Using Moving Frames to Construct Equivariant Maps

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Abstract. The equivariant method of moving frames is applied to formulate a systematic method for explicitly determining general equivariant maps.

1. Introduction.

The equivariant method of moving frames introduced in [**10**] — see [**20**] for a recent survey — provides a systematic method for explicitly constructing generating sets of invariants and differential invariants of smooth Lie group actions on manifolds. In this approach, a *moving frame* is defined as an equivariant map from the manifold back to the group. The equivariant method was extended to Lie pseudo-group actions in [**22**], and, more recently, to finite and discrete group actions in [**21**]. In the latter case, the generating invariants are piecewise analytic, and, in a sense, "simpler" — meaning in number, in their construction and/or in their explicit formulas — than the traditional polynomial invariants, [**4**, **16**, **17**], and the less traditional, but better behaved rational invariants, [**2**, **8**, **24**, **26**].

In this paper, we apply the equivariant method of moving frames to formulate a similarly systematic method for explicitly determining general equivariant maps. Our method applies to general Lie group actions and, along similar lines as in [**21**], to finite and discrete groups, although we will not explicitly develop the latter here. (See [**7**] for an alternative approach to the case of finite groups.) The moving frame method presented here is both simpler and more direct than those appearing in [**1**, **12**, **29**]; moreover it applies to essentially any (suitably prolonged) Lie group action, while the latter are restricted to certain classical linear group actions. In particular, the moving frame formulas produce

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a nonlinear generalization of what has been called the Malgrange formula for equivariant linear maps, [**1**, **25**].

As in the original equivariant moving frame theory, the algorithms in this paper are formulated in a differential geometric framework, relying on the Implicit Function Theorem, which, of course, may well be non-constructive. On the other hand, it is a remarkable fact that the required maps and invariants can be explicitly constructed in the vast majority of the examples of mathematical and practical importance. Nonetheless, it would be of great interest to reformulate the constructions here in a fully algebraic manner, based on the methods developed by Hubert and Kogan, [**8**, **9**].

This paper was motivated by the emerging importance of equivariant maps in machine learning, [**1**, **11**, **15**, **28**]. It is expected that these constructions will have significant impact in this active field of modern research. See also Gatermann, [**12**], for potential applications to dynamical systems. Further recent papers on equivariant maps in algebra and geometry include [**7**, **6**, **14**].

2. Equivariant Moving Frames.

We begin by describing the equivariant moving frame construction for Lie group actions, referring to [**10**, **18**, **20**] for details and proofs. Let G be an r-dimensional Lie group acting smoothly on an m-dimensional manifold M.

Definition 1. A *moving frame* is a smooth, G-equivariant map $\rho : M \to G$.

There are two principal types of equivariance:

$$\rho(g \cdot z) = \begin{cases} g \cdot \rho(z), & \text{left moving frame,} \\ \rho(z) \cdot g^{-1}, & \text{right moving frame.} \end{cases} \tag{1}$$

If $\rho(z)$ is any right-equivariant moving frame then $\widetilde{\rho}(z) = \rho(z)^{-1}$ is left-equivariant and conversely. All classical moving frames, cf. [**5**], are left-equivariant, but the right versions are often easier to compute, and will be the ones primarily used here.

It is not difficult to establish the basic requirements for the existence of an equivariant moving frame, [**10**].

Theorem 2. *A moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.*

Recall that G acts *freely* if the isotropy subgroup $G_z = \{ g \in G \mid g \cdot z = z \}$ of each point $z \in M$ is trivial: $G_z = \{e\}$. Slightly less restrictively, the group acts *locally freely* if the isotropy subgroups G_z are all discrete, or, equivalently, that the orbits all have the same dimension, r, as G itself. In particular, if $\dim G > \dim M$, the action cannot be (locally) free. Locally free actions lead to locally equivariant moving frames; in fact most classical moving frames, [**5**], are only locally equivariant. *Regularity* requires that, in addition, the orbits form a regular foliation; the latter is a global restriction that is satisfied in all examples of practical import. There are two principal intrinsic means of converting a non-free action into a (locally) free action. Prolonging to a jet space, [**10**] — the method used to produce all classical moving frames — leads to differential invariants;

extending the action to a Cartesian product space, [**18**], leads to joint invariants, and is more relevant to machine learning applications; see the examples below.

The explicit construction of a moving frame relies on the choice of a (local) *cross-section* to the group orbits, meaning an $(m-r)$-dimensional submanifold $K \subset M$ that intersects each orbit transversally and at most once. Such cross-sections are plentiful, and, when dealing with Lie group actions, easy to find.

Theorem 3. *Let G act freely and regularly on M, and let $K \subset M$ be a cross-section. Given $z \in M$, let $g = \rho(z)$ be the unique group element that maps z to the cross-section: $g \cdot z = \rho(z) \cdot z \in K$. Then $\rho\colon M \to G$ is a right moving frame.*

For simplicity, we will always assume that K is a *coordinate cross-section*, obtained by setting r of the coordinates on M to constants. By possibly relabelling, we can assume that, writing

$$z = (z_1, \ldots, z_m) = (x_1, \ldots, x_{m-r}, y_1, \ldots, y_r) = (x, y),$$

we have

$$K = \{ y_1 = c_1, \ \ldots \ y_r = c_r \}, \tag{2}$$

where, as always, $r = \dim G$. In this case, the associated right moving frame $g = \rho(z)$ is obtained by solving the *normalization equations*

$$Y_1(g, z) = c_1, \quad \ldots \quad Y_r(g, z) = c_r, \tag{3}$$

for the group parameters $g = (g_1, \ldots, g_r)$ in terms of the coordinates $z = (z_1, \ldots, z_m)$, where $Y_j(g, z)$ denote the final r components of the group transformation formulas

$$g \cdot z = Z(g, z) = (X(g, z), Y(g, z)) = \big(X_1(g, z), \ldots, X_{m-r}(g, z), Y_1(g, z), \ldots, Y_r(g, z) \big).$$

Transversality combined with the Implicit Function Theorem implies the existence of a local solution $g = \rho(z)$ to the algebraic normalization equations (3).

By definition, if ρ is such a right moving frame, $\rho(z) \cdot z \in K$, and hence, separating into x and y constituents,

$$\rho(z) \cdot z = \big(I(z), c\big) = \big(I_1(z), \ldots, I_{m-r}(z), c_1, \ldots, c_r\big), \tag{4}$$

where $c_1, \ldots, c_r$ are the normalization constants that defin=e the cross-section (2), while $I_1(z), \ldots, I_{m-r}(z)$ form a complete system of functionally independent *invariants* for the action. Indeed, if $J(z)$ is any invariant, then it can be explicitly expressed in terms of the fundamental invariants by the *Replacement Rule*

$$J(z_1, \ldots, z_m) = J(I_1(z), \ldots, I_{m-r}(z), c_1, \ldots, c_r), \tag{5}$$

obtained by replacing each z_i appearing in the formula for J by corresponding fundamental invariant or normalization constant. (The latter are known as the *phantom invariants*.) The proof of formula (5) relies on the fact that any invariant is constant on the orbits of G and hence uniquely determined by its values on the cross-section.

3. Equivariant Maps.

Now suppose that our Lie group G also acts on an n-dimensional manifold N.

Definition 4. A function $F: M \to N$ is called G-*equivariant* if

$$F(g \cdot z) = g \cdot F(z) \qquad \text{for all} \qquad g \in G, \quad z \in M. \tag{6}$$

Note that a moving frame is a particular example of an equivariant map when $N = G$ and G acts on itself by right or left multiplication. The goal of this section is to use moving frames to construct a formula for a general equivariant map.

Consider the Cartesian product action $(z, w) \longmapsto (g \cdot z, g \cdot w)$ of G on $M \times N$. Observe that if the action of G is on M (locally) free, then the Cartesian product action is also (locally) free. We will concentrate on this case, noting that non-free actions can be dealt with by similar constructions, perhaps involving partial moving frames, [**19**].

The first result is immediate.

Lemma 5. *A function $F: M \to N$ is G-equivariant if and only if its graph $\Gamma_F = \{ (z, F(z)) \mid z \in M \}$ is a G-invariant m-dimensional submanifold of $M \times N$.*

Given a cross-section $K \subset M$ to the action of G on M, let us use the Cartesian product cross-section $K \times N \subset M \times N$ for the product action. By transversality, the intersection

$$\Sigma_F = \Gamma_F \cap (K \times N) \tag{7}$$

is an $(m - r)$-dimensional submanifold of the Cartesian product cross-section. Moreover, since Γ_F is G-invariant, we can reconstruct the graph

$$\Gamma_F = G \cdot \Sigma_F = \{ g \cdot (z, w) \mid \ (z, w) \in \Sigma_F \} \tag{8}$$

as the union of all the G orbits passing through Σ_F. Conversely, given an $(m - r)$-dimensional submanifold $\Sigma_F \subset K \times N$ that is transverse to the vertical fibers, then (8) allows us to construct a corresponding m-dimensional G-invariant submanifold that locally agrees with the graph of a G equivariant function $F: M \to N$.

The preceding cross-section induces a right moving frame $\widehat{\rho}: M \times N \to G$ such that $\widehat{\rho}(z, w) = \rho(z)$, where $\rho: M \to G$ is the right moving frame on M corresponding to K. Note that, in view of (4),

$$\rho(z) \cdot (z, w) = \big(I_1(z), \ldots, I_{m-r}(z), c_1, \ldots, c_r, J_1(z, w), \ldots, J_n(z, w) \big) = \big(I(z), c, J(z, w) \big), \tag{9}$$

where $I = (I_1, \ldots, I_{m-r})$ are the fundamental invariants on M that were derived above, while $J = (J_1, \ldots, J_n)$ are further invariants on $M \times N$, and such that the combined list forms a complete system of functionally independent invariants on $M \times N$. In particular, each J_ν must depend explicitly on w as otherwise, in view of the Replacement Rule (5), it would be function of the I_j's and hence not independent.

Now let us explicitly reconstruct the graph of an equivariant function from its intersection with the cross-section (7). The latter can be written as

$$\Sigma_F = \{ (x, c, w) \mid \ w = H(x) \} \tag{10}$$

for some vector-valued function $H(x_1, \ldots, x_{m-r})$. Thus, $(z, w) \in \Gamma_F$ if and only if

$$\rho(z) \cdot (z, w) = \big(I(z), c, J(z, w) \big) \in \Sigma_F. \tag{11}$$

Combining the last two equations, we conclude that the equivariant function F with graph $\Gamma_F = \{ w = F(z) \}$ is defined implicitly by

$$J(z, w) = H\big(I(z)\big). \tag{12}$$

We can then solve (12) for

$$w = F\big(z, H\big(I(z)\big) \big). \tag{13}$$

Since $H(x)$ is arbitrary, this provides an explicit formula for a general equivariant function $F\colon M \to N$. As we will see, (13) reduces to the Malgrange formula for equivariant linear maps, cf. [**1**, **25**], when G is a classical group acting linearly on a Cartesian product space.

In fact, the equivariant function F can be explicitly determined by using the corresponding left equivariant moving frame $\widetilde{\rho}\colon M \to G$, which can be simply obtained from the right equivariant moving frame by composing with the group inversion map: $\widetilde{\rho}(z) = \rho(z)^{-1}$. Indeed, given $(z_0, w_0) = (x_0, y_0, w_0) \in M \times N$, let us write out the coordinate formulas for the action of the corresponding left moving frame $\widetilde{\rho}(z) = \rho(z)^{-1}$ explicitly as

$$\rho(z)^{-1} \cdot (z_0, w_0) = \big(\varphi(z; z_0), \psi(z; z_0, w_0) \big).$$

Note that these formulas can be obtained by inverting the group transformations and then substituting for the group parameters $g \in G$ using the moving frame $g = \rho(z)$. In particular, if $(z_0, w_0) = \rho(z) \cdot (z, w) = \big(I(z), c, J(z, w) \big) \in K$, then

$$\rho(z)^{-1} \cdot \big(I(z), c, J(z, w) \big) = \big(\varphi(z; I(z), c), \psi(z; I(z), c, J(z, w)) \big) = (z, w). \tag{14}$$

This implies that we can write (13) in the form

$$w = \psi\big(z; I(z), c, H\big(I(z)\big) \big) = F\big(z, H\big(I(z)\big) \big). \tag{15}$$

Observe that ψ is a universal function of its arguments, uniquely prescribed by the group action on $M \times N$, while H is an arbitrary function of the fundamental invariants on M. However, in simple contexts, it is easier to just directly solve (12) for w.

Example 6. We start with an almost trivial example. Consider the action of the n-dimensional abelian group $G = \mathbb{R}^n$ on $N = \mathbb{R}^n$ by translation: $w \mapsto w + a$ for $a \in G$. Let M be the k-fold Cartesian product space $M = N \times \cdots \times N = \mathbb{R}^{kn}$, subject to the diagonal action, consisting of simultaneous translations:

$$x_i \longmapsto x_i + a \qquad \text{for} \qquad i = 1, \ldots, k.$$

To construct equivariant maps $F\colon M \to N$, so $w = F(x_1, \ldots, x_k)$, by the preceding moving frame construction, we use the cross-section $K = \{ x_k = 0 \} \subset M \times N$. Solving the normalization equation $x_k + a = 0$ produces the right moving frame

$$\rho(z) = \rho(x_1, \ldots, x_k) = -x_k.$$

Thus,

$$\rho(z)\cdot(z,w)=\rho(x_1,\ldots,x_k)\cdot(x_1,\ldots,x_k,w)=(x_1-x_k,\ldots,x_{k-1}-x_k,0,w-x_k),$$

whose non-constant entries are the fundamental translation invariants.

A transverse submanifold of dimension $kn-r=(k-1)n$ of the cross-section can be written in the form

$$w=H(x_1,\ldots,x_{k-1}).$$

Thus, using formula (12), the corresponding equivariant map $w=F(x_1,\ldots,x_k)$ is given implicitly by

$$w-x_k=H(x_1-x_k,\ldots,x_{k-1}-x_k)$$

and hence, explicitly,

$$w=F(x_1,\ldots,x_k)=x_k+H(x_1-x_k,\ldots,x_{k-1}-x_k), \tag{16}$$

where H is an arbitrary function of the invariants.

Example 7. Using the same notation as in Example 6, we consider the diagonal action of the orthogonal group $G=\mathrm{O}(n)$ on $M\times N$:

$$x_i \longmapsto Qx_i, \qquad w \longmapsto Qw, \qquad x_1,\ldots,x_k,w\in N=\mathbb{R}^n, \qquad Q\in\mathrm{O}(n).$$

Freeness of the diagonal action requires that $k\geq n$, and that the vectors $x_1,\ldots,x_k$ span $\mathbb{R}^n$. To compute a moving frame, we work in the dense open subset $M_0\subset M$ where the first n vectors $x_1,\ldots,x_n$ form a basis for $\mathbb{R}^n$. Note that if $x_1,\ldots,x_k$ lie in the open dense subset of the Cartesian product space where G acts freely, we can relabel, i.e., permute, the vectors to ensure that we are in M_0. Assembling the vectors into the corresponding $n\times k$ *data matrix*[†] $X=(x_1,\ldots,x_k)=(X_0,X_1)$, where X_0 is square, of size $n\times n$, and, by our assumption, invertible, while X_1 has size $n\times(k-n)$.

A convenient cross-section is obtained through use of the QR factorization of X_0, cf. [**13, 23**]. Namely, we choose as cross-section the subset $K\subset M$ containing all vectors $r_1,\ldots,r_n\in\mathbb{R}^n$ such that $r_{ii}>0$ and $r_{ji}=0$ when $i<j\leq n$. In other words, the corresponding $n\times n$ matrix $R_0=(r_1,\ldots,r_n)$ is upper triangular with positive entries on the main diagonal. The normalization equations take the matrix form $QX_0=R_0$, or, equivalently, $X_0=Q^TR_0$, which is thus its QR factorization. The right moving frame is given by

$$\rho(x_1,\ldots,x_k)=\rho(X)=Q,$$

and the nonzero entries of

$$R=(R_0,R_1)=\rho(X)\cdot X$$

form a complete system of functionally independent invariants of the action of $\mathrm{O}(n)$ on M, while the zero entries lying below the diagonal in R_0 are the phantom invariants.

[†] In applications, the transpose of X is more commonly referred to as the "data matrix", cf. [**23**].

Remark: The classical dot product orthogonal invariants $I_{ij} = x_i \cdot x_j$ generate the algebra of orthogonal invariants, [**29**], but are not functionally independent. Moreover, the Replacement Rule (5) allows us to easily rewrite any invariant function $J = J(x_1, \ldots, x_k) = J(X)$ in terms of the fundamental invariants, namely, $J = J(r_1, \ldots, r_k) = J(R)$. In particular, $I_{ij} = r_i \cdot r_j$ are the entries of the positive definite Gram matrix $X^T X = R^T R$. Thus, the entries of R are functions of the entries of the Gram matrix, i.e., the dot product invariants. Using this and the Replacement Rule allows us to conclude that any invariant can be written as a (non-unique) function of the dot product invariants I_{ij}.

To construct equivariant functions, we use the moving frame formula to construct the additional invariants on $M \times N$, which are the components of

$$v = \rho(X)\, w = Q\, w,$$

or, equivalently, the columns of the extended factorization

$$Q\,\widehat{X} = \widehat{R} = (R, v), \qquad \text{where} \qquad \widehat{X} = (X, w) = (Q^T R, Q^T v).$$

Consider a submanifold of the cross-section of the form

$$v = H(R), \qquad \text{so that} \qquad H(R) = \big(\, H_1(R), \ldots, H_n(R) \,\big)^T$$

is a vector-valued invariant function, i.e., its entries are functions of the nonzero entries of R. This implies that

$$w = Q^T H(R) = X\, R_0^{-1}\, H(R) = \sum_{i=1}^{n} h_i(R)\, x_i, \tag{17}$$

depending on the invariant functions

$$h(R) = \big(\, h_1(R), \ldots, h_k(R) \,\big) = R_0^{-1}\, H(R). \tag{18}$$

Malgrange's representation of $\mathrm{O}(n)$ equivariant functions, cf. [**25**], has the form (17), but with the sum extending from 1 to k. However, we can write $x_{n+1}, \ldots, x_k$ as orthogonally invariant linear combinations of the basis vectors $x_1, \ldots, x_n$, and hence Malgrange's formula can be easily converted into the reduced moving frame formula (17).

Example 8. Combining Examples 6 and 7, let us consider the diagonal action of the Euclidean group $G = \mathrm{E}(n) = \mathrm{O}(n) \ltimes \mathbb{R}^n$ on $M \times N$, where $N = \mathbb{R}^n$ and $M = \mathbb{R}^{(k+1)\,n}$ is the $(k+1)$-fold Cartesian product of N. We assume $k \geq n$ and the first $n+1$ points† $x_0, \ldots, x_n$ do not lie on an affine hyperplane or, equivalently, the volume of the simplex of which they are vertices is nonzero.

In this case, we can construct a cross-section by first translating the points to positions $y_i = x_i - x_k$. Let $Y = (Y_0, Y_1) = (y_1, \ldots, y_k)$ be the corresponding translated data matrix, omitting the first column $y_0 = 0$, and where Y_0 is square and nonsingular. We then use the orthogonal transformation so that $Q\,Y_0 = R_0$ is upper triangular, as before, and the

† Here we label the points starting at 0.

nonzero entries of $R = (R_0, R_1) = Q\,Y$ are the fundamental invariants. An alternative is to translate the points so that they have zero mean, so $y_i = x_i - \overline{x}$, where $\overline{x} = (x_1 + \cdots + x_k)/k$, and then perform the $Q\,R$ factorization of the resulting normalized data matrix Y, with the nonzero entries of R providing a slightly different set of fundamental invariants. Let us focus on the latter version, which is more computationally stable, in the rest of the construction.

Applying the moving frame to $w \in N$ produces the additional invariants

$$v = \rho(z) \cdot w = Q\,(w - \overline{x}).$$

The submanifold $v = H(R)$ of the cross-section produces the Euclidean analog of the orthogonal Malgrange formula (17), namely that every Euclidean equivariant map has the form

$$w = Q^T H(R) + \overline{x} = Y_0\, R_0^{-1}\, H(R) + \overline{x} = \overline{x} + \sum_{i=1}^{n} h_i(R)\,(x_i - \overline{x}), \tag{19}$$

where, as in (18), the $h_i(R)$ are functions of the fundamental invariants. Alternatively, one can write the h_i as functions of the interpoint distances $\|\, x_i - x_j \,\|$, keeping in mind that the latter invariants are not functionally independent; see [**18**]. As in Example 7, we can also take the sum to be from 1 to n because the latter vectors $y_{n+1}, \ldots, y_k$ are Euclidean-invariant linear combinations of $y_1, \ldots, y_n$.

Example 9. Consider the standard action $x \mapsto A\,x$ of the general linear group of invertible $n \times n$ matrices $A \in \mathrm{GL}(n)$ on $x \in N = \mathbb{R}^n$. As above, we consider the Cartesian product action on $k \geq n$ copies of N, denoted by M, and seek equivariant functions from $M \to N$. To compute a moving frame, we work in the dense open subset $M_0 \subset M$ defined above, using the $n \times k$ data matrix $X = (x_1, \ldots, x_k) = (X_0, X_1)$, where X_0 is invertible. The action of $\mathrm{GL}(n)$ is then given by $X \mapsto A\,X$.

We choose the cross-section $K = \{\, X_0 = \mathrm{I} \,\} \subset M_0$. The normalization equations are simply $A\,X_0 = \mathrm{I}$, and hence the right equivariant moving frame is $\rho(X) = X_0^{-1}$. The invariants are the entries of $X_0^{-1} X_1$. We can view the rows $a_1, \ldots, a_n$ of the inverse matrix X_0^{-1} as the dual basis vectors for the basis $x_1, \ldots, x_n$, meaning $a_i\, x_j = \delta_{ij}$ for $i, j = 1, \ldots, n$. Thus, the joint $\mathrm{GL}(n)$ invariants $I = (\ \ldots\ I_{ij}\ \ldots\ )$ are the products $I_{ij} = a_i\, x_j$ for $i = 1, \ldots, n$, $j = n+1, \ldots, k$, between the dual basis vectors and the remainder.

To construct equivariant maps, we work as above. Now $v = \rho(X)\, w = X_0^{-1} w$. Given a submanifold of the cross-section defined by $v = H(Y)$, the corresponding equivariant map is implicitly given by $X_0^{-1} w = H(X_0^{-1} X_1) = H(I)$, or, in explicit form,

$$w = X_0\, H(I) = \sum_{i=1}^{n} h_i(I)\, x_i, \tag{20}$$

where the h_i are scalar invariant functions, thus establishing the corresponding (reduced) Malgrange representation.

Example 10. Consider next the special linear group $\mathrm{SL}(n)$ consisting of all unimodular (unit determinant) $n \times n$ matrices, and corresponding to volume-preserving maps $x \mapsto A x$ for $x \in N = \mathbb{R}^n$ and $A \in \mathrm{SL}(n)$. let M be as before, and we again assume the first n vectors $x_1, \ldots, x_n$ form a basis.

Now the cross-section can be taken to be $K = \{ X_0 = c\,\mathrm{I} \} \subset M_0$ where $c \in \mathbb{R}$ is arbitrary. The corresponding right moving frame is $\rho(X) = (\det X_0)\, X_0^{-1} = \operatorname{cof} X_0$, the latter denoting the cofactor matrix. The invariants are the non-constant entries of $\rho(X)\, X = (\det X_0)(\mathrm{I}, X_0^{-1} X_1)$, namely† $I = (I_0 \ \ldots \ I_{ij} \ \ldots\ )$, where $I_0 = \det X_0$ and $I_{ij} = a_i\, x_j$ are the entries of $X_0^{-1} X_1$.

Given a submanifold of the cross-section defined by $v = H(c, Y)$, the corresponding equivariant map is implicitly given by $(\det X_0) X_0^{-1} w = H(I)$, or, in explicit form,

$$w = (\det X_0)^{-1} X_0\, H(I) = \sum_{i=1}^{n} \widetilde{h}_i(I)\, x_i, \tag{21}$$

where $\widetilde{h}_i(I) = h_i(I)/I_0$ are scalar invariants, which is again the Malgrange formula.

Remark: Both of the preceding actions can be extended by including translations, leading to the affine group $\mathrm{A}(n) = \mathrm{GL}(n) \ltimes \mathbb{R}^n$ and the equi-affine group $\mathrm{SA}(n) = \mathrm{SL}(n) \ltimes \mathbb{R}^n$. These are treated in the same fashion as the Euclidean group. One uses the translation to normalize the data points $x_1, \ldots, x_k$ so that they have mean zero, and then the invariants are the general or special linear invariants of the normalized data matrix. In both cases, the reduced Malgrange representation takes the form

$$w = \overline{x} + \sum_{i=1}^{n} h_i(I)\, (x_i - \overline{x}), \tag{22}$$

where I denotes, respectively, the preceding general or special linear invariants of the normalized data matrix.

Example 11. Finally, let us study the projective group action

$$x \longmapsto y = \frac{A x + b}{c^T x + d}, \qquad \text{where} \qquad \begin{pmatrix} A & b \\ c^T & d \end{pmatrix} \in \mathrm{GL}(n+1, \mathbb{R}), \qquad x \in \mathbb{R}^n. \tag{23}$$

Here A is an $n \times n$ matrix, $b, c \in \mathbb{R}^n$, and $d \in \mathbb{R}$. The most important case for image processing is when $n = 2$ in which the action comes from the projection of three-dimensional objects onto a camera plane, [**3**].

The Cartesian product action will be free on a suitable dense open subset of M provided $k \geq n + 2$. The joint invariants are found, using moving frames, in [**18**; Example 3.9]. Define

$$V(x_1, \ldots, x_{n+1}) = \frac{1}{n!} \det(x_2 - x_1, \ \ldots\ , x_{n+1} - x_1) \tag{24}$$

† One can, of course, drop the I_0 factor from all but the first invariants; however, the resulting invariants $\widetilde{I} = (I_0 \ \ldots \ I_{ij} \ \ldots\ )$ will not satisfy the moving frame Replacement Rule.

to be the volume of the simplex with the indicated vertices, which is assumed to be nonzero. Then a complete system of fundamental joint invariants are given by the *volume cross ratios*

$$I_{ij}(x_1, \ldots, x_k) = \frac{V(x_1, \ldots, x_{n+1})\, V(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, x_{n+2}, x_j)}{V(x_1, \ldots, x_n, x_{n+2})\, V(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, x_{n+1}, x_j)}, \qquad i = 1, \ldots, n, \quad j = n+3, \ldots, k. \tag{25}$$

We note that all volume cross ratios are projectively invariant, cf. [**27**], and the particular ones listed in (25) are functionally independent. Syzygies among the various cross ratios can be readily found using the Replacement Rule (5); see [**18**]. For example, when $n = 1$, so we are dealing with the projective line $N = \mathbb{R} \subset \mathbb{RP}^1$, the volume cross ratios (25) reduce to the standard cross ratio invariants

$$I_j = \frac{(x_2 - x_1)\,(x_j - x_3)}{(x_3 - x_1)\,(x_j - x_2)}, \qquad j = 4, \ldots, k. \tag{26}$$

Similarly, in the planar case $n = 2$, the fundamental projective invariants are the area cross ratios

$$I_{1j} = \frac{V(x_1, x_2, x_3)\, V(x_2, x_4, x_j)}{V(x_1, x_2, x_4)\, V(x_2, x_3, x_j)}, \quad I_{2j} = \frac{V(x_1, x_2, x_3)\, V(x_1, x_4, x_j)}{V(x_1, x_2, x_4)\, V(x_1, x_3, x_j)}, \quad j = 5, \ldots, k. \tag{27}$$

To construct projectively equivariant maps $F\colon M \to N$ on the k fold Cartesian product, we work as before, producing them in implicit form

$$J_i(x_1, \ldots, x_k, w) = H_i(I), \qquad i = 1, \ldots, n, \tag{28}$$

where $J_i = I_{iw}$ denotes the volume cross ratio invariant (25) in which w replaces x_j, while I represents the complete systems of projective invariants (25) depending on $x_1, \ldots, x_k$. The implicit form (28) can be readily solved for w as a function of the x_j's and the $H_i(I)$'s. Indeed, each J_i is a linear fractional function of w, and hence, clearing denominators, (28) is equivalent to an inhomogeneous linear system of equations in w whose coefficients and right hand side depend on $x_1, \ldots, x_k$ and the projective invariants I. The explicit solution to this system, $w = F(x_1, \ldots, x_k, I)$, even in the two-dimensional case, does not look particularly enlightening. We will interpret (28), or its explicit solution, as the nonlinear analog of the Malgrange formula in this case.

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