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EQUIVALENCE OF DIFFERENTIAL OPERATORS*

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Abstract. Two versions of the equivalence problem—determining when two second-order differential operators on the line are the same under a change of variables—are solved completely using the Cartan method of equivalence.

Key words. differential operator, Cartan equivalence method, invariant, symmetry, Lie algebra

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1. Introduction. The basic equivalence problem to be treated here is to determine when two second-order differential operators on the real line can be transformed into each other by an appropriate change of variables. There are two different possible interpretations of the notion of equivalence, depending on whether we wish to preserve the differential expression corresponding to the operator or the Lie bracket between operators. In this paper we treat both versions of the equivalence problem for second-order operators on the line. The problems here are related to the more general equivalence problem for second-order ordinary differential equations [4], [5], [11], but are specialized by linearity. We employ the equivalence method of Cartan, which gives necessary and sufficient conditions for equivalence. Although the simplest of the possible equivalence problems arising in the study of differential operators, these problems provide a good illustration of the power and ease of use of Cartan's equivalence method, which offers a straightforward algorithm for solving these and other equivalence problems important in applications. Extensions to higher-order or higher-dimensional operators can be readily done using the methods of this paper, although the intervening calculations will, as a rule, become much more complicated. In the proof of the theorem, we assume that the reader has a basic familiarity with the Cartan equivalence method as explained, for instance, in [2], [3], [6], and [7], although the reader can certainly understand the final results without all the intervening machinery.

This paper originated in answer to a question raised by Levine [9], who asked when a differential operator can be expressed as a bilinear combination of first-order differential operators that generate a finite-dimensional Lie algebra. This problem has applications to scattering theory in molecular dynamics and quantum chemistry. Indeed, there are now a number of well-established methods for dealing with such operators, where the calculation of eigenvalues, spectra, and dynamics is considerably simplified. The companion paper [8] applies the results of this paper to solving Levine's problem completely.

2. Equivalence problems for differential operators. Consider a second-order differential operator

$$\mathscr{D} = f(x)D^2 + g(x)D + h(x), \tag{2.1}$$

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where f, g, h, are analytic functions of the real variable $x \in \mathbb{R}$, and $D = d/dx$. If we apply $\mathcal{D}$ to a scalar-valued function $u(x)$, we obtain the expression

$$\mathcal{D}[u] = fu'' + gu' + hu. \tag{2.2}$$

In particular, we can look at the linear, homogeneous second-order ordinary differential equation $\mathcal{D}[u] = 0$, or the eigenvalue problem $\mathcal{D}[u] = \lambda u$, or the Schrödinger equation $u_t = i\mathcal{D}[u]$, in which $\mathcal{D}$ plays the role of the Hamiltonian.

We will be concerned with the problem of when two such differential operators can be mapped into each other by an appropriate change of coordinates. It turns out that two natural classes of transformations can be employed to change the differential operator. Clearly, as far as the independent and dependent variables are concerned, the appropriate pseudogroup consists of the fiber-preserving transformations that are linear in the fiber variable u:

$$\bar{x} = \varphi(x), \qquad \bar{u} = \psi(x)u. \tag{2.3}$$

The total derivative operators are related by the chain rule formula[1]

$$\bar{D} = \frac{1}{\varphi'(x)} D. \tag{2.4}$$

In the first of our two equivalence problems, we identify the two differential expressions $\bar{\mathcal{D}}[\bar{u}] = \mathcal{D}[u]$ (cf. (2.2)), where

$$\bar{\mathcal{D}} = \bar{f}(\bar{x})\bar{D}^2 + \bar{g}(\bar{x})\bar{D} + \bar{h}(\bar{x})$$

is another second-order differential operator. The explicit formulae for the new coefficient functions $\bar{f}, \bar{g}, \bar{h}$, in terms of the original coefficients f, g, h of $\mathcal{D}$, can be determined using the transformation rule

$$\bar{\mathcal{D}} = \mathcal{D} \cdot \frac{1}{\psi(x)}, \tag{2.5}$$

together with the chain rule (2.4). The first of our equivalence problems for differential operators then amounts to determining conditions on the two differential operators such that there exists a transformation (2.3) that maps one to the other according to (2.5).

The transformation rule (2.5) has the disadvantage of not preserving either the eigenvalue problem or the Schrödinger equation associated with the operator. For instance, $\mathcal{D}[u] = \lambda u$ does not imply $\bar{\mathcal{D}}[\bar{u}] = \lambda \bar{u}$, since we are missing a factor of $\psi(x)$. To rectify this situation, we need to premultiply by $\psi(x)$ and use the alternative transformation rule

$$\bar{\mathcal{D}} = \psi(x) \cdot \mathcal{D} \cdot \frac{1}{\psi(x)}. \tag{2.6}$$

This transformation rule leads to slightly different formulae expressing the new coefficients $\bar{f}, \bar{g}, \bar{h}$, in terms of f, g, h. The transformations (2.6) enjoy the additional property of preserving the standard commutator Lie bracket $[\mathcal{D}, \mathcal{E}] = \mathcal{D} \cdot \mathcal{E} - \mathcal{E} \cdot \mathcal{D}$ between differential operators. The second equivalence problem is to determine conditions on two differential operators such that there exists a transformation (2.3) mapping one to the other according to (2.6).

[1] For simplicity, we will explicitly denote the pull-back maps only in the statements of the theorems.

We will solve both equivalence problems in this paper. Incidentally, if we try to combine all the transformations (2.3), (2.5), (2.6) (for different functions $\psi(x)$), we end up with the trivial result that all second-order differential operators are equivalent under this largest pseudogroup.

To apply Cartan's algorithm to either equivalence problem, we need to recast the transformation rules (2.3), and (2.5) or (2.6), in the language of differential forms. The appropriate space to work in will be the second jet space J^2, which has coordinates x, u, p, q. Here p represents the derivative u', and q the derivative u''. The immediate goal is to construct an appropriate coframe, or pointwise basis for the cotangent space T^*J^2, that will encode the relevant transformation rules for our problem(s). The first remark is that as long as $u \neq 0$, the pseudogroup of transformations (2.3) is uniquely prescribed by imposing the 1-form equations

$$d\bar{x} = \alpha\, dx, \tag{2.7}$$

$$\frac{d\bar{u}}{\bar{u}} = \frac{du}{u} + \beta\, dx. \tag{2.8}$$

Here α and β are functions J^2, whose precise form does not need to be specified in advance. Indeed, the first equation implies that $\bar{x} = \varphi(x)$, with $\alpha = \varphi'$, while the second necessarily requires the linearity of the transformation in u, so that $\bar{u} = \psi(x)u$, with $\beta = \psi'/\psi$. Note that the restriction to $u \neq 0$, which means that we are restricting our attention to either the positive or negative real u-axis, is inessential as far as the differential operator itself is concerned. (Indeed, analytic continuation will extend our results across the apparent singular subspace $u = 0$.)

For the derivative variables p and q to transform correctly, we need to preserve the *contact ideal* on J^2, which is the differential ideal generated by the pair of 1-forms $du - p\,dx$, $dp - q\,dx$. In general, a diffeomorphism $\Phi: J^2 \to J^2$ determines a contact transformation if and only if

$$d\bar{u} - \bar{p}\, d\bar{x} = \lambda(du - p\,dx), \tag{2.9}$$

$$d\bar{p} - \bar{q}\, d\bar{x} = \mu(du - p\,dx) + \nu(dp - q\,dx), \tag{2.10}$$

where λ, μ, ν, are functions on J^2. Equations (2.7)–(2.9) by themselves already constitute part of an overdetermined equivalence problem on J^2. There is an algorithm, due to Cartan, to reduce this to an equivalence problem of standard form, but in our case, we can do this by inspection. It is easy to see that the 1-form $(du - p\,dx)/u$ is invariant, so the identification

$$\frac{d\bar{u} - \bar{p}\, d\bar{x}}{\bar{u}} = \frac{du - p\,dx}{u} \tag{2.11}$$

can replace both (2.8) and (2.9). The reader can check that the 1-form identities (2.7), (2.10), (2.11), are equivalent to requiring that the transformation on J^2 be the prolongation of a point transformation of the special form (2.3), with the derivative variables p, q, transforming correctly. Therefore, we take as the first three elements of our eventual coframe the 1-forms

$$\omega_1 = dx, \quad \omega_2 = \frac{du - p\,dx}{u}, \quad \omega_3 = dp - q\,dx, \tag{2.12}$$

with the transformation rules

$$\bar{\omega}_1 = A\omega_1, \quad \bar{\omega}_2 = \omega_2, \quad \bar{\omega}_3 = B\omega_2 + C\omega_3, \quad A, C \neq 0,$$

where A, B, C, are functions on J^2. This much of the coframe is the same for each equivalence problem. To complete the coframe, we need to supplement these 1-forms with an additional 1-form, which will encode the action of the transformation rule (2.5) or (2.6) on the differential operator itself.

In both cases, there is an obvious invariant function for the problem. For the equivalence problem (2.5), the invariant is the differential expression (2.2), i.e.,

$$I(x, u, p, q) = \mathscr{D}[u] = f(x)q + g(x)p + h(x)u. \tag{2.13}$$

For the second problem (2.6), the invariant is slightly more complicated:

$$I(x, u, p, q) = \frac{\mathscr{D}[u]}{u} = \frac{f(x)q + g(x)p}{u} + h(x), \tag{2.14}$$

since we need to take care of the extra factor of ψ. In either case, we have $I(x, u, p, q) = \bar{I}(\bar{x}, \bar{u}, \bar{p}, \bar{q})$ under the identification (2.5) or (2.6). We therefore take our final 1-form to be the differential $\omega_4 = dI$, so that for the equivalence problem (2.5) we have

$$\omega_4 = f\,dq + g\,dp + h\,du + \{f'q + g'p + h'u\}\,dx, \tag{2.15}$$

whereas for the alternative problem (2.6) we take

$$\omega_4 = \frac{f}{u}\,dq + \frac{g}{u}\,dp - \frac{fq + gp}{u^2}\,du + \left\{\frac{f'q + g'p}{u} + h'\right\}dx. \tag{2.16}$$

In both cases, the four 1-forms $\omega_1, \omega_2, \omega_3, \omega_4$, provide a coframe on the subset

$$\Omega^* = \{(x, u, p, q) \in J^2 \mid u \neq 0 \text{ and } f(x) \neq 0\}. \tag{2.17}$$

From now on, we restrict our attention to a connected component $\Omega \subset \Omega^*$ of the subset (2.17); note that on such a component, the signs of $f(x)$ and u are fixed. We require only that the last coframe elements agree up to contact, i.e.,

$$\bar{\omega}_4 = D\omega_2 + E\omega_3 + \omega_4.$$

We therefore define the structure group

$$G = \left\{ \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & B & C & 0 \\ 0 & D & E & 1 \end{pmatrix} : A, B, C, D, E \in \mathbb{R},\ A \cdot C \neq 0 \right\},$$

which happens to be the same for both equivalence problems, even though the two coframes are different.

As a consequence of these preliminary considerations, we have successfully encoded our equivalence problem in terms of a coframe, and have shown the following.

PROPOSITION 2.1. *Let* $\mathscr{D}$ *and* $\bar{\mathscr{D}}$ *be second-order differential operators. Let* $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ *and* $\{\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3, \bar{\omega}_4\}$ *be the corresponding coframes, on open subsets* Ω *and* $\bar{\Omega}$ *of the second jet space, given by* (2.12) *and* (2.15) *or* (2.16), *the choice of* ω_4 *and* $\bar{\omega}_4$ *depending on the equivalence problem under consideration. The differential operators are equivalent under the pseudogroup* (2.3) *according to the respective transformation rule* (2.5) *or* (2.6) *if and only if there is a diffeomorphism* $\Phi: \Omega \to \bar{\Omega}$ *that satisfies*

$$\Phi^*(\bar{\omega}_i) = \sum_{j=1}^{4} g_{ij}\omega_j, \qquad i = 1, \cdots, 4,$$

where $g = (g_{ij})$ *is a* G*-valued function on* J^2, *and* Φ^* *denotes the pull-back map on differential forms.*

To apply Cartan's algorithm for this equivalence problem, we must "lift" the coframes to the space $J^2 \times G$. The *lifted coframe* takes the form

$$\theta_1 = A\omega_1, \quad \theta_2 = \omega_2, \quad \theta_3 = B\omega_2 + C\omega_3, \quad \theta_4 = D\omega_2 + E\omega_3 + \omega_4, \tag{2.18}$$

where the coefficients A, B, C, D, E are now interpreted as coordinates in the structure group G. We then have the standard reformulation of the equivalence condition of Proposition 2.1.

PROPOSITION 2.2. *Under the setup of Proposition* 2.2, *two differential operators* $\mathcal{D}$ *and* $\bar{\mathcal{D}}$ *are equivalent if and only if there is a diffeomorphism* $\Psi: \Omega \times G \to \bar{\Omega} \times G$ *that commutes with the natural left action of G, and maps the appropriate lifted coframe elements to each other*:

$$\Psi^*(\bar{\theta}_i) = \theta_i, \qquad i = 1, \cdots, 4. \tag{2.19}$$

3. Solution of the first equivalence problem. To keep our presentation as short as possible, we will assume that the reader has some familiarity with the mechanics of Cartan's equivalence method as discussed, for instance, in [2], [3], and [6]. We will solve both equivalence problems, beginning with the setup (2.5), corresponding to the coframe element (2.15). The solutions are fairly similar intrinsically, although the parametric formulae differ. We present this case in detail, and briefly indicate how the other problem goes in the following section.

We begin with the lifted coframe (2.18), based on the base coframe (2.12), (2.15). The basic tool in Cartan's method is the invariance of the exterior derivative operation under smooth maps, so we begin by computing the differentials $d\theta_i$. They are found to have the form

$$\begin{aligned} d\theta_1 &= \alpha \wedge \theta_1 + \sigma_1, \\ d\theta_2 &= \sigma_2, \\ d\theta_3 &= \beta \wedge \theta_2 + \gamma \wedge \theta_3 + \sigma_3, \\ d\theta_4 &= \delta \wedge \theta_2 + \varepsilon \wedge \theta_3 + \sigma_4. \end{aligned}$$

Here $\alpha, \beta, \gamma, \delta, \varepsilon$, form a basis for the right-invariant 1-forms on the Lie group G, and the torsion terms take the form

$$\sigma_i = \sum_{j<k} \tau_{ijk}\theta_j \wedge \theta_k, \qquad i = 1, \cdots, 4.$$

In the absorption part of Cartan's process, we are allowed to replace each 1-form $\alpha, \beta, \gamma, \delta, \varepsilon$ by an expression of the form $\alpha + \Sigma z_j\theta_j$, etc., where the functions z_j are chosen so as to make as many of the torsion coefficients τ_{ijk} vanish as possible. In the present setup we can readily "absorb" all the torsion components except

$$\tau_{212} = -\frac{B+Cp}{ACu}, \quad \tau_{213} = \frac{1}{ACu}, \quad \tau_{314} = \frac{C}{Af}, \quad \tau_{414} = \frac{E}{Af}.$$

These components are invariants of the problem. Since they depend on the group parameters, the next step in the process is to normalize them to as simple a form as possible through a suitable choice of the group parameters. There are two possible normalizations for these torsion components, depending on $\kappa_1 = \text{sign}\,(f(x) \cdot u)$, leading to two different branches for the equivalence problem. (However, as we remarked above, as far as the differential operator itself is concerned, the division into two branches is not essential, since we can always change the sign of u by restricting our attention to a different connected component Ω of the domain (2.17). It is nevertheless convenient to retain the sign through our analysis.) We normalize the torsion components to 0, κ_1, 1, 0, respectively, by setting

$$(3.1)\qquad A=\frac{\sigma_1}{\sqrt{|fu|}},\quad B=-\sigma_2 p\sqrt{\left|\frac{f}{u}\right|},\quad C=\sigma_2\sqrt{\left|\frac{f}{u}\right|},\quad E=0.$$

Here $\sigma_1=\pm 1$ is an undetermined sign that must be left ambiguous (even with the specification of κ_1), and $\sigma_2=\sigma_1\cdot(\operatorname{sign} f)$. (See [7] for a detailed discussion of these types of signs.) The normalizations (3.1) have the effect of reducing the original Lie group G to a one-parameter subgroup, with D the only remaining undetermined parameter.

In the second loop through the equivalence procedure, we substitute (3.1) into the formulas for the lifted coframe (2.18) and recompute the differentials. The unabsorbable torsion component $\tau_{413}=D$ can then be normalized to zero by setting $D=0$. Note that we could have avoided adding in contact terms in our definition of θ_4, since ω_4 turns out to already be an invariant form.

By normalizing the torsion components, we have managed to eliminate all of the group parameters. This has had the effect of (a) reducing the structure group to the identity, and (b) reducing the lifted invariant coframe to an invariant coframe on the base space J^2, known as an $\{e\}$-*structure* or local parallelism. The explicit formula for the invariant coframe comes from (2.18), (3.1), and we have

$$(3.2)\qquad \begin{aligned} \theta_1&=\frac{\sigma_1\,dx}{\sqrt{|fu|}},\\ \theta_2&=\frac{du-p\,dx}{u},\\ \theta_3&=\sigma_2\sqrt{\left|\frac{f}{u}\right|}\left\{-\frac{p}{u}(du-p\,dx)+(dp-q\,dx)\right\},\\ \theta_4&=f\,dq+g\,dp+h\,du+(f'q+g'p+h'u)\,dx. \end{aligned}$$

Indeed, as the reader can check, these 1-forms do satisfy the invariance conditions

$$\bar{\theta}_i=\theta_i,\qquad i=1,2,3,4,$$

under the pseudogroup of transformations (2.3), (2.5). Applying the exterior derivative to the invariant coframe elements, and re-expressing the resulting 2-forms in terms of the coframe, we find that the structure equations for our problem take the form

$$(3.3)\qquad \begin{aligned} d\theta_1&=\tfrac{1}{2}\theta_1\wedge\theta_2,\\ d\theta_2&=\kappa_1\theta_1\wedge\theta_3,\\ d\theta_3&=-I\theta_1\wedge\theta_2+\kappa_1 J\theta_1\wedge\theta_3+\theta_1\wedge\theta_4+\tfrac{1}{2}\theta_2\wedge\theta_3,\\ d\theta_4&=0, \end{aligned}$$

where

$$(3.4)\qquad I=fq+gp+hu,\qquad J=\sigma_2\sqrt{\left|\frac{u}{f}\right|}\left(\frac{1}{2}f'-\frac{3}{2}p\frac{f}{u}-g\right).$$

Because the coframe is invariant, the functions I and J are the fundamental invariants of the problem. Note that we have recovered our original invariant (2.13) as one of the torsion components in the structure equations (3.3).

The *covariant derivatives* $F_{,\theta_i}$ of a function F with respect to the coframe (3.2) are defined by expressing the differential of F in terms of the invariant coframe:

$$dF = F_{,\theta_1}\theta_1 + F_{,\theta_2}\theta_2 + F_{,\theta_3}\theta_3 + F_{,\theta_4}\theta_4. \tag{3.5}$$

Explicitly,

$$\begin{aligned} F_{,\theta_1} &= \sigma_1\sqrt{|fu|}\hat{D}_x F, \\ F_{,\theta_2} &= uF_u + pF_p - \frac{pg+hu}{f}F_q, \\ F_{,\theta_3} &= \sigma_2\sqrt{\left|\frac{u}{f}\right|}\left(F_p - \frac{g}{f}F_q\right), \\ F_{,\theta_4} &= \frac{1}{f}F_q. \end{aligned} \tag{3.6}$$

Here $\hat{D}_x$ denotes the differential operator

$$\hat{D}_x = \frac{\partial}{\partial x} + p\frac{\partial}{\partial u} + q\frac{\partial}{\partial p} + R\frac{\partial}{\partial q}, \tag{3.7}$$

where

$$R = -\frac{gq + hp + f'q + g'p + h'u}{f}. \tag{3.8}$$

Note that if we differentiate the invariant equation $I = \text{constant}$ (which is the same as the ordinary differential equation $\mathscr{D}[u] = \text{constant}$) with respect to x and solve for the third-order derivative $r = u'''$, we recover (3.8). In this sense, $\hat{D}_x$ can be identified with the total derivative operator on J^2.

The covariant derivatives of any of the fundamental invariants (3.4), called the *derived invariants*, are also clearly invariants. Since the differentials of I and J are of the form

$$dI = \theta_4, \qquad dJ = \kappa_1 K\theta_1 + \tfrac{1}{2}J\theta_2 - \tfrac{3}{2}\kappa_1\theta_3,$$

the only independent derived invariant is

$$\begin{aligned} K &= \kappa_1 J_{,\theta_1} = \kappa_1\sigma_1\sqrt{|fu|}\hat{D}_x J \\ &= -\frac{3}{2}fq + \frac{3}{4}f\frac{p^2}{u} - \frac{1}{2}p(f'+g) + u\frac{2ff'' - f'^2 + 2f'g - 4fg'}{4f}. \end{aligned} \tag{3.9}$$

(Note that K does not have an ambiguous sign.) We can continue differentiating to deduce higher-order derived invariants; for example,

$$dK = L\theta_1 + (K - \tfrac{3}{2}I)\theta_2 - J\theta_3 - \tfrac{3}{2}\theta_4,$$

so we have one second-order derived invariant

$$L = K_{,\theta_1} = \kappa_1 J_{,\theta_1,\theta_1} = \sigma_1\sqrt{|fu|}\hat{D}_x K,$$

which we avoid writing out explicitly.

Given an $\{e\}$-structure as above, we define its *rank* to be the number of functionally independent invariants (including all possible derived invariants). The *order* of the $\{e\}$-structure is the highest-order derived invariant required to complete the independent set of invariants. According to the standard Jacobian criterion for functional independence, the particular $\{e\}$-structure given by the coframe (3.3) will have rank 4 and order 2, provided $dI \wedge dJ \wedge dK \wedge dL \neq 0$, whereby I, J, K, L are a complete set of functionally independent invariants. Exceptional cases with lower rank (and lower order) can occur if this wedge product vanishes.

To investigate the structure of the invariants in more detail, we proceed as follows. Note first that since $f \cdot u \neq 0$, the invariants I and J are always functionally independent. We can eliminate p and q from the original equations (3.4):

$$p = \frac{f'-2g}{3f}u - \frac{2}{3}\kappa_1\sigma_2 J\sqrt{\left|\frac{u}{f}\right|},$$

$$q = -\frac{gp+hu-I}{f} = \frac{2g^2-f'g-3fh}{3f^2}u - \frac{2}{3}\kappa_1\sigma_1 J\frac{g|u|^{1/2}}{|f|^{3/2}} + \frac{I}{f}.$$

Substituting these into (3.9), we find that we can write

$$K = a(x)u + \frac{1}{3}\kappa_1 J^2 - \frac{3}{2}I,$$

where

$$a = \frac{5gf'-2f'^2-2g^2}{6f} + \frac{3}{2}h - g' + \frac{1}{2}f''. \tag{3.10}$$

If the function $a(x) \equiv 0$, then K is a function of I and J. An easy chain rule argument shows that in this case, besides I and J, there are no further independent derived invariants. Therefore, if $a \equiv 0$, the rank of the $\{e\}$-structure is 2, and the order is zero.

Otherwise, if a does not vanish identically, we can take

$$\tilde{K} = a(x)u$$

as a new independent invariant, and compute its derived invariant:

$$\tilde{L} = \tilde{K}_{,\theta_1} = \sigma_1\sqrt{|fu|}\,\hat{D}_x\tilde{K} = \kappa_1 b(x)|\tilde{K}|^{3/2} - \frac{2}{3}J\tilde{K},$$

where

$$b(x) = \sigma_1\frac{3a'f+af'-2ag}{3\sqrt{|fa^3|}}. \tag{3.11}$$

Note that b is an invariant that depends only on x. If b is constant, then $I, J, \tilde{K}$ form a complete set of functionally independent invariants; the rank is 3 and the order 2. Otherwise, for b not constant, we have an $\{e\}$-structure of maximal rank, with $I, J, \tilde{K}, b$ comprising our four fundamental independent invariants. In this case, we complete the solution to the equivalence problem by computing one final derived invariant:

$$b_{,\theta_1} = \sigma_1\sqrt{|fu|}\,b' = c(x)|\tilde{K}|^{1/2},$$

where

$$c(x) = \sigma_1\sqrt{|f/a|}\,b'(x) \tag{3.12}$$

is also an invariant. In the case of an $\{e\}$-structure of maximal rank, the *determining function* F for our equivalence problem is prescribed by re-expressing c in terms of b, i.e., we write

$$c(x) = F[b(x)]. \tag{3.13}$$

Note that F may be a multiply-valued function.

Example 3.1. Let us consider the case of a simple operator

$$\mathcal{D} = D^2 + h(x) \tag{3.14}$$

of Sturm–Liouville type, i.e., $f = 1$, $g = 0$. Such operators play a key role in quantum mechanics, scattering theory, and the theory of the Korteweg–de Vries equation. Here

$$a(x) = \tfrac{3}{2} h(x),$$

so we have a structure of rank 2 if and only if $h \equiv 0$ and $\mathcal{D} = D^2$. As a result of our construction, we deduce that a second-order differential operator (2.1) is equivalent to the differential operator D^2 if and only if

$$3ff'' - 2f'^2 + 5f'g - 6fg' - 2g^2 + 9fh = 0. \tag{3.15}$$

Continuing, if $h \neq 0$, then

$$b = \pm \sqrt{\frac{3}{2}} \frac{h'}{|h|^{3/2}}.$$

Therefore b is constant if and only if $h(x) = (cx + d)^{-2}$, i.e., we have either a translate of the radial Laplace operator

$$\mathcal{D} = D^2 + \frac{k}{x^2}, \tag{3.16}$$

when $b = -\sqrt{6/k} \neq 0$, or $\mathcal{D} = D^2 + k$, when $b = 0$. Note that $k \neq 0$ can be scaled to 1.

Finally, in the case when b is not constant, then

$$c = \frac{hh'' - \frac{3}{2}h^2}{h^3}.$$

The determining function F will be found by writing

$$\frac{h''}{h^2} = \tilde{F}\left(\frac{h'}{h^{3/2}}\right), \tag{3.17}$$

from which

$$F(t) = \tilde{F}(t) - \tfrac{3}{2} t^2.$$

We note that the ordinary differential equation (3.17) can be solved explicitly by quadratures, owing to the presence of an obvious two-parameter symmetry group of translations in t and scalings; see [10].

In essence, the collection of all the invariants and their derived invariants will completely solve our equivalence problem, providing explicit necessary and sufficient conditions for two differential operators to be equivalent under a transformation (2.5). The following theorem is a consequence of general results on the equivalence of $\{e\}$-structures [6], [7].

THEOREM 3.2. *Let $\mathcal{D}$ and $\bar{\mathcal{D}}$ be real-analytic differential operators. Define the function $a(x)$ by* (3.10). *If $a \neq 0$, then define the functions $b(x)$, $c(x)$ by* (3.11), (3.12). *Then $\mathcal{D}$ and $\bar{\mathcal{D}}$ are equivalent under a change of variables* (2.3), (2.5) *if and only if*:

(i) *$a \equiv \bar{a} \equiv 0$, in which case both $\mathcal{D}$ and $\bar{\mathcal{D}}$ are equivalent to the operator D^2; or*

(ii) *Both a and $\bar{a}$ do not vanish identically and $b = \bar{b}$ are constant, in which case both $\mathcal{D}$ and $\bar{\mathcal{D}}$ are equivalent to either the radial Laplace operator* (3.16) *or the operator $D^2 + 1$; or*

(iii) *Both a and $\bar{a}$ do not vanish identically, both b and $\bar{b}$ are not constant, the determining functions prescribed by* (3.13) *are identical, $F = \bar{F}$, and the equation $b(x) = \bar{b}(\bar{x})$ has a real solution branch.* (*For complex equivalence, the last statement is unnecessary.*)

The change of variables required to map one operator into the other is implicitly given as the solution to the equations

$$b(x)=\bar{b}(\bar{x}), \qquad \bar{a}(\bar{x})\bar{u}=a(x)u, \tag{3.18}$$

restricted so that the signs $\kappa_1=\text{sign}\,(f\cdot u)$ and $\bar{\kappa}_1=\text{sign}\,(\bar{f}\cdot\bar{u})$ agree: $\kappa_1=\bar{\kappa}_1$.

In fact, the connection with the operator (3.14) can be used to complete the solution to the equivalence problem.

THEOREM 3.3. *If $\mathcal{D}$ is a second-order differential operator, then there is a transformation* (2.3), (2.5) *taking $\mathcal{D}$ into the operator $\bar{D}^2+\frac{2}{3}a(\bar{x})$, where the potential $a(\bar{x})$ is given by the* (*relative*) *invariant* (3.10) *when $\bar{x}=\varphi(x)$. Moreover, two differential operators are equivalent if and only if their corresponding potentials differ by the rescaling and translation group $\bar{a}(\bar{x})=\lambda^2 a(\lambda\bar{x}+\delta)$.*

In other words, the equivalence class of a differential operator under (2.6) is completely determined by its potential; moreover, two potentials are equivalent if and only if they are rescaled translates of each other.

4. Solution of the second equivalence problem. In this case, we begin as before with the lifted coframe (2.18), now based on the base coframe (2.12), (2.16). In the first loop through the equivalence procedure, we are left with the unabsorbable torsion components

$$\tau_{212}=-\frac{B+Cp}{ACu}, \quad \tau_{213}=\frac{1}{ACu}, \quad \tau_{314}=\frac{C}{Afu}, \quad \tau_{414}=\frac{E}{Afu}.$$

Again, there are two branches, depending on $\kappa_1=\text{sign}\,f$. Here the sign restriction is more essential than in the previous equivalence problem, since we cannot change the sign of f by a transformation of type (2.6). We normalize the torsion components to 0, κ_1, 1, 0, respectively, by setting

$$A=\frac{\sigma_1}{u\sqrt{|f|}}, \quad B=-\sigma_2 p\sqrt{|f|}, \quad C=\sigma_2\sqrt{|f|}, \quad E=0,$$

where σ_1 is an ambiguous sign, and $\sigma_2=\sigma_1\kappa_1$. In the second loop through the equivalence procedure, the unabsorbable torsion components $\tau_{413}=-\tau_{312}=D$ can both be normalized to zero by setting $D=0$. The final invariant coframe is now given by

$$\begin{aligned}
\theta_1&=\frac{\sigma_1\,dx}{u\sqrt{|f|}},\\
\theta_2&=\frac{du-p\,dx}{u},\\
\theta_3&=\sigma_2\sqrt{|f|}\left\{(dp-q\,dx)-\frac{p}{u}(du-p\,dx)\right\},\\
\theta_4&=\frac{f}{u}\,dq+\frac{g}{u}\,dp+\frac{fq+gp}{u^2}\,du+\left\{\frac{f'q+g'p}{u}+h'\right\}dx.
\end{aligned} \tag{4.1}$$

The structure equations take a slightly different form:

$$\begin{aligned}
d\theta_1&=0,\\
d\theta_2&=\kappa_1\theta_1\wedge\theta_3,\\
d\theta_3&=-2J\theta_1\wedge\theta_3+\theta_1\wedge\theta_4,\\
d\theta_4&=0,
\end{aligned} \tag{4.2}$$

where

$$J = \frac{\sigma_1}{4\sqrt{|f|}}\left(2g - f' + \frac{4pf}{u}\right) \tag{4.3}$$

is a fundamental invariant of the problem. Interestingly, the original invariant I given in (2.14) does *not* appear among the structure functions of the adapted coframe. Indeed, it is easy to see that it cannot appear even among the derived invariants of the structure functions, since only the derivative h' appears in the coframe (4.1), so it would be impossible to recover the function h, which appears in the expression for I, by differentiation. Thus, the invariant coframe (4.1) must be supplemented by the additional invariant I to effect the correct solution to the problem. Although we have come up with a nonstandard equivalence problem, Cartan himself was already aware of such possibilities. Indeed, in his original treatment of the equivalence method, he allows for the incorporation of additional function invariants into an equivalence problem, and, as he says, "Rien n'est changé à la solution..." [1, p. 725]. Here, we have one invariant provided by the structure functions, and one additional invariant, both of whose derived invariants must be taken into account when discussing the solution to the problem.

The *covariant derivatives* of a function F with respect to the coframe (4.1) are

$$\begin{aligned} F_{,\theta_1} &= \sigma_1\sqrt{|f|}\,\hat{D}_x F, \\ F_{,\theta_2} &= uF_u + pF_p - \frac{fq + pg(1-u)}{f}F_q, \\ F_{,\theta_3} &= \frac{\sigma_2 u}{\sqrt{|f|}}\left(F_p - \frac{g}{f}F_q\right), \\ F_{,\theta_4} &= \frac{u}{f}F_q. \end{aligned} \tag{4.4}$$

Here

$$\hat{D}_x = \frac{\partial}{\partial x} + p\frac{\partial}{\partial u} + q\frac{\partial}{\partial p} + R\frac{\partial}{\partial q}$$

is similar to the total derivative operator (3.7), but

$$R = \frac{fqp + gp^2}{u} - (gq + f'q + g'p + h'u) \tag{4.5}$$

is different. As in § 3, if we differentiate the equation $I = \text{constant}$ with respect to x and solve for the third order derivative $r = u'''$, then we recover the expression (4.5).

Since the differentials of I and J are of the form

$$dI = \theta_4, \qquad dJ = K\theta_1 + \theta_3,$$

the only independent derived invariant is

$$\begin{aligned} K = J_{,\theta_1} &= \sigma_1\sqrt{|f|}\,\hat{D}_x J \\ &= f\frac{qu - p^2}{u^2} + f'\frac{p}{2u} - \frac{2ff'' - f'^2 + 2f'g - 4fg'}{8f}. \end{aligned} \tag{4.6}$$

Furthermore,

$$dK = L\theta_1 - 2J\theta_3 + \theta_4,$$

so we have only one further second-order derived invariant:

$$L = K_{,\theta_1} = J_{,\theta_1,\theta_1} = \sigma_1\sqrt{|f|}\,\hat{D}_x K.$$

Note that in the case of the transformation rule (2.6), there is always a one-parameter symmetry group of any differential operator, namely, the scaling $u \to \lambda u$. Since the invariants must respect this symmetry, there can be at most three functionally independent invariants. Thus, the rank of this $\{e\}$-structure can be at most 3, and this will happen when $dI \wedge dJ \wedge dK \neq 0$.

To investigate the structure of the invariants in more detail, we proceed as before. We solve (2.14), (4.3), for p and q:

$$p = u\left(\frac{\sigma_2 J}{\sqrt{|f|}} + \frac{f' - 2g}{4f}\right),$$

$$q = \frac{uI - uh - pg}{f} = u\left(\frac{I}{f} - \sigma_1\frac{gJ}{|f|^{3/2}} + \frac{2g^2 - f'g - 4fh}{4f^2}\right).$$

Thus

$$K = -a(x) + I - \kappa_1 J^2, \tag{4.7}$$

where

$$a = \frac{8gf' - 3f'^2 - 4g^2}{16f} + h - \frac{1}{2}g' + \frac{1}{4}f''. \tag{4.8}$$

The only degenerate case is when a is constant, so the rank is 2 and the order zero. Otherwise the rank is 3, and we can take $a(x)$ as a new invariant. The final derived invariant is

$$b = a_{,\theta_1} = \sigma_1\sqrt{|f|}\,a'. \tag{4.9}$$

The *determining function* is found by re-expressing b in terms of a:

$$b(x) = F[a(x)]. \tag{4.10}$$

Note that since b has the ambiguous sign σ_1, the determining function F is only prescribed up to an ambiguous $\pm$ sign. (Indeed, the orientation reversing change of variables $x \to -x$, $u \to u$, will change the sign of f.) One solution to this annoying complication is to replace the invariant b by its square $b^2 = \kappa_1 f a'^2$.

THEOREM 4.1. *Let $\mathcal{D}$ and $\bar{\mathcal{D}}$ be real-analytic differential operators. Define the functions $a(x)$, $b(x)$, by* (4.8), (4.9). *Then $\mathcal{D}$ and $\bar{\mathcal{D}}$ are equivalent under a change of variables* (2.3), (2.6) *if and only if the signs $\kappa_1 = \operatorname{sign}(f) = \bar{\kappa}_1 = \operatorname{sign}(\bar{f})$ agree, and either*:

(i) *$a = \bar{a} = k$ are constant, in which case both $\mathcal{D}$ and $\bar{\mathcal{D}}$ are equivalent to the operator $D^2 + k$; or*

(ii) *Both a and $\bar{a}$ are not constant, the determining functions prescribed by* (4.10) *are identical, $F = \bar{F}$, and the equation $a(x) = \bar{a}(\bar{x})$ has a real solution branch.*

Example 4.2. Let us consider the case of the operator $\mathcal{D} = D^2 + h(x)$. In this case, $a(x) = h(x)$; hence we have a structure of rank 2 if and only if h is constant. A differential operator (2.1) is equivalent to the differential operator $D^2 + k$ via (2.6) if and only if $a = k$, i.e.,

$$4ff'' - 3f'^2 + 8f'g - 8fg' - 4g^2 + 16fh = 16kf. \tag{4.11}$$

Otherwise, since $b = h'$, the determining function will be found by writing

$$h' = F(h). \tag{4.12}$$

For a fixed determining function F, the general solution of (4.12) are just the translates of h, i.e., $\tilde{h}(x)=h(x-\delta)$. We conclude that two operators of the form (3.14) are equivalent under the transformation group (2.6) if and only if their potentials are translates of each other.

Conversely, given a determining function F, we can always construct a corresponding potential $h(x)$ by solving the elementary first-order ordinary differential equation (4.12). We thus recover the classical result that a general second-order differential operator can always be transformed into an operator of the form (3.14).

THEOREM 4.3. *If $\mathscr{D}$ is a second-order differential operator, then there is a transformation* (2.6) *taking $\mathscr{D}$ into the operator $\pm\bar{D}^2+a(\bar{x})$, where the potential $a(\bar{x})$ is given by the invariant* (4.8) *when $\bar{x}=\varphi(x)$, and the sign in front of $\bar{D}^2$ is determined by the sign of f, the coefficient of D^2 in $\mathscr{D}$. Moreover, two differential operators are equivalent if and only if their signs are the same and the corresponding potentials differ by a translation*: $\bar{a}(x)=a(x+\delta)$.

In other words, outside singular points where $f(x)=0$, the equivalence class of a differential operator under (2.6) is completely determined by its potential and the sign of its leading coefficient; moreover, two potentials are equivalent if and only if they are translates of each other.

5. Symmetries of differential operators. We will call a group of transformations of the form (2.3) a symmetry group of the differential operator $\mathscr{D}$ if the corresponding transformation (2.5) or (2.6) leaves the operator unchanged. (This is more restrictive than the concept of a symmetry group of a differential equation [10].) It is interesting to see what the corresponding infinitesimal symmetry criteria are.

PROPOSITION 5.1. *Given a vector field $\mathbf{v}=\xi(x)(\partial/\partial x)+\eta(x)u(\partial/\partial u)$ that generates a one-parameter group of transformations of the form* (2.3) *on $\mathbb{R}^2$, define a corresponding first-order differential operator $\mathscr{V}=\xi(x)D+\eta(x)$. The group generated by $\mathbf{v}$ is a symmetry group of the differential operator $\mathscr{D}$ of the type* (2.5) *or of the type* (2.6) *if and only if the operator equation*

$$[\mathscr{V},\mathscr{D}]+\eta\cdot\mathscr{D}=0 \tag{5.1}$$

or, respectively,

$$[\mathscr{V},\mathscr{D}]=0 \tag{5.2}$$

holds.

In either case, the proof is straightforward. In the second case, the scaling vector field with $\eta=1$ always generates a symmetry group.

Cartan's method gives us a complete handle on the symmetry group of an $\{e\}$-structure. If the structure has rank r and the underlying space has dimension n, then the symmetry group forms a Lie group of dimension $n-r$. For the differential operator equivalence problems, then, $n=4$, and so the symmetry group will have dimension $4-r$, where r is the number of functionally independent invariants. This leads to the following results.

THEOREM 5.2. *Let $\mathscr{D}$ be a real-analytic differential operator, and consider the symmetries of the type* (2.5). *Define the functions $a(x)$, $b(x)$, $c(x)$ as in* § 3. *Then*:

(i) *$\mathscr{D}$ admits a two-parameter symmetry group if and only if $a\equiv 0$.*

(ii) *$\mathscr{D}$ admits a one-parameter symmetry group if and only if a does not vanish identically, and b is constant.*

(iii) *If b is not constant, then $\mathscr{D}$ can admit only a discrete symmetry group.*

Thus a differential operator (2.1) is equivalent to the differential operator D^2 if and only if it admits a two-parameter group of symmetries which is also equivalent to the condition (3.15). The two-parameter symmetry group for the operator D^2 is, of course, generated by translations $(x, u) \to (x+k, u)$, and the scaling transformations $(x, u) \to (\lambda x, \lambda^2 u)$. Similarly, we have the result that the differential operator is equivalent to the radial Laplace operator (3.16) or the operator D^2+1 if and only if it admits a one-parameter group of symmetries. For the radial Laplace operator (3.16), the symmetry group is the scaling group $(x, u) \to (\lambda x, \lambda^2 u)$; for D^2+1 the translation group remains. Finally, for any other differential operator, the symmetry group is at most a discrete subgroup.

THEOREM 5.3. *Let $\mathcal{D}$ be a real-analytic differential operator, and consider the symmetries of the type* (2.6). *Let $a(x)$ be the corresponding potential* (4.8). *Then $\mathcal{D}$ always admits the one-parameter scaling symmetry group $(x, u) \to (x, \lambda u)$. Moreover, $\mathcal{D}$ admits a two-parameter symmetry group if and only if a is constant; otherwise there is only the possibility of additional discrete symmetries.*

See Hsu and Kamran [4] for more detailed information on the use of Cartan's equivalence method for determining the possible symmetry groups of general second-order ordinary differential equations.

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