrence relations are computed similarly; the final results are

$$
\begin{aligned}
dI_{xxx} &= \left(I_{xxxx} + \frac{3\,I_{xxy}^2}{\kappa_1 - \kappa_2} - 3\kappa_1^3 \right)\varpi^1 + \left(I_{xxxy} + \frac{3\,I_{xxy} I_{xyy}}{\kappa_1 - \kappa_2} \right)\varpi^2 \\
&\qquad\qquad + \vartheta_{xxx} + \frac{3\,I_{xxy}}{\kappa_1 - \kappa_2}\vartheta_{xy} - 3\,\kappa_1^2 \vartheta_x, \\
dI_{xxy} &= \left(I_{xxxy} - \frac{(I_{xxx} - 2\,I_{xyy})\,I_{xxy}}{\kappa_1 - \kappa_2} \right)\varpi^1 + \left(I_{xxyy} - \frac{(I_{xxx} - 2\,I_{xyy})\,I_{xyy}}{\kappa_1 - \kappa_2} - \kappa_1 \kappa_2^2 \right)\varpi^2 \\
&\qquad\qquad + \vartheta_{xxy} - \frac{I_{xxx} - 2\,I_{xyy}}{\kappa_1 - \kappa_2}\vartheta_{xy} - \kappa_1 \kappa_2 \vartheta_y, \\
dI_{xyy} &= \left(I_{xxyy} - \frac{(2\,I_{xxy} - I_{yyy})\,I_{xxy}}{\kappa_1 - \kappa_2} - \kappa_1^2 \kappa_2 \right)\varpi^1 + \left(I_{xyyy} - \frac{(2\,I_{xxy} - I_{yyy})\,I_{xyy}}{\kappa_1 - \kappa_2} \right)\varpi^2 \\
&\qquad\qquad + \vartheta_{xyy} - \frac{2\,I_{xxy} - I_{yyy}}{\kappa_1 - \kappa_2}\vartheta_{xy} - \kappa_1 \kappa_2 \vartheta_x, \\
dI_{yyy} &= \left(I_{xyyy} - \frac{3\,I_{xxy}\,I_{xyy}}{\kappa_1 - \kappa_2} \right)\varpi^1 + \left(I_{yyyy} - \frac{3\,I_{xyy}^2}{\kappa_1 - \kappa_2} - 3\kappa_2^3 \right)\varpi^2 \\
&\qquad\qquad + \vartheta_{yyy} - \frac{3\,I_{xxy}}{\kappa_1 - \kappa_2}\vartheta_{xy} - 3\,\kappa_2^2 \vartheta_y.
\end{aligned}
\tag{66}
$$

In view of $(49, 61)$, this implies

$$
\begin{aligned}
I_{xxxx} &= \kappa_{1,1,1} - \frac{3\,\kappa_{1,2}^2}{\kappa_1 - \kappa_2} + 3\,\kappa_1^3, \\
I_{xxxy} &= \kappa_{1,1,2} - \frac{3\,\kappa_{1,2}\,\kappa_{2,1}}{\kappa_1 - \kappa_2} = \kappa_{1,2,1} + \frac{\kappa_{1,1}\,\kappa_{1,2} - 2\,\kappa_{1,2}\,\kappa_{2,1}}{\kappa_1 - \kappa_2}, \\
I_{xxyy} &= \kappa_{1,2,2} + \frac{\kappa_{1,1}\,\kappa_{2,1} - 2\,\kappa_{2,1}^2}{\kappa_1 - \kappa_2} + \kappa_1 \kappa_2^2 = \kappa_{2,1,1} + \frac{2\,\kappa_{1,2}^2 - \kappa_{1,2}\,\kappa_{2,2}}{\kappa_1 - \kappa_2} + \kappa_1^2 \kappa_2, \\
I_{xyyy} &= \kappa_{2,1,2} + \frac{2\,\kappa_{1,2}\,\kappa_{2,1} - \kappa_{2,1}\,\kappa_{2,2}}{\kappa_1 - \kappa_2} = \kappa_{2,2,1} + \frac{3\,\kappa_{1,2}\,\kappa_{2,1}}{\kappa_1 - \kappa_2}, \\
I_{yyyy} &= \kappa_{2,2,2} + \frac{3\,\kappa_{2,1}^2}{\kappa_1 - \kappa_2} + 3\,\kappa_2^3.
\end{aligned}
\tag{67}
$$

where $\kappa_{1,1,1} = \mathcal{D}_1^2 \kappa_1$, $\kappa_{1,1,2} = \mathcal{D}_2 \mathcal{D}_1 \kappa_1$, etc. There are two formulae for $I_{xxxy}, I_{xxyy}, I_{xyyy}$, because they appear twice in (66). The two expressions for I_{xxxy} and I_{xyyy} agree due to the commutator formula (65) for the invariant differential operators $\mathcal{D}_1, \mathcal{D}_2$, whereas the two expressions for I_{xxyy} yield the celebrated *Codazzi syzygy*

$$
\kappa_{1,2,2} - \kappa_{2,1,1} + \frac{\kappa_{1,1}\,\kappa_{2,1} - 2\,\kappa_{2,1}^2 - 2\,\kappa_{1,2}^2 + \kappa_{1,2}\,\kappa_{2,2}}{\kappa_1 - \kappa_2} - \kappa_1 \kappa_2 \,(\kappa_1 - \kappa_2) = 0, \tag{68}
$$

which can be re-written compactly in terms of the commutator invariants (63) as

$$K = \kappa_1 \kappa_2 = (\mathcal{D}_1 - Y_2) Y_2 - (\mathcal{D}_2 + Y_1) Y_1. \tag{69}$$

The latter is the key identity employed by Guggenheimer, [**3**; (10-52)], for a short proof of Gauss' Theorema Egregium, that the Gauss curvature is *intrinsic* meaning that it depends only on the surface's metric (first fundamental form).

Remark: The argument presented in Guggenheimer, [**3**; p. 234] is slightly flawed. Using his notation, the one-forms π^1, π^2 are not uniquely prescribed by the first fundamental form, but only up to a rotational parameter, which is subsequently fixed by diagonalizing the second fundamental form. Thus, his invariants ρ_1, ρ_2, which are, up to sign, our commutator invariants, also depend on this rotational parameter, and hence, in contrast to his stated claim, do *not* depend only on the first fundamental form. Nevertheless, the combination represented by the right hand side of (69) or, equivalently, equation (10-52) in [**3**] *is* independent of the rotational parameter; once this is verified, either by Guggenheimer's earlier argument, or directly, the identity (69) does provide a proof of Gauss' Theorem.

Finally, the recurrence formulae for the lowest order invariant contact forms are

$$\begin{aligned}
d\vartheta &= d\iota(\theta) = \iota(dx \wedge \theta_x + dy \wedge \theta_y) + \nu^5 \wedge \iota(u_x \theta) + \nu^6 \wedge \iota(u_y \theta) = \varpi^1 \wedge \vartheta_x + \varpi^2 \wedge \vartheta_y, \\
d\vartheta_x &= d\iota(\theta_x) = \iota(dx \wedge \theta_{xx} + dy \wedge \theta_{xy}) - \nu^4 \wedge \iota(\theta_y) + \nu^5 \wedge \iota(2 u_x \theta_x + u_{xx}\theta) \\
&\qquad\qquad + \nu^6 \wedge \iota(u_y \theta_x + u_x \theta_y + u_{xy}\theta) \\
&= \varpi^1 \wedge \big(\vartheta_{xx} - Y_1 \vartheta_y - \kappa_1^2 \vartheta \big) + \varpi^2 \wedge \big(\vartheta_{xy} - Y_2 \vartheta_y \big) + \frac{\vartheta_{xy} \wedge \vartheta_y}{\kappa_1 - \kappa_2} - \kappa_1 \vartheta_x \wedge \vartheta, \\
d\vartheta_y &= d\iota(\theta_y) = \iota(dx \wedge \theta_{xy} + dy \wedge \theta_{yy}) + \nu^4 \wedge \iota(\theta_x) + \nu^5 \wedge \iota(u_y \theta_x + u_x \theta_y + u_{xy}\theta) \\
&\qquad\qquad + \nu^6 \wedge \iota(2 u_y \theta_y + u_{yy}\theta) \\
&= \varpi^1 \wedge \big(\vartheta_{xy} + Y_1 \vartheta_x \big) + \varpi^2 \wedge \big(\vartheta_{yy} + Y_2 \vartheta_x - \kappa_2^2 \vartheta \big) + \frac{\vartheta_{xy} \wedge \vartheta_y}{\kappa_1 - \kappa_2} - \kappa_2 \vartheta_y \wedge \vartheta.
\end{aligned} \tag{70}$$

The first two summands correspond to the invariant horizontal differential of each invariant contact form, and, again via (49), produce the formulas for the higher order invariant contact forms in terms of the invariant Lie derivatives of the order zero invariant contact form (48):

$$\begin{aligned}
\vartheta_x &= \mathcal{D}_1 \vartheta, \\
\vartheta_y &= \mathcal{D}_2 \vartheta,
\end{aligned} \qquad
\begin{aligned}
\vartheta_{xx} &= \mathcal{D}_1^2 \vartheta + Y_1 \mathcal{D}_2 \vartheta + \kappa_1^2 \vartheta, \\
\vartheta_{xy} &= \mathcal{D}_2 \mathcal{D}_1 \vartheta + Y_2 \mathcal{D}_2 \vartheta = \mathcal{D}_1 \mathcal{D}_2 \vartheta - Y_1 \mathcal{D}_1 \vartheta, \\
\vartheta_{yy} &= \mathcal{D}_2^2 \vartheta - Y_2 \mathcal{D}_1 \vartheta + \kappa_2^2 \vartheta.
\end{aligned} \tag{71}$$

The two expressions for ϑ_{xy} agree as a consequence of the commutation formula (65).

Higher order recurrence formulas can all be constructed algorithmically in a similar fashion. We have, in this manner, produced the entire differential-algebraic structure of the differential invariants and invariant variational quasi-tricomplex for Euclidean surfaces. See

[**4**, **8**] for applications to Euclidean-invariant variational problems, including the Willmore problem, and surface flows, including mean and Gauss curvature flows.

A classical result, [**3**], that also follows immediately from the moving frame recurrence relations, is that the algebra of Euclidean surface differential invariants is generated by the principal curvatures or, equivalently, the mean and Gauss curvature invariants through invariant differentiation. Surprisingly, I recently discovered, [**9**, **13**], that, for generic surfaces, the mean curvature alone generates the differential invariant algebra, because the Gauss curvature can in fact be expressed as an explicit rational function of the invariant derivatives of order ≤ 4 of the mean curvature: $K = \Phi(\mathcal{D}^{(4)}H)$. Let us present a slight refinement of this result.

Definition 2. A surface $S \subset \mathbb{R}^3$ is *mean curvature degenerate* if, for any non-umbilic point $z_0 \in S$, there exist scalar functions $f_1(t), f_2(t)$, such that

$$\mathcal{D}_1 H = f_1(H), \qquad \mathcal{D}_2 H = f_2(H), \tag{72}$$

at all points $z \in S$ in a suitable neighborhood of z_0.

Clearly any constant mean curvature surface is mean curvature degenerate, with $f_1(t) \equiv f_2(t) \equiv 0$. Surfaces with non-constant mean curvature that admit a one-parameter group of Euclidean symmetries, i.e., non-cylindrical or non-spherical surfaces of rotation, non-planar surfaces of translation, or helicoid surfaces, obtained by, respectively, rotating, translating, or screwing a plane curve, are also mean curvature degenerate since, by the signature characterization of symmetry groups, [**1**], they have exactly one non-constant functionally independent differential invariant, namely their mean curvature H and hence any other differential invariant, including the invariant derivatives of H — as well as the Gauss curvature K — must be functionally dependent upon H. There also exist surfaces without continuous symmetries that are, nevertheless, mean curvature degenerate since it is entirely possible that (72) holds, but the Gauss curvature remains functionally independent of H. However, I do not know whether there is a good intrinsic geometric characterization of such surfaces. Indeed, their geometric properties have not, as far as I know, been studied to date.

Theorem 3. *If a surface is mean curvature nondegenerate then the algebra of differential invariants is generated entirely by the mean curvature and its successive invariant derivatives.*

Proof: Following the arguments in [**9**], in view of the Codazzi formula (69), it suffices to write the commutator invariants Y_1, Y_2 in terms of the mean curvature. To this end, we note that the commutator identity (65) can be applied to any differential invariant. In particular,

$$\mathcal{D}_1\mathcal{D}_2 H - \mathcal{D}_2\mathcal{D}_1 H = Y_1\, \mathcal{D}_1 H + Y_2\, \mathcal{D}_2 H, \tag{73}$$

and, furthermore, for $j = 1$ or 2,

$$\mathcal{D}_1\mathcal{D}_2\mathcal{D}_j H - \mathcal{D}_2\mathcal{D}_1\mathcal{D}_j H = Y_1\, \mathcal{D}_1\mathcal{D}_j H + Y_2\, \mathcal{D}_2\mathcal{D}_j H. \tag{74}$$

Provided the nondegeneracy condition

$$(\mathcal{D}_1 H)\,(\mathcal{D}_2\mathcal{D}_j H) \neq (\mathcal{D}_2 H)\,(\mathcal{D}_1\mathcal{D}_j H), \qquad \text{for } j = 1 \text{ or } 2, \tag{75}$$

holds, we can solve (73–74) to write the commutator invariants Y_1, Y_2 as explicit rational functions of invariant derivatives of H. Plugging these expressions into the right hand side of the Codazzi identity (69) produces an explicit formula for the Gauss curvature as a rational function of the invariant derivatives, of order ≤ 4, of the mean curvature, valid for all surfaces satisfying the nondegeneracy condition (75).

Thus it remains to show that (75) is equivalent to mean curvature nondegeneracy of the surface. First, if (72) holds, then

$$\mathcal{D}_i\mathcal{D}_j H = \mathcal{D}_i f_j(H) = f_j'(H)\,\mathcal{D}_i H = f_j'(H)\,f_i(H), \qquad i, j = 1, 2.$$

This immediately implies

$$(\mathcal{D}_1 H)\,(\mathcal{D}_2\mathcal{D}_j H) = (\mathcal{D}_2 H)\,(\mathcal{D}_1\mathcal{D}_j H), \qquad j = 1, 2, \tag{76}$$

proving mean curvature degeneracy. Vice versa, in view of formula (49), the degeneracy condition (76) implies that, for each $j = 1, 2$, the differentials $d_{\mathcal{H}} H$, $d_{\mathcal{H}}(\mathcal{D}_j H)$ are linearly dependent everywhere on S, which, by the general characterization theorem for functional dependency, [**5**; Theorem 2.16], implies that, locally, $\mathcal{D}_j H$ can be written as a function of H, thus establishing the condition (72). *Q.E.D.*

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