Differential Invariants of Maximally Symmetric Submanifolds

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Abstract. Let G be a Lie group acting smoothly on a manifold M. A closed, nonsingular submanifold $S \subset M$ is called *maximally symmetric* if its symmetry subgroup $G_S \subset G$ has the maximal possible dimension, namely $\dim G_S = \dim S$, and hence $S = G_S \cdot z_0$ is an orbit of G_S. Maximally symmetric submanifolds are characterized by the property that all their differential invariants are constant. In this paper, we explain how to directly compute the numerical values of the differential invariants of a maximally symmetric submanifold from the infinitesimal generators of its symmetry group. The equivariant method of moving frames is applied to significantly simplify the resulting formulae. The method is illustrated by examples of curves and surfaces in various classical geometries.

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1. Introduction.

Suppose G is a Lie group acting smoothly on a manifold M, and let $\mathfrak{g}$ denote the corresponding Lie algebra of infinitesimal generators. The *symmetry group* of a closed submanifold $S \subset M$ is, by definition, the subgroup $G_S = \{ g \in G \mid g \cdot S = S \}$ consisting of all group transformations that map S to itself. A submanifold S is called *nonsingular* if G_S acts locally freely on S, [**17**]. A nonsingular submanifold is called *maximally symmetric* if $\dim G_S = \dim S$, and hence coincides with an orbit of its symmetry group: $S = G_S \cdot z_0$, for some $z_0 \in M$. According to a theorem of É. Cartan, [**3, 5**], a nonsingular submanifold is maximally symmetric if and only if all its differential invariants are constant. The goal of this paper is to develop effective formulae for computing the values of the differential invariants of such a maximally symmetric orbit $G_S \cdot z_0$ directly from the infinitesimal generators of its symmetry group, namely the symmetry subalgebra $\mathfrak{g}_S \subset \mathfrak{g}$.

As a simple example illustrating our concern, suppose $G = \mathrm{SE}(3)$ is the special Euclidean group consisting of the orientation-preserving rigid motions of $M = \mathbb{R}^3$. The fundamental differential invariants of a space curve $C \subset \mathbb{R}^3$ are its curvature κ and torsion τ. A (topologically closed) curve is maximally symmetric if and only if it is the orbit, $C = \left\{ \exp(t\,\mathbf{v}) z_0 \right\}$, of a one-parameter subgroup generated by a Euclidean vector field $\mathbf{v} \in \mathfrak{se}(3)$. All such curves are circles or helices, and our task is to determine the curvature and torsion of the orbit curve directly from the infinitesimal generator $\mathbf{v}$.

Interestingly, the formulae that we derive for the differential invariants of maximally symmetric submanifolds shed new light on the classical prolongation formula for vector fields on jet bundles, [**16**]. However, they turn out to be quite intricate even for fairly simple geometries. A significant simplification can be effected by normalizing some or all of the parameters — base point and subgroup — by applying an adapted group element. The most effective means of producing compact formulae is to appeal to the new equivariant formulation, [**5, 19**], of the classical Cartan method of moving frames, which we review in Section 4. Our methods and results will be illustrated by a number of examples from classical geometries. This investigation was motivated by several applications arising in geometry, [**12, 21**], differential equations, [**16**], and computer vision, [**19**].

2. Jets of Orbits.

Let M be a smooth m-dimensional manifold. For fixed $0 < p < m$, let $\mathrm{J}^n = \mathrm{J}^n(M,p)$ denote the associated n^{th} order (extended) jet space of p-dimensional submanifolds $S \subset M$; see [**16**] for details. We use $\mathrm{j}_n S|_z$ to denote the jet of S at the point $z \in M$.

Let $z = (x,u) = (x^1, \ldots, x^p, u^1, \ldots, u^q)$ be local coordinates on M, where we view the first p as independent variables, and the latter $q = m - p$ as dependent variables. We locally identify submanifolds with graphs of functions $u = f(x)$. (This omits submanifolds that are not transversal to the vertical fibers $\{x = c\}$, which can behandled by suitably changing coordinates, e.g., switching the roles of the independent and dependent variables.) The induced local coordinates on J^n are denoted $z^{(n)} = (x, u^{(n)}) = (\ \ldots \ x^i \ \ldots \ u^\alpha_J \ \ldots \ )$, with u^α_J, for $0 \le \#J \le n$ and $1 \le \alpha \le q$, representing the partial derivatives $\partial^J f^\alpha / \partial x^J$ of the graphing function.

Let H be a p-dimensional Lie group that acts† smoothly on M, and let $\mathfrak{h}$ denote the Lie algebra containing its infinitesimal generators, which are smooth vector fields on M. As usual, the prolonged action of H on the jet space J^n will be denoted by $H^{(n)}$. If H acts (locally) freely at $z_0 \in M$ (see below), the orbit $S = H \cdot z_0$ of H through z_0 is a smooth p-dimensional submanifold with tangent space $TS|_{z_0} = \mathfrak{h}|_{z_0}$. The aim of this section is to write out formulas for the jets (derivatives) of the orbit in terms of the infinitesimal generators of H.

Before dealing with the general situation, let us first look at the simplest case. Let M be two-dimensional manifold with local coordinates (x, u). Restricting to curves given by the graphs of functions $u = f(x)$, the associated jet space $\mathrm{J}^n(M, 1)$ has local coordinates $(x, u^{(n)}) = (x, u, u_x, u_{xx}, \ldots, u_n)$ representing the derivatives of f at the point x.

Let

$$\mathbf{v} = \xi(x,u)\,\frac{\partial}{\partial x} + \varphi(x,u)\,\frac{\partial}{\partial u} \tag{2.1}$$

be the infinitesimal generator of a one-parameter transformation group $\exp(t\,\mathbf{v})$ acting on M. Let $z_0 = (x_0, u_0) \in M$. We assume that $\xi(x_0, u_0) \neq 0$, as otherwise the curve has a vertical tangent, or, if $\varphi(x_0, u_0) = 0$ also, degenerates to a point. Our goal is to compute the derivatives

$$u_{k,0} = \frac{d^k u}{dx^k}\,(x_0), \qquad\qquad k = 1, 2, \ \ldots \ ,$$

of the curve $C = \exp(t\,\mathbf{v}) \cdot z_0$ at the point z_0. To this end, let

$$\mathbf{v}^{(n)} = \xi(x,u)\,\frac{\partial}{\partial x} + \sum_{k=0}^{n} \varphi_k(x, u^{(k)})\,\frac{\partial}{\partial u_k} \tag{2.2}$$

denote the corresponding prolonged vector field, generating the prolonged action on J^n, so that $\exp(t\,\mathbf{v}^{(n)}) = \exp(t\,\mathbf{v})^{(n)}$. The coefficients of $\mathbf{v}^{(n)}$ are prescribed by the well-known prolongation formula, [**16**; Theorem 4.16]:

$$\varphi_k = D_x^k \big[\, \varphi(x,u) - u_x\, \xi(x,u) \,\big] + u_{k+1}\, \xi(x,u), \tag{2.3}$$

where

$$D_x = \frac{\partial}{\partial x} + u_x\,\frac{\partial}{\partial u} + u_{xx}\,\frac{\partial}{\partial u_x} + \cdots \tag{2.4}$$

is the total derivative with respect to the independent variable x.

Proposition 2.1. *Under the above assumptions, the jet coordinates*

$$u_0^{(n)} = (u_0, u_{1,0}, \ldots, u_{n,0})$$

of the orbit curve $C = \big\{ \exp(t\mathbf{v}) \cdot (x_0, u_0) \big\}$ at the base point x_0 are given by the recursive formula

$$u_{k,0} = \frac{\varphi_{k-1}(x_0, u_0^{(k-1)})}{\xi(x_0, u_0)}\,, \qquad\qquad k = 1, \ldots, n. \tag{2.5}$$

† To simplify the exposition, we will assume that group actions are global. However, all results and formulas apply equally well to local group actions.

Furthermore, define the modified total derivative operator

$$\widehat{D}_x = \frac{\partial}{\partial x} \;+\; \frac{\varphi(x,u)}{\xi(x,u)}\,\frac{\partial}{\partial u}\;, \tag{2.6}$$

obtained by replacing u_1 *by the right hand side of* (2.5) *with* $k=1$ *and truncating. Set*

$$\psi_k(x,u) = \widehat{D}_x^k u. \tag{2.7}$$

Then one has the alternative formula

$$u_{k,0} = \psi_k(x_0,u_0) = \frac{\varphi_{k-1}(x_0,u_0,\psi_1(x_0,u_0),\ldots,\psi_{k-1}(x_0,u_0))}{\xi(x_0,u_0)}\,. \tag{2.8}$$

Thus, if $F(x,u^{(n)})$ *is any differential function, then its value on the orbit curve at the base point can be computed as*

$$F(x_0,u_0^{(n)}) = F(x_0,u_0,\psi_1(x_0,u_0),\;\ldots\;,\psi_n(x_0,u_0)). \tag{2.9}$$

Proof: The curve jet

$$\mathrm{j}_n C = \bigl\{\,\exp(t\,\mathbf{v}^{(n)})\cdot z_0^{(n)}\,\bigr\} \subset \mathrm{J}^n(M,1), \qquad \text{where} \qquad z_0^{(n)} = (x_0,u_0^{(n)}) = \mathrm{j}_n C|_{z_0},$$

is obtained by integrating the ordinary differential equations

$$\frac{dx}{dt} = \xi(x,u), \qquad \frac{du}{dt} = \varphi(x,u), \qquad \frac{du_1}{dt} = \varphi_1(x,u,u_1), \qquad \ldots, \qquad \frac{du_n}{dt} = \varphi_n(x,u^{(n)}),$$

prescribing the prolonged flow of the vector field $\mathbf{v}^{(n)}$ on J^n. Thus, by implicit differentiation,

$$u_{k+1} = \frac{du_k}{dx} = \frac{du_k/dt}{dx/dt} = \frac{\varphi_k}{\xi}\,,$$

which immediately establishes the first formula.

Moreover, given any smooth function $F(x,u)$, its total derivative, when restricted to the first order curve jet $\mathrm{j}_1 C$, is given by

$$D_x F(x,u,u_1) = \frac{\partial F}{\partial x} + u_1\,\frac{\partial F}{\partial u} = \frac{\partial F}{\partial x} + \frac{\varphi}{\xi}\,\frac{\partial F}{\partial u} = \widehat{D}_x F,$$

proving (2.8) when $k=1$. A simple induction on k establishes the general formula. The final formula (2.9) is an immediate corollary. *Q.E.D.*

Remark: The fact that (2.5) and (2.8) are equal provides us with an interesting new perspective on the classical prolongation formula (2.3). For example, taking $k=2$, we find

$$\varphi_1\left(x,u,\frac{\varphi}{\xi}\right) = \xi\,\left(\frac{\partial}{\partial x} + \frac{\varphi}{\xi}\,\frac{\partial}{\partial u}\right)^2 u = \xi\,\left(\frac{\partial}{\partial x} + \frac{\varphi}{\xi}\,\frac{\partial}{\partial u}\right)\left(\frac{\varphi}{\xi}\right)\,, \tag{2.10}$$

which can be verified by direct computation.

The preceding formulas can be straightforwardly adapted to curves in higher dimensional manifolds. We merely replace u by the various dependent variables u^α, and $\varphi(x,u)$ by the corresponding infinitesimal generator coefficients $\varphi^\alpha(x,u^1,\ldots u^q)$. The second term in the modified total derivative (2.6) becomes a summation over $\alpha = 1,\ldots,q$. Rather than write out the resulting formulas in detail, let us turn to the general case.

Let H be a p-dimensional Lie group acting smoothly on M. Let

$$\mathbf{v}_\kappa = \sum_{i=1}^{p} \xi^i_\kappa(x,u)\,\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha_\kappa(x,u)\,\frac{\partial}{\partial u^\alpha}\;, \qquad \kappa = 1,\ldots,p, \tag{2.11}$$

be a basis for its infinitesimal generators, spanning a p-dimensional Lie algebra $\mathfrak{h}$ of vector fields on M. Their prolongations to the submanifold jet space $\mathrm{J}^n(M,p)$ are given, in local coordinates, by

$$\mathbf{v}^{(n)}_\kappa = \mathbf{v}_\kappa + \sum_{\alpha=1}^{q} \sum_{1\le k=\#J\le n} \varphi^\alpha_{J,\kappa}(x,u^{(k)})\,\frac{\partial}{\partial u^\alpha_J}\;, \tag{2.12}$$

where, for each symmetric multi-index $J = (j_1,\ldots,j_k)$ with $1 \le j_\nu \le p$ and $k = \#J$,

$$\varphi^\alpha_{J,\kappa} = D_J\left(\varphi^\alpha_\kappa - \sum_{i=1}^{p} \xi^i_\kappa\, u^\alpha_i\right) + \sum_{i=1}^{p} \xi^i_\kappa\, u^\alpha_{J,i}, \tag{2.13}$$

with $D_J = D_{j_1}\cdots D_{j_k}$ denoting the corresponding iterated total derivative.

Let

$$\Xi(x,u) = \big(\,\xi^i_\kappa(x,u)\,\big) = \begin{pmatrix} \xi^1_1(x,u) & \ldots & \xi^p_1(x,u) \\ \vdots & \ddots & \vdots \\ \xi^1_p(x,u) & \ldots & \xi^p_p(x,u) \end{pmatrix} \tag{2.14}$$

be the $p\times p$ matrix formed by the independent variable coefficients, whose κ^{th} row contains the independent variable coefficients of the κ^{th} generator $\mathbf{v}_\kappa$. If

$$\det \Xi(x_0,u_0) \ne 0 \tag{2.15}$$

then the Implicit Function Theorem implies that the orbit $S = H\cdot z_0$ is a p-dimensional submanifold $N \subset M$ that is transverse to the vertical fibers at $z_0 = (x_0,u_0)$, and hence can be locally represented as the graph of a function $u = f(x)$. Under this assumption, the following result generalizes Proposition 2.1:

Theorem 2.2. *Under the above assumptions, at z_0 the jet coordinates of the group orbit $S = H\cdot z_0$ are provided by the recursive formula*

$$u^\alpha_{K,0} = \Psi^\alpha_K(x_0,u_0^{(k-1)}), \qquad 1 \le k = \#K \le n, \qquad \alpha = 1,\ldots,q, \tag{2.16}$$

where the functions $\Psi^\alpha_K(x,u^{(k-1)})$ are given by

$$\begin{pmatrix} \Psi^\alpha_{J1}(x,u^{(k-1)}) \\ \vdots \\ \Psi^\alpha_{Jp}(x,u^{(k-1)}) \end{pmatrix} = \Xi(x,u)^{-1} \begin{pmatrix} \varphi^\alpha_{J,1}(x,u^{(k-1)}) \\ \vdots \\ \varphi^\alpha_{J,p}(x,u^{(k-1)}) \end{pmatrix}, \qquad \begin{matrix} \#J = k-1, \\ \alpha = 1,\ldots,q. \end{matrix} \tag{2.17}$$

In particular, when $k = 1$, we use the quantities

$$\begin{pmatrix} \psi_1^\alpha(x,u) \\ \vdots \\ \psi_p^\alpha(x,u) \end{pmatrix} = \begin{pmatrix} \Psi_1^\alpha(x,u) \\ \vdots \\ \Psi_p^\alpha(x,u) \end{pmatrix} = \Xi(x,u)^{-1} \begin{pmatrix} \varphi_1^\alpha(x,u) \\ \vdots \\ \varphi_p^\alpha(x,u) \end{pmatrix}, \qquad \alpha = 1, \ldots, q, \tag{2.18}$$

to define the modified total derivatives

$$\widehat{D}_i = \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \psi_i^\alpha(x,u) \frac{\partial}{\partial u^\alpha}, \qquad i = 1, \ldots, p. \tag{2.19}$$

Let

$$\psi_K^\alpha(x,u) = \widehat{D}_K(u^\alpha), \qquad 0 \le \#K \le n, \qquad \alpha = 1, \ldots, q, \tag{2.20}$$

which is consistent with (2.18). *Then we have the alternative formula*

$$u_{K,0}^\alpha = \psi_K^\alpha(x_0, u_0), \tag{2.21}$$

which thus implies that

$$\psi_K^\alpha(x_0,u_0) = \Psi_K^\alpha(x_0, \psi^{(k-1)}(x_0,u_0)), \tag{2.22}$$

where the notation means that we use (2.21) *to replace all the derivatives $u_{J,0}^\alpha$ for $\#J \le k-1$ that appear on the right hand side of* (2.16). *Thus, the value of a differential function $F(x, u^{(n)})$ on the orbit at the base point is obtained by replacing each derivative coordinates by the corresponding function* (2.22)*:*

$$F(x_0, u_0^{(n)}) = F(x_0, u_0, \ \ldots \ , \psi_K^\alpha(x_0,u_0), \ \ldots \ ). \tag{2.23}$$

Theorem 2.2 is proved by a straightforward adaption of the two-dimensional argument, and so, for brevity, will be omitted. Examples will appear below.

Remark: Transversality of the orbit implies that we can choose a basis $\mathbf{v}_1, \ldots, \mathbf{v}_p$ of the infinitesimal generators of H with the property that

$$\mathbf{v}_\kappa|_{(x_0,u_0)} = \left. \frac{\partial}{\partial x^\kappa} \right|_{(x_0,u_0)} + \sum_{\alpha=1}^{q} \varphi_\kappa^\alpha(x_0,u_0) \left. \frac{\partial}{\partial u^\alpha} \right|_{(x_0,u_0)}, \qquad \kappa = 1, \ldots, p. \tag{2.24}$$

This choice of basis serves to trivialize the matrix $\Xi(x_0,u_0) = \mathrm{I}$ in (2.14), and hence allows us to inductively determine the orbit jet coordinates $u_0^{(n)}$ directly from the infinitesimal generator coefficients:

$$u_{Ji,0}^\alpha = \psi_{Ji}^\alpha(x_0,u_0) = \Psi_{Ji}^\alpha(x_0, u_0^{(k-1)}) = \varphi_{J,i}^\alpha(x_0, \psi^{(k-1)}(x_0,u_0)), \tag{2.25}$$

which is valid for any multi-index J with $\#J = k-1$ and any $i = 1, \ldots, p$.

Remark: It is easy to see that the modified total derivatives (2.19) mutually commute: $[\, \widehat{D}_i, \widehat{D}_j \,] = 0$ for all i, j. Indeed, their construction coincides with the initial step in the elementary proof of the Frobenius Theorem described in [**16**; p. 422].

Remark: As above, the identification (2.22) provides a new interpretation of the standard prolongation formula (2.13). Furthermore, observe that, on the left hand side of (2.25), $J\,i = (j_1, \ldots, j_{k-1}, i)$ denotes a *symmetric* multi-index, whereas on the right hand side $\varphi^\alpha_{J,i}$ refers to the coefficient of $\partial/\partial u^\alpha_J$ in the prolonged vector field $\mathbf{v}_i^{(n)}$, and thus is *not fully symmetric* in J, i. It is striking that, *provided the vector fields* $\mathbf{v}_1, \ldots, \mathbf{v}_r$ *span a Lie algebra*, the formula yields the same result for all permutations of the indices in $K = J\,i$.

3. Maximally Symmetric Submanifolds.

Let G be a Lie group that acts (locally) on an m-dimensional manifold M. Let us review some basic terminology.

Definition 3.1. The *isotropy subgroup* of a subset $S \subset M$ is

$$\widehat{G}_S = \{\, g \in G \mid \; g \cdot S \subset S \,\}. \tag{3.1}$$

The *global isotropy subgroup*

$$G^*_S = \bigcap_{z \in S} \widehat{G}_z = \{\, g \in G \mid \; g \cdot z = z \text{ for all } z \in S \,\} \subset \widehat{G}_S$$

consists of those group elements which fix *all* points in S.

Definition 3.2. The group G acts

- *freely* if $\widehat{G}_z = \{e\}$ for all $z \in M$,
- *locally freely* if $\widehat{G}_z$ is a discrete subgroup for all $z \in M$,
- *effectively* if $G^*_M = \{e\}$,
- *locally effectively* if G^*_M is a discrete subgroup,
- *effectively on subsets* if $G^*_U = \{e\}$ for every open $U \subset M$,
- *locally effectively on subsets* if G^*_U is a discrete subgroup for every open $U \subset M$.

If $S \subset M$ is a *closed* submanifold, then its *symmetry group* G_S, by definition, coincides with its isotropy subgroup: $G_S = \widehat{G}_S$. For non-closed submanifolds, there is a distinction between them as we allow local invariance of the submanifold under its symmetry group. For instance, if $G = \mathbb{R}^2$ acts by translations on $M = \mathbb{R}^2$, then any non-infinite open horizontal line segment, e.g., $S = \{\, (x, 0) \mid -1 < x < 1 \,\}$, has trivial isotropy subgroup, $\widehat{G}_S = \{e\}$, but is locally invariant under the one-parameter subgroup of horizontal translations generated by ∂_x, and we wish to encode this fact in the symmetry group. One approach is to first define the symmetry subalgebra to consist of all infinitesimal generators $\mathbf{v} \in \mathfrak{g}$ that are everywhere tangent to the submanifold:

$$\mathfrak{g}_S = \{\, \mathbf{v} \in \mathfrak{g} \mid \; \mathbf{v}|_z \in TS|_z \quad \text{for all} \quad z \in S \,\}.$$

Then $G_S \subset G$ will be the connected subgroup having subalgebra $\mathfrak{g}_S \subset \mathfrak{g}$. Of course, this fails to address the question of discrete symmetries of non-closed submanifolds. An alternative approach would be to recast the construction using the more general machinery of groupoids, [**25**], but, for simplicity, we will not pursue this direction any further here.

The following theorem is due to Ovsiannikov, [**23**], and was slightly corrected in [**17**].

Theorem 3.3. *If G acts locally effectively on subsets of M, then, for $n \gg 0$ sufficiently large, the prolonged action $G^{(n)}$ is locally free on a dense open subset $\mathcal{V}^n \subset \mathrm{J}^n$.*

Remark: Any analytic action can be made effective by dividing by the global isotropy normal subgroup G_M^*. Although all known examples of prolonged effective group actions are, in fact, free on an open subset of a sufficiently high order jet space, there is, frustratingly, as yet no general proof, nor known counterexample, to this more general result.

The open subset $\mathcal{V}^n \subset \mathrm{J}^n$ described in Theorem 3.3, which consists of all prolonged group orbits of dimension equal to $r = \dim G$, is called the *regular subset*, and its elements $z^{(n)} \in \mathcal{V}^n$ are the *regular jets*. The *singular subset* $\mathcal{S}^n = \mathrm{J}^n \setminus \mathcal{V}^n$ is the remainder, containing the *singular jets*.

Definition 3.4. A submanifold $S \subset M$ is *order n regular* if $\mathrm{j}_n S \subset \mathcal{V}^n$. A submanifold $S \subset M$ is *totally singular* if $\mathrm{j}_n S \subset \mathcal{S}^n$ for all $n = 0, 1, \ldots$.

In [**17**; Theorem 7.6], the following geometric characterization of totally singular submanifolds was established. Section 8 of [**17**] contains further Lie algebra-theoretic characterizations of totally singular submanifolds of homogeneous spaces.

Theorem 3.5. *A submanifold $S \subset M$ is totally singular if and only if its symmetry subgroup G_S does not act locally freely on S itself.*

A real-valued function[†] $I \colon \mathrm{J}^n \to \mathbb{R}$ is known as a *differential invariant* if it is unaffected by the prolonged group transformations, so $I(g^{(n)} \cdot z^{(n)}) = I(z^{(n)})$ for all $z^{(n)} \in \mathrm{J}^n$ and all $g \in G$ such that both $z^{(n)}$ and $g^{(n)} \cdot z^{(n)}$ lie in the domain of I. Any finite-dimensional group action admits an infinite number of functionally independent differential invariants of progressively higher and higher order. The *Basis Theorem*, [**16**; Theorem 5.49], states that they can all be generated by repeated invariant differentiation of a finite number of low order invariants.

Theorem 3.6. *Given a finite-dimensional Lie group G acting on p-dimensional submanifolds $S \subset M$, then, locally, there exist a finite collection of* generating differential invariants *$I_1, \ldots, I_\ell$, along with exactly p invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$, having the property that every differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives: $\mathcal{D}_J I_\kappa = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_k} I_\kappa$.*

When restricted to a given submanifold, the differential invariants will no longer be functionally independent. As shown by Cartan, [**3**] — see also [**5**; Theorem 14.7] — the dimension of the symmetry group of a regular submanifold $S \subset M$ is completely determined by the number of functionally independent restricted differential invariants.

Theorem 3.7. *Let $S \subset M$ be a regular p-dimensional submanifold. Then the number k of functionally independent differential invariants on S is equal to the codimension of its symmetry group: $k = p - \dim G_S$.*

[†] Throughout, functions, maps, etc., may only be defined on an open subset of their indicated domain.

Thus, the *maximally symmetric* regular p-dimensional submanifolds are those possessing a p-dimensional symmetry group. As an immediate corollary of the preceding Theorem, we deduce Cartan's characterization of maximally symmetric submanifolds, [**3**].

Theorem 3.8. *A closed, regular p-dimensional submanifold $S \subset M$ is maximally symmetric, with p-dimensional symmetry group $H = G_S \subset G$, if and only if all its differential invariants are constant if and only if $S \subset H \cdot z_0$ is an open submanifold of an H orbit.*

In Section 5, we will apply the results of Section 2 to compute the values of the constant differential invariants of maximally symmetric submanifolds. The resulting expressions will typically turn out to be quite complicated.

4. Moving Frames.

In order to make additional progress, we will appeal to the equivariant method of moving frames initiated in [**5**, **19**]. We restrict our attention to the case of finite-dimensional Lie group actions; recent extensions of the moving frame approach to infinite-dimensional pseudo-groups can be found in [**22**].

Definition 4.1. An n^{th} order *moving frame* is a smooth, G-equivariant map†

$$\rho^{(n)} : \mathrm{J}^n \longrightarrow G.$$

The group G acts on itself by left or right multiplication; thus a right moving frame satisfies

$$\rho^{(n)}(g^{(n)} \cdot z^{(n)}) = \rho^{(n)}(z^{(n)}) \cdot g^{-1}. \tag{4.1}$$

Local equivariance allows one to restrict this condition to group elements g near the identity. All classical moving frames, e.g., those appearing in [**3**, **7**, **8**, **9**, **11**], can be regarded as left equivariant maps, although the right equivariant versions are often easier to compute, [**5**]. If $\rho(z)$ is any left-equivariant moving frame then $\widetilde{\rho}(z) = \rho(z)^{-1}$ is right-equivariant and conversely.

The existence of a moving frame imposes certain constraints on the group action:

Theorem 4.2. *A moving frame exists in a neighborhood of a point $z^{(n)} \in \mathrm{J}^n$ if and only if G acts freely and regularly near $z^{(n)}$.*

Therefore, a (locally equivariant) moving frame exists in a neighborhood of any regular jet $z^{(n)} \in \mathcal{V}^n$. In practice, one constructs a moving frame by the process of *normalization*, relying on the choice of a local cross-section $K^n \subset \mathrm{J}^n$ to the prolonged group orbits. The corresponding value of the right moving frame at a jet $z^{(n)} \in \mathrm{J}^n$ is the unique group element $g = \rho^{(n)}(z^{(n)}) \in G$ that maps it to the cross-section:

$$\rho^{(n)}(z^{(n)}) \cdot z^{(n)} = g^{(n)} \cdot z^{(n)} \in K^n. \tag{4.2}$$

† As noted earlier, the notation allows $\rho^{(n)}$ to be only defined on an open subset of J^n.

The moving frame $\rho^{(n)}$ clearly depends on the choice of cross-section, which is usually designed so as to simplify the required computations as much as possible. In most situations, one selects a *coordinate cross-section* defined by setting a number of the coordinate functions to specified constant values:

$$z_\kappa = c_\kappa, \qquad \kappa = 1, \ldots, r, \tag{4.3}$$

where $z_1, \ldots, z_r$ are r coordinates selected from among the jet variables x^i, u^α_J, and the constants $c_1, \ldots, c_r$ chosen so that (4.3) defines a bona fide cross-section. Extending the constructions to non-coordinate cross-sections is straightforward, [**10**, **15**, **21**].

The moving frame engenders an invariantization process ι that maps each differential function $F\colon \mathrm{J}^n \to \mathbb{R}$ to a differential invariant $I = \iota(F)$, defined as the unique differential invariant that coincides with F on the cross-section. Thus, invariantization does not affect invariants, $\iota(I) = I$, and, moreover, defines a morphism projecting the algebra of differential functions onto the algebra of differential invariants. In particular, the *normalized differential invariants* induced by the moving frame are obtained by invariantization of the basic jet coordinates:

$$H^i = \iota(x^i), \qquad I^\alpha_J = \iota(u^\alpha_J). \tag{4.4}$$

These naturally split into two classes: Those coming from the coordinates used to define the cross-section (4.3) will be constant, and are known as the *phantom differential invariants*. The remainder, known as the *basic differential invariants*, form a complete system of functionally independent differential invariants. Once the normalized differential invariants are known, the invariantization process is implemented by simply replacing each jet coordinate by the corresponding normalized differential invariant (4.4), so that

$$\iota\big[\,F(x,u^{(n)})\,\big] = \iota\big[\,F(\;\ldots\;x^i\;\ldots\;u^\alpha_J\;\ldots\;)\,\big] = F(\;\ldots\;H^i\;\ldots\;I^\alpha_J\;\ldots\;) = F(H,I^{(n)}). \tag{4.5}$$

Since differential invariants are not affected by invariantization, this implies the powerful (albeit trivial) *Replacement Theorem*:

$$J(x,u^{(n)}) = J(H,I^{(n)}) \quad \text{whenever } J \text{ is a differential invariant.} \tag{4.6}$$

5. Computing Differential Invariants.

We now turn to the main task at hand — determining the values of the differential invariants of maximally symmetric submanifolds. Let's begin with the case of curves, so $p = 1$, lying in a manifold M of dimension $m = 1 + q$. For simplicity, let us assume G acts transitively on M, i.e., we are looking at curves in a (locally) homogeneous space. It is known†, that, except for a handful of group actions that pseudostabilize under prolongation, there are precisely $q = m - 1$ generating differential invariants with the property that all others are found by differentiation with respect to a group-invariant arc length element. For a maximally symmetric curve, the q generating differential invariants are constant, and so all the higher order differentiated invariants are automatically zero.

The first main result follows as a direct corollary of Proposition 2.1.

† This follows from the solution to [**16**; Exercise 5.35], which can be effected by using the moving frame recurrence formulae, [**5**]; see also [**20**; Theorem 7.2].

Theorem 5.1. *Let $C = \{ \exp(t\,\mathbf{v}) \cdot z_0 \}$ be the maximally symmetric curve through the point $z_0 \in M$ generated by $\mathbf{v} \in \mathfrak{g}$. Then we can evaluate the value of any (necessarily constant) differential invariant $I = F(x, u^{(n)})$ by replacing each derivative coordinate u^α_k by the corresponding function* (2.8)*:*

$$I = F(x_0, u_0, \;\ldots\; \psi^\alpha_k(x_0,u_0) \;\ldots\;). \tag{5.1}$$

Example 5.2. Consider the equi-affine group $\mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$ acting on plane curves $C \subset M = \mathbb{R}^2$ via

$$g \cdot (x,u) = (\alpha\, x + \beta\, u + a, \gamma\, x + \delta\, u + b), \qquad \alpha\,\delta - \beta\,\gamma = 1. \tag{5.2}$$

The fundamental differential invariant is the *equi-affine curvature*

$$\kappa = \iota(u_{xxxx}) = \frac{u_{xx}u_{xxxx} - \frac{5}{3}u_{xxx}^2}{u_{xx}^{8/3}}\,. \tag{5.3}$$

All other differential invariants are (locally) expressible as functions of the curvature and its derivatives with respect to the equi-affine arc length

$$\omega = ds = u_{xx}^{1/3}\, dx. \tag{5.4}$$

The totally singular curves are the straight lines, which admit a three-dimensional symmetry group. The maximally symmetric curves are the conic sections, [**2**, **17**], and our task is to determine their equi-affine curvature.

A basis for the infinitesimal generators of the action is provided by the vector fields

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_u, \qquad \mathbf{v}_3 = x\,\partial_u, \qquad \mathbf{v}_4 = -\,x\,\partial_x + u\,\partial_u, \qquad \mathbf{v}_5 = u\,\partial_x, \tag{5.5}$$

and so the general infinitesimal generator is

$$\mathbf{v} = \sum_{\kappa=1}^{5} a_\kappa \mathbf{v}_\kappa = (a_1 - a_4 x + a_5 u)\,\frac{\partial}{\partial x} + (a_2 + a_3 x + a_4 u)\,\frac{\partial}{\partial u}\,. \tag{5.6}$$

To compute the equi-affine curvature of the corresponding nondegenerate conic section $C = \{ \exp(t\,\mathbf{v}) \cdot (x,u) \}$, we first introduce the modified total derivative operator (2.6)

$$\widehat{D}_x = \frac{\partial}{\partial x} \;+\; \frac{a_2 + a_3 x + a_4 u}{a_1 - a_4 x + a_5 u}\,\frac{\partial}{\partial u}\,.$$

Thus, according to (2.7–8), the curve's jet coordinates are given by

$$\begin{aligned}
u_x &= \psi_1(x,u) = \widehat{D}_x u = \frac{a_2 + a_3 x + a_4 u}{a_1 - a_4 x + a_5 u}\,,\\
u_{xx} &= \psi_2(x,u) = \widehat{D}_x \psi_1(x,u)\\
&= (a_1 - a_4 x + a_5 u)^{-3}\big[\, a_1^2\, a_3 + 2\, a_1\, a_2\, a_4 - a_2^2\, a_5\\
&\qquad\qquad + (a_3\, a_5 + a_4^2)\big(-2\, a_2\, x + 2\, a_1\, u - a_3\, x^2 - 2\, a_4\, x\, u + a_5\, u^2\big)\big]\,,
\end{aligned}$$

and, in general,

$$u_k = \psi_k(x,u) = \widehat{D}_x \psi_{k-1}(x,u),$$

where the higher order expressions are too unwieldy to reproduce in print. Then, specializing (5.1), the equi-affine curvature (5.3) of the orbit $\left\{ \exp(t\mathbf{v}) \cdot (x,u) \right\}$ generated by (5.6) equals

$$\kappa = \frac{\psi_2(x,u)\psi_4(x,u) - \frac{5}{3}\,\psi_3(x,u)^2}{\psi_2(x,u)^{8/3}}\,, \tag{5.7}$$

which is a rather long explicit formula, but eminently computable.

One of the immediate lessons from such examples is that the expressions following from Theorem 5.1 tend to be rather complicated. A significant simplification can be effected by applying a preliminary group transformation. Observe that a group element $g \in G$ will map the maximally symmetric curve $C = \left\{ \exp(t\mathbf{v}) \cdot z_0 \right\}$ generated by $\mathbf{v} \in \mathfrak{g}$ and based at $z_0 \in M$ to another maximally symmetric curve $\widehat{C} = g \cdot C = \left\{ \exp(t\widehat{\mathbf{v}}) \cdot \widehat{z}_0 \right\}$ generated by[†] $\widehat{\mathbf{v}} = \mathrm{Ad}\, g(\mathbf{v})$, and based at the image point $\widehat{z}_0 = g \cdot z_0$. Moreover, C and $\widehat{C}$ have the same differential invariants. For example, if g represents a translation that maps (x_0,u_0) to 0, then its adjoint effect on other infinitesimal generators is to replace (x,u) by $(x-x_0, u-u_0)$.

An alternative approach is to use the moving frame cross-section to place the curve in a normal form. With this restriction, the differential invariants are found to have particularly simple expressions in terms of the corresponding "compatible" infinitesimal generators. To this end, let $K^n \subset \mathrm{J}^n|_{z_0}$ be a cross-section to the prolonged group orbits, which we assume to be entirely based at the point $z_0 = (x_0,u_0) \in M$, i.e., its first $m = p+q$ defining equations are given by $x^i = x^i_0$, $u^\alpha = u^\alpha_0$ for $i = 1,\ldots,p$, $\alpha = 1,\ldots,q$.

Definition 5.3. The maximally symmetric orbit curve $C = \left\{ \exp(t\mathbf{v}) \cdot z_0 \right\}$ through z_0 is *compatible* with the moving frame cross-section provided its jet at z_0 lies in the cross-section: $\mathrm{j}_n C|_{z_0} \in K^n$.

The next result is stated for coordinate cross-sections, with an evident modification in the non-coordinate version.

Proposition 5.4. *Let* $\mathbf{v} = \xi(x,u)\partial_x + \sum_\alpha \varphi^\alpha(x,u)\partial_{u^\alpha} \in \mathfrak{g}$. *Then the induced orbit curve* $C = \left\{ \exp(t\mathbf{v}) \cdot (x_0,u_0) \right\}$ *is compatible with the moving frame cross-section if and only if* $\xi(x_0,u_0) \neq 0$, *and, for each cross-section equation* $u^{\alpha_\kappa}_{k_\kappa} = c_\kappa$, *the corresponding function* (2.7) *satisfies*

$$\psi^{\alpha_\kappa}_{k_\kappa}(x_0,u_0) = c_\kappa. \tag{5.8}$$

Thus, applying Proposition 2.1, we deduce a simpler formula for the differential invariants of a compatible maximally symmetric curve.

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† Here, $\mathrm{Ad}\, g$ denotes the adjoint action of the group element g on the Lie algebra $\mathfrak{g}$.

Theorem 5.5. *If the maximally symmetric curve* $C = \{\exp(t\,\mathbf{v}) \cdot (x_0, u_0)\}$ *is compatible with the moving frame cross-section, then its basic differential invariants* $(H, I^{(n)}) = (x_0, \ldots, u_0^\alpha, \ldots, I_k^\alpha, \ldots)$*, as in (4.4), are given by*

$$I_k^\alpha = \psi_k^\alpha(x_0, u_0) = \frac{\varphi_{k-1}^\alpha(x_0, I^{(k-1)})}{\xi(x_0, u_0)}\,. \tag{5.9}$$

We remark that the second formula gives a simple recursive rule for generating the differential invariants directly from the prolonged infinitesimal generator coefficients. In particular, all the cross-section variables appearing in $I^{(k-1)}$ are equal to the constant values prescribed by the cross-section equations (4.3).

We emphasize that, by the cross-section construction of the moving frame, *any* maximally symmetric curve can be made compatible by applying the corresponding right moving frame element to it. Namely, given any regular orbit curve $C = \{\exp(t\,\mathbf{v}) \cdot z\}$, the transformed curve $\widehat{C} = \rho^{(n)}(\mathrm{j}_n C|_z) \cdot C$ will be compatible, and generated by $\widehat{\mathbf{v}} = \mathrm{Ad}\,\rho^{(n)}(\mathrm{j}_n C|_z)\,\mathbf{v}$. Thus, having a moving frame already in hand leads to a significant simplification of the formulae. While the algebraic manipulations required to compute a moving frame *ab initio* might offset any computational advantages offered by this approach, there are many other compelling reasons for finding the moving frame, [**19**], that could motivate its adoption.

Example 5.6. Let us return to the equi-affine group SA(2) acting on plane curves, as treated in Example 5.2. To define the classical equi-affine moving frame, [**9**, **14**], we select the coordinate cross-section

$$x = u = u_x = 0, \qquad u_{xx} = 1, \qquad u_{xxx} = 0. \tag{5.10}$$

The fourth order prolongation of the general infinitesimal generator (5.6) is

$$\begin{aligned}\mathbf{v}^{(4)} = {} & (a_1 - a_4 x + a_5 u)\,\frac{\partial}{\partial x} + (a_2 + a_3 x + a_4 u)\,\frac{\partial}{\partial u} + (a_3 + 2\,a_4 u_x - a_5 u_x^2)\,\frac{\partial}{\partial u_x} + \\ & + (3\,a_4 u_{xx} - 3\,a_5 u_x u_{xx})\,\frac{\partial}{\partial u_{xx}} + \big(\,4\,a_4 u_{xxx} - a_5(4\,u_x u_{xxx} + 3\,u_{xx}^2)\,\big)\,\frac{\partial}{\partial u_{xxx}} + \\ & + \big(\,5\,a_4 u_{xxxx} - a_5(5\,u_x u_{xxxx} + 10\,u_{xx} u_{xxx})\,\big)\,\frac{\partial}{\partial u_{xxxx}}\,.\end{aligned} \tag{5.11}$$

Thus, according to (5.9), at the base point $(x_0, u_0) = (0,0) = \mathbf{0}$, the relevant functions (2.8) are given by

$$\psi_1(0,0) = \frac{\varphi(0,0)}{\xi(0,0)} = \frac{a_2}{a_1}\,, \qquad \psi_2(0,0) = \frac{\varphi_1(0,0,\psi_1(0,0))}{\xi(0,0)} = \frac{a_1^2 a_3 + 2\,a_1 a_2 a_4 - a_2^2 a_5}{a_1^3}\,,$$

$$\psi_3(0,0) = \frac{\varphi_2(0,0,\psi_1(0,0),\psi_2(0,0))}{\xi(0,0)} = \frac{3(a_1 a_4 - a_2 a_5)(a_1^2 a_3 + 2\,a_1 a_2 a_4 - a_2^2 a_5)}{a_1^5}\,,$$

and so on. Thus, for the orbit generated by $\mathbf{v}$ through the base point to be compatible with the cross-section (5.10), we require[†]

$$0 = \psi_1(0,0) = \frac{\varphi(0,0)}{\xi(0,0)} = \frac{a_2}{a_1}\,, \qquad 1 = \psi_2(0,0) = \frac{\varphi_1(0,0,0)}{\xi(0,0)} = \frac{a_3}{a_1}\,,$$
$$0 = \psi_3(0,0) = \frac{\varphi_2(0,0,0,1)}{\xi(0,0)} = \frac{3\,a_4}{a_1}\,,$$

and so compatibility requires that

$$a_2 = 0, \qquad a_3 = a_1, \qquad a_4 = 0.$$

For such infinitesimal generators $\mathbf{v} \in \mathfrak{sa}(2)$, the equi-affine curvature of the conic section $C = \left\{ \exp(t\mathbf{v}) \cdot \mathbf{0} \right\}$ is given by

$$\kappa = \psi_4(0,0) = \frac{\varphi_3(0,0,0,1,0)}{\xi(0,0)} = -\,\frac{3\,a_5}{a_1}\,. \tag{5.12}$$

Incidentally, the higher order differential invariants $\kappa_s, \kappa_{ss}, \ldots$, are clearly all zero. This can be reconfirmed using the general formula (5.9) and the recurrence formulas relating the normalized and differentiated invariants [**5**]; for example,

$$0 = \kappa_s = \iota(u_{xxxxx}) = \psi_5(0,0) = \frac{\varphi_4(0,0,0,1,0,\kappa)}{\xi(0,0)}\,.$$

Example 5.7. Consider the projective group PSL(3) acting on curves $C \subset M = \mathbb{RP}^2$ via

$$(x,u) \longmapsto \left(\frac{\alpha x + \beta u + \gamma}{\rho x + \sigma u + \tau}, \frac{\lambda x + \mu u + \nu}{\rho x + \sigma u + \tau} \right), \qquad \det \begin{vmatrix} \alpha & \beta & \gamma \\ \lambda & \mu & \nu \\ \rho & \sigma & \tau \end{vmatrix} = 1. \tag{5.13}$$

The classical moving frame, [**3**], relies on the following cross-section equations:

$$x = u = u_x = 0, \qquad u_{xx} = 1, \qquad u_{xxx} = u_{4x} = 0, \qquad u_{5x} = 1, \qquad u_{6x} = 0. \tag{5.14}$$

The fundamental differential invariant is the projective curvature $\kappa = \iota(u_{7x})$, which is a rather complicated seventh order differential function, [**4, 16**].

The maximally symmetric curves, i.e., those with constant projective curvature, are the W curves studied by Lie and Klein, [**13**]. We can then use formula (5.1) to compute their projective curvatures. However, since the resulting formula is much too complicated to display, we will only compute the value for compatible nondegenerate W curves. Adopting the following basis

$$\begin{gathered} \mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_u, \qquad \mathbf{v}_3 = x\,\partial_x, \qquad \mathbf{v}_4 = u\,\partial_x, \qquad \mathbf{v}_5 = x\,\partial_u, \qquad \mathbf{v}_6 = u\,\partial_u, \\ \mathbf{v}_7 = x^2\,\partial_x + x\,u\,\partial_u, \qquad \mathbf{v}_8 = x\,u\partial_x + u^2\,\partial_u, \end{gathered} \tag{5.15}$$

[†] Note that these expressions can simply be computed directly from the infinitesimal generator formula (5.11), and do not require the more complicated expressions listed just above.

of $\mathfrak{sl}(3)$, and applying prolongation as before, we find that the general infinitesimal generator $\mathbf{v} = a_1 \mathbf{v}_1 + \cdots + a_8 \mathbf{v}_8$ generates a compatible W curve $C = \left\{ \exp(t\,\mathbf{v}) \cdot \mathbf{0} \right\}$passing through the origin if and only if

$$a_2 = a_3 = a_6 = 0, \qquad a_1 = a_5 = -6\,a_8, \qquad a_4 = -a_7.$$

The projective curvature of such a curve is equal to

$$\kappa = \frac{a_7}{2\,a_8}\,. \tag{5.16}$$

Example 5.8. Consider the Euclidean group $\mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$ acting by rigid motions on space curves $C \subset M = \mathbb{R}^3$. We use coordinates $z = (x, u, v)$ on M and, for simplicity, assume that the curve is realized as the graph of the functions $u = u(x)$, $v = v(x)$. The classical moving frame, [**9**], relies on the equations

$$x = 0, \qquad u = 0, \qquad v = 0, \qquad u_x = 0, \qquad v_x = 0, \qquad v_{xx} = 0, \tag{5.17}$$

which serve to define a coordinate cross-section provided $u_{xx} \neq 0$. (Indeed, the classical moving frame is not defined at inflection points of the space curve, [**6**, **9**].) The generating differential invariants are the curvature and torsion, which are obtained by invariantizing

$$\begin{aligned} \kappa = \iota(u_{xx}) &= \frac{\left|\,(1+v_x^2)u_{xx}^2 - 2\,u_x v_x u_{xx} v_{xx} + (1+u_x^2)v_{xx}^2\,\right|}{(1+u_x^2+v_x^2)^{3/2}}\,, \\ \tau = \frac{\iota(v_{xxx})}{\iota(u_{xx})} &= \frac{u_{xx}v_{xxx} - v_{xx}u_{xxx}}{1+u_x^2+v_x^2}\,, \end{aligned} \tag{5.18}$$

cf. [**18**]. All other differential invariants are (locally) expressible as functions of the curvature, torsion, and their derivatives with respect to the Euclidean arc length

$$\omega = ds = \sqrt{1+u_x^2+v_x^2}\; dx. \tag{5.19}$$

The totally singular curves are the straight lines, which have a two-dimensional Euclidean symmetry group. The maximally symmetric curves are the circles and circular helices.

Introducing the basis vector fields

$$\begin{aligned} &\mathbf{v}_1 = \partial_x, && \mathbf{v}_2 = \partial_u, && \mathbf{v}_3 = \partial_v, \\ &\mathbf{v}_4 = v\,\partial_u - u\,\partial_v, && \mathbf{v}_5 = -\,u\,\partial_x + x\,\partial_u, && \mathbf{v}_6 = -\,v\,\partial_x + x\,\partial_v, \end{aligned} \tag{5.20}$$

the general infinitesimal generator $\mathbf{v} = a_1 \mathbf{v}_1 + \cdots + a_6 \mathbf{v}_6$ has second order prolongation

$$\begin{aligned} \mathbf{v}^{(2)} = (a_1 - a_5 u - a_6 v)\,\frac{\partial}{\partial x} + (a_2 + a_4 v + a_5 x)\,\frac{\partial}{\partial u} + (a_3 - a_4 u + a_6 x)\,\frac{\partial}{\partial u} &+ \\ + \left(a_4 v_x + a_5 (1+u_x^2) + a_6 u_x v_x \right) \frac{\partial}{\partial u_x} + \left(-\,a_4 u_x + a_5 u_x v_x + a_6 (1+v_x^2) \right) \frac{\partial}{\partial v_x} &+ \\ + \left(a_4 v_{xx} + 3\,a_5 u_x u_{xx} + a_6 (2\,u_{xx} v_x + u_x v_{xx}) \right) \frac{\partial}{\partial u_{xx}} &+ \\ + \left(-\,a_4 u_{xx} + a_5 (u_{xx} v_x + 2\,u_x v_{xx}) + 3\,a_6 v_x v_{xx} \right) \frac{\partial}{\partial v_{xx}}&\,. \end{aligned}$$

Thus, taking into account (5.9), the circular or helical orbit $C = \{\exp(t\mathbf{v})\cdot\mathbf{0}\}$ will be compatible with the cross-section (5.10) provided

$$a_2 = 0, \qquad a_3 = 0, \qquad a_6 = 0.$$

Using formula (5.9) and replacing the lower order derivatives appearing in the infinitesimal generator coefficients by their cross-section values (5.17), (5.18), we find that its curvature and torsion are given by

$$\kappa = \frac{a_5}{a_1}, \qquad \tau = \frac{-\,a_4\,\kappa}{a_1\,\kappa} = -\,\frac{a_4}{a_1}.$$

In particular, the curve is a circle if and only if $a_4 = 0$; otherwise, it is a helix.

To compute the curvature and torsion of a general orbit, we set

$$\widehat{D}_x = \frac{\partial}{\partial x} + \frac{a_2 + a_4 v + a_5 x}{a_1 - a_5 u - a_6 v}\,\frac{\partial}{\partial u} + \frac{a_3 - a_4 u + a_6 x}{a_1 - a_5 u - a_6 v}\,\frac{\partial}{\partial v},$$

in accordance with (2.6). Thus, in view of (2.7–8), the orbit jet coordinates are

$$u_x = \psi^u_1(x,u,v) = \widehat{D}_x u = \frac{a_2 + a_4 v + a_5 x}{a_1 - a_5 u - a_6 v}, \quad v_x = \psi^v_1(x,u,v) = \widehat{D}_x v = \frac{a_3 - a_4 u + a_6 x}{a_1 - a_5 u - a_6 v},$$

and, in general,

$$u_k = \psi^u_k(x,u,v) = \widehat{D}_x \psi^u_{k-1}(x,u,v), \qquad u_k = \psi^u_k(x,u,v) = \widehat{D}_x \psi^u_{k-1}(x,u,v).$$

Then, specializing (5.9) to the expressions (5.18), the curvature and torsion of the maximally symmetric curve (helix or circle) $\exp(t\mathbf{v})(x,u,v)$ are equal to

$$\begin{aligned}
\kappa &= \frac{\left|\begin{array}{c}(1+\psi^v_1(x,u,v)^2)\psi^u_2(x,u,v)^2 - 2\,\psi^u_1(x,u,v)v_x\psi^u_2(x,u,v)\psi^v_2(x,u,v) + \\ +\,(1+\psi^u_1(x,u,v)^2)\psi^v_2(x,u,v)^2\end{array}\right|}{(1+\psi^u_1(x,u,v)^2+\psi^v_1(x,u,v)^2)^{3/2}}, \\
\tau &= \frac{\psi^u_2(x,u,v)\psi^v_3(x,u,v) - \psi^v_2(x,u,v)\psi^u_3(x,u,v)}{1+\psi^u_1(x,u,v)^2+\psi^v_1(x,u,v)^2}.
\end{aligned} \tag{5.21}$$

Finally, let us extend our method to maximally symmetric submanifolds of higher dimension $p \geq 2$. Let $z_0 \in M$, and let $\mathfrak{h} \subset \mathfrak{g}$ be a p-dimensional Lie subalgebra whose orbit $S = \exp(\mathfrak{h})\cdot z_0$ is a regular p-dimensional submanifold; this requires that $\dim \mathfrak{h}|_{z_0} = p = \dim \mathfrak{h}$. The orbit is *compatible* with the moving frame cross-section provided $\mathrm{j}_n S|_{z_0} \in K^n$.

Theorem 5.9. *Let* $\mathfrak{h} \subset \mathfrak{g}$ *be a* p*-dimensional Lie subalgebra. If the orbit* $S = \exp(\mathfrak{h})\cdot z_0$ *is compatible with the moving frame cross-section, then its constant differential invariants are prescribed by the values of the functions defined in* (2.22)*:*

$$I^\alpha_{Ji} = \psi^\alpha_{Ji}(x_0,u_0) = \phi^\alpha_{J,i}(x_0, I^{(k-1)}). \tag{5.22}$$

Again, the values of the cross-section variables appearing in (5.22) can be replaced by the corresponding constants, (4.3). The resulting formulas (5.22) can be used recursively to determine the constant values of the generating differential invariants. As before, the higher order differentiated invariants all vanish.

Example 5.10. Consider the Euclidean group $\mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$ acting on surfaces $S \subset M = \mathbb{R}^3$. We use coordinates $z = (x, y, u)$, and assume that the surface is given by the graph of a function $u = f(x, y)$. The classical moving frame, [**9**], relies on the equations

$$x = 0, \qquad y = 0, \qquad u = 0, \qquad u_x = 0, \qquad u_y = 0, \qquad u_{xy} = 0, \tag{5.23}$$

which serve to define a coordinate cross-section provided $u_{xx} \neq u_{yy}$, i.e., we are not at an umbilic point on the surface. The fundamental differential invariants are the principle curvatures $\kappa^1 = \iota(u_{xx})$, $\kappa^2 = \iota(u_{yy})$, or, equivalently the mean curvature $H = \frac{1}{2}(\kappa^1 + \kappa^2)$ and the Gauss curvature $K = \kappa^1 \kappa^2$. As is well known, the Gauss and mean curvature generate the algebra of Euclidean surface differential invariants via invariant differentiation with respect to the induced Frenet frame. Less well known is the recent observation, [**21**], that, for suitably non-degenerate surfaces, the differential invariant algebra can, in fact, be generated by the mean curvature alone via invariant differentiation.

The totally singular surfaces are the planes and spheres; each is totally umbilic and, moreover, has a non-freely acting three-dimensional Euclidean symmetry group. The maximally symmetric surfaces are the cylinders, with isotropy subgroup consisting of a translation along the cylinder's axis and a rotation around it.

Introducing the basis vector fields

$$\begin{aligned} &\mathbf{v}_1 = \partial_x, && \mathbf{v}_2 = \partial_y, && \mathbf{v}_3 = \partial_u, \\ &\mathbf{v}_4 = y\,\partial_x - x\,\partial_y, && \mathbf{v}_5 = -u\,\partial_x + x\,\partial_u, && \mathbf{v}_6 = -u\,\partial_y + y\,\partial_u, \end{aligned} \tag{5.24}$$

the general infinitesimal generator $\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_6\mathbf{v}_6 \in \mathfrak{se}(3)$ has second order prolongation

$$\begin{aligned} \mathbf{v}_a^{(2)} = {} & (a_1 + a_4 y - a_5 u)\,\frac{\partial}{\partial x} + (a_2 - a_4 x - a_6 u)\,\frac{\partial}{\partial y} + (a_3 + a_5 x + a_6 y)\,\frac{\partial}{\partial u} + \\ & + \big(\,a_4 u_y + a_5(1 + u_x^2) + a_6 u_x u_y\,\big)\,\frac{\partial}{\partial u_x} + \big(-a_4 u_x + a_5 u_x u_y + a_6(1 + u_y^2)\big)\,\frac{\partial}{\partial u_y} + \\ & + \big(\,2a_4 u_{xy} + 3a_5 u_x u_{xx} + a_6(u_y u_{xx} + 2u_x u_{xy})\big)\,\frac{\partial}{\partial u_{xx}} + \\ & + \big(\,a_4(u_{yy} - u_{xx}) + a_5(u_y u_{xx} + 2u_x u_{xy}) + a_6(2u_y u_{xy} + u_x u_{yy})\big)\,\frac{\partial}{\partial u_{xy}} + \\ & + \big(-2a_4 u_{xy} + a_5(2u_y u_{xy} + u_x u_{yy}) + 3a_6 u_y u_{yy}\,\big)\,\frac{\partial}{\partial u_{yy}}\;. \end{aligned} \tag{5.25}$$

Let $\mathfrak{h} \subset \mathfrak{se}(3)$ be a two-dimensional subalgebra with basis $\mathbf{v}_a, \mathbf{v}_b$ for $a, b \in \mathbb{R}^6$. This requires that $[\,\mathbf{v}_a, \mathbf{v}_b\,] = c_a\mathbf{v}_a + c_b\mathbf{v}_b$ for some $c_a, c_b \in \mathbb{R}$, which imposes certain quadratic constraints on the coefficients a, b. Indeed, by the classification of subalgebras of $\mathfrak{se}(3)$, [**1**], we can, in fact, assume that $\mathbf{v}_a = a_1\partial_x + a_2\partial_y + a_3\partial_u$ generates a one-parameter translation subgroup in a direction $\widehat{a} = (a_1, a_2, a_3) \neq 0$, while either $\mathfrak{h}$ is abelian, and so $\mathbf{v}_b$ generates a translation subgroup in a second, independent direction, or $\mathfrak{h}$ is non-abelian, and $\mathbf{v}_b$ generates a one-parameter subgroup consisting of rotations around a line $\{\, z_0 + t\,\widehat{a} \mid t \in \mathbb{R}\,\}$ parallel to the translation direction $\widehat{a}$. The abelian case leads to a totally singular orbit —

a plane — and so we assume from here on that $\mathfrak{h} \subset \mathfrak{se}(3)$ is a two-dimensional non-abelian subalgebra.

With this proviso, we consider the orbit $S = \exp(\mathfrak{h}) \cdot \mathbf{0}$ through the origin. (We can employ translations to place any other orbit there, invoking the adjoint action discussed above to adapt the final formulas.) As in (2.24), we can take our basis $\mathbf{v}_a, \mathbf{v}_b$ such that $a_1 = b_2 = 1$ while $a_2 = b_1 = 0$. In view of (2.25), (5.25), we find that the orbit will be compatible with the cross-section (5.23) provided

$$a_1 = b_2 = 1, \qquad a_2 = a_3 = a_4 = a_6 = b_1 = b_3 = b_4 = b_5 = 0.$$

Then formula (5.22) implies that the principle curvatures of the compatible cylindrical orbit are given, respectively, by the coefficients of ∂_{u_x} in $\mathbf{v}_a$ and of ∂_{u_y} in $\mathbf{v}_b$, namely,

$$\kappa^1 = \varphi_{x,a}(0,0,0,0,0) = a_5, \qquad \kappa^2 = \varphi_{y,b}(0,0,0,0,0) = b_6.$$

One of these is necessarily zero (this follows from the Lie algebra condition), while the other is the reciprocal of the radius of the cylindrical cross-section. For more general orbits, one can either employ the adjoint action induced by the moving frame to make them compatible, or resort to substituting the induced functions (2.22) into the formulas for the principal curvatures.

Example 5.11. Consider the equi-affine group $\mathrm{SA}(3) = \mathrm{SL}(3) \ltimes \mathbb{R}^3$ acting on surfaces $S \subset M = \mathbb{R}^3$. As in the preceding example, we use coordinates $z = (x, y, u)$, and assume that the surface is given by the graph of a function $u = f(x, y)$. There are two non-degenerate cases, depending on the sign of the Hessian determinant $H = u_{xx}u_{yy} - u_{xy}^2$. We concentrate on the hyperbolic case, where $H < 0$, here; the elliptic case $H > 0$ follows from a simple change of some signs, while parabolic points, with $H = 0$ are degenerate, and require a higher order moving frame. For a hyperbolic surface, the classical moving frame, [**9**, **21**], relies on the (non-coordinate) cross-section K^3 defined by the equations

$$\begin{gathered} x = y = u = 0, \qquad u_x = 0, \qquad u_y = 0, \qquad u_{xy} = 0, \qquad u_{xx} = 1, \qquad u_{yy} = -1, \\ u_{xxy} = 0, \qquad u_{yyy} = 0, \qquad u_{xxx} = u_{xyy}. \end{gathered} \tag{5.26}$$

There is a single independent third order differential invariant

$$P = \iota(u_{xxx}) = \iota(u_{xyy}), \tag{5.27}$$

whose square, P^2, is traditionally known as the *Pick invariant*, [**24**]. In [**21**], it was proved that, for suitably non-degenerate surfaces, the algebra of differential invariants can be generated by invariant differentiation of the Pick invariant alone.

Omitting the details of the computation, which follow the same lines as in the preceding example, we introduce the following basis for the infinitesimal generators in $\mathfrak{sa}(3)$:

$$\begin{gathered} \mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_y, \qquad \mathbf{v}_3 = \partial_u. \qquad \mathbf{v}_4 = x\,\partial_x - u\,\partial_u, \qquad \mathbf{v}_5 = y\,\partial_y - u\,\partial_u, \\ \mathbf{v}_6 = y\,\partial_x, \qquad \mathbf{v}_7 = u\,\partial_x, \qquad \mathbf{v}_8 = x\,\partial_y, \qquad \mathbf{v}_9 = u\,\partial_y, \qquad \mathbf{v}_{10} = x\,\partial_u, \qquad \mathbf{v}_{11} = y\,\partial_u. \end{gathered}$$

When restricted to the cross-section (5.26), the prolongation of the general infinitesimal generator $\mathbf{v}_a = a_1 \mathbf{v}_1 + \cdots + a_{11}\mathbf{v}_{11}$ to second order is given by

$$\begin{aligned} \mathbf{v}^{(2)}|_{K^3} = a_1 \partial_x + a_2 \partial_y + a_3 \partial_u + a_{10} \partial_{u_x} + a_{11} \partial_{u_y} - \\ - (3a_4 + a_5)\partial_{u_{xx}} + (a_8 - a_6)\partial_{u_{xy}} + (a_4 + 3a_5)\partial_{u_{yy}}. \end{aligned} \tag{5.28}$$

We assume that two such generators $\mathbf{v}_a, \mathbf{v}_b \in \mathfrak{sa}(3)$ span a two-dimensional subalgebra $\mathfrak{h} \subset \mathfrak{sa}(3)$. We further assume, as in (2.24), without loss of generality, that $a_1 = b_2 = 1$, $a_2 = b_1 = 0$. With this fixed, we can use (2.25) and (5.28) to find the compatibility conditions and determine the value of the Pick invariant P. The compatibility equations are listed in the same order as the cross-section equations (5.26–27) (omitting the first set $x = y = u = 0$):

$$\begin{gathered} a_3 = b_3 = a_{11} = b_{10} = 0, \qquad a_{10} = 1, \qquad b_{11} = -1, \\ -3b_4 - b_5 = a_8 - a_6 = 0, \qquad\quad b_4 + 3b_5 = 0, \\ P = -3a_4 - a_5 = b_8 - b_6 = a_4 + 3a_5. \end{gathered} \tag{5.29}$$

Note that, in some cases, there are multiple expressions for the derivatives, which is the result of the non-symmetry of the indices on the φ's in (5.22). It is worth emphasizing that the requirement that $\mathbf{v}_a, \mathbf{v}_b$ span a Lie subalgebra ensures that the various expressions agree.

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