Lectures on Moving Frames

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Abstract. This article presents the equivariant method of moving frames for finite-dimensional Lie group actions, surveying a variety of applications, including geometry, differential equations, computer vision, numerical analysis, the calculus of variations, and invariant flows.

1. Introduction.

According to Akivis, [**1**], the method of moving frames originates in work of the Estonian mathematician Martin Bartels (1769–1836), a teacher of both Gauss and Lobachevsky. The field is most closely associated with Élie Cartan, [**21**], who forged earlier contributions by Darboux, Frenet, Serret, and Cotton into a powerful tool for analyzing the geometric properties of submanifolds and their invariants under the action of transformation groups. In the 1970's, several researchers, cf. [**24**, **36**, **37**, **48**], began the process of developing a firm theoretical foundation for the method. The final crucial step, [**31**], is to define a moving frame simply as an equivariant map from the manifold back to the transformation group. All classical moving frames can be reinterpreted in this manner. Moreover, the equivariant approach is completely algorithmic, and applies to very general group actions.

Cartan's normalization construction of a moving frame can be interpreted as the choice of a cross-section to the group orbits. This enables one to algorithmically construct an equivariant moving frame along with a complete systems of invariants through the induced invariantization process. The existence of an equivariant moving frame requires freeness of the underlying group action, i.e., the isotropy subgroup of any single point is trivial.

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Classically, non-free actions are made free by prolonging to jet space, leading to differential invariants and the solution to equivalence and symmetry problems via the differential invariant signature. Alternatively, applying the moving frame method to Cartesian product actions leads to the classification of joint invariants and joint differential invariants, [**77**]. Finally, an amalgamation of jet and Cartesian product actions dubbed *multi-space* was proposed in [**78**] to serve as the basis for the geometric analysis of numerical approximations, and systematic construction of invariant numerical algorithms, [**54**].

With the basic moving frame machinery in hand, a plethora of new, unexpected, and significant applications soon appeared. In [**75**, **7**, **55**, **56**], the theory was applied to produce new algorithms for solving the basic symmetry and equivalence problems of polynomials that form the foundation of classical invariant theory. In [**20**, **10**, **2**, **6**, **91**, **72**], the characterization of submanifolds via their differential invariant signatures was applied to the problem of object recognition and symmetry detection, [**16**, **17**, **30**, **87**]. Applications to the classification of joint invariants and joint differential invariants appear in [**31**, **77**, **11**]. In computer vision, joint differential invariants have been proposed as noise-resistant alternatives to the standard differential invariant signatures, [**28**, **69**]. The approximation of higher order differential invariants by joint differential invariants and, generally, ordinary joint invariants leads to fully invariant finite difference numerical schemes, [**19**, **20**, **10**, **78**, **54**]. The all-important recurrence formulae lead to a complete characterization of the differential invariant algebra of group actions, and lead to new results on minimal generating invariants, even in very classical geometries, [**79**, **43**, **81**, **47**, **44**]. The general problem from the calculus of variations of directly constructing the invariant Euler-Lagrange equations from their invariant Lagrangians was solved in [**57**]. Applications to the evolution of differential invariants under invariant submanifold flows, leading to integrable soliton equations and signature evolution in computer vision, can be found in [**80**, **50**].

Applications of equivariant moving frames that are being developed by other research groups include the computation of symmetry groups and classification of partial differential equations [**60**, **70**]; geometry of curves and surfaces in homogeneous spaces, with applications to integrable systems, [**61**, **62**, **63**]; symmetry and equivalence of polygons and point configurations, [**12**, **49**], recognition of DNA supercoils, [**90**], recovering structure of three-dimensional objects from motion, [**6**], classification of projective curves in visual recognition, [**41**]; construction of integral invariant signatures for object recognition in 2D and 3D images, [**32**]; determination of invariants and covariants of Killing tensors, with applications to general relativity, separation of variables, and Hamiltonian systems, [**27**, **67**, **66**]; further developments in classical invariant theory, [**7**, **55**, **56**]; computation of Casimir invariants of Lie algebras and the classification of subalgebras, with applications in quantum mechanics, [**13**, **14**]. A rigorous, algebraically-based reformulation of the method, suitable for symbolic computations, has been proposed by Hubert and Kogan, [**45**, **46**].

Finally, in recent work with Pohjanpelto, [**83**, **84**, **85**], the theory and algorithms have recently been extended to the vastly more complicated case of infinite-dimensional Lie pseudo-groups. Applications to infinite-dimensional symmetry groups of partial differential equations can be found in [**22**, **23**, **71**, **94**], and to the classification of Laplace invariants and factorization of linear partial differential operators in [**92**].

2. Equivariant Moving Frames.

We begin by describing the general equivariant moving frame construction. Let G be an r-dimensional Lie group acting smoothly on an m-dimensional manifold M.

Definition 2.1. A *moving frame* is a smooth, G-equivariant map $\rho : M \to G$.

There are two principal types of equivariance:

$$\rho(g \cdot z) = \begin{cases} g \cdot \rho(z) & \text{left moving frame} \\ \rho(z) \cdot g^{-1} & \text{right moving frame} \end{cases} \tag{2.1}$$

If $\rho(z)$ is any right-equivariant moving frame then $\widetilde{\rho}(z) = \rho(z)^{-1}$ is left-equivariant and conversely. All classical moving frames are left-equivariant, but the right versions are often easier to compute. In classical geometrical situations, one can identify left-equivariant moving frames with the usual frame-based versions, cf. [**39**].

It is not difficult to establish the basic requirements for the existence of an equivariant moving frame.

Theorem 2.2. *A moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.*

Recall that G acts *freely* if the isotropy subgroup $G_z = \{ g \in G \mid g \cdot z = z \}$ of each point $z \in M$ is trivial: $G_z = \{e\}$. This implies *local freeness*, meaning that the isotropy subgroups G_z are all discrete, or, equivalently, that the orbits all have the same dimension, r, as G itself. *Regularity* requires that, in addition, the orbits form a regular foliation.

The explicit construction of a moving frame relies on the choice of a (local) *cross-section* to the group orbits, meaning an $(m-r)$-dimensional submanifold $\mathcal{K} \subset M$ that intersects each orbit transversally and at most once.

Theorem 2.3. *Let G act freely and regularly on M, and let $\mathcal{K} \subset M$ be a cross-section. Given $z \in M$, let $g = \rho(z)$ be the unique group element that maps z to the cross-section: $g \cdot z = \rho(z) \cdot z \in \mathcal{K}$. Then $\rho : M \to G$ is a right moving frame.*

Given local coordinates $z = (z_1, \ldots, z_m)$ on M, suppose the cross-section $\mathcal{K}$ is defined by the r equations

$$Z_1(z) = c_1, \qquad \ldots \qquad Z_r(z) = c_r, \tag{2.2}$$

where $Z_1, \ldots, Z_r$ are scalar-valued functions, while $c_1, \ldots, c_r$ are suitably chosen constants. In many applications, the Z_λ are merely coordinate functions. The associated right moving frame $g = \rho(z)$ is obtained by solving the *normalization equations*

$$Z_1(g \cdot z) = c_1, \qquad \ldots \qquad Z_r(g \cdot z) = c_r, \tag{2.3}$$

for the group parameters $g = (g_1, \ldots, g_r)$ in terms of the coordinates $z = (z_1, \ldots, z_m)$. Transversality combined with the Implicit Function Theorem implies the existence of a local solution to these algebraic equations.

The specification of a moving frame by choice of a cross-section induces an invariantization process that maps functions to invariants.

Definition 2.4. The *invariantization* of a function $F: M \to \mathbb{R}$ is the unique invariant function $I = \iota(F)$ that agrees with F on the cross-section: $I \mid \mathcal{K} = F \mid \mathcal{K}$.

In practice, the invariantization of a function $F(z)$ is obtained by first transforming it according to the group, $F(g \cdot z)$ and then replacing the group parameters by their moving frame formulae $g = \rho(z)$, so that $\iota[F(z)] = F(\rho(z) \cdot z)$. In particular, invariantization of the coordinate functions yields the *fundamental invariants*: $I_\nu(z) = \iota(z_\nu)$. Once these have been computed, the invariantization of a general function $F(z)$ is simply given by

$$\iota\big[\, F(z_1, \ldots, z_n) \,\big] = F(I_1(z), \ldots, I_n(z)). \tag{2.4}$$

In particular, the functions defining the cross-section (2.2) have constant invariantization, $\iota(Z_\nu) = c_\nu$, and are known as the *phantom invariants*. Thus, there are precisely $m - r$ functionally independent fundamental invariants. Moreover, if $I(z)$ is any invariant, then clearly $\iota(I) = I$, which implies the elegant and powerful *Replacement Rule*

$$I(z_1, \ldots, z_n) = I(I_1(z), \ldots, I_n(z)), \tag{2.5}$$

that can be used to immediately rewrite $I(z)$ in terms of the fundamental invariants.

Of course, most interesting group actions are *not* free, and therefore do not admit moving frames in the sense of Definition 2.1. There are two well-known methods that convert a non-free (but effective) action into a free action. The first is to look at the Cartesian product action of G on several copies of M, which leads to joint invariants. The second is to prolong the group action to jet space, which is the natural setting for the traditional moving frame theory, leading to differential invariants. Combining the two methods of jet prolongation and Cartesian product results in joint differential invariants. In applications of symmetry constructions to numerical approximations of derivatives and differential invariants, one requires a unification of these different actions into a common framework, called multispace, [**54**, **78**]. These are discussed in turn in the following sections.

3. Moving Frames on Jet Space and Differential Invariants.

Traditional moving frames are obtained by prolonging the group action to the n^{th} order submanifold jet bundle $\mathrm{J}^n = \mathrm{J}^n(M, p)$, which is defined as the set of equivalence classes of p-dimensional submanifolds $S \subset M$ under the equivalence relation of n^{th} order contact at a single point; see [**73**; Chapter 3] for details. Since G preserves the contact equivalence relation, it induces an action on the jet space J^n, known as its n^{th} order *prolongation* and denoted by $G^{(n)}$. The formulas for the prolonged group action are found by implicit differentiation.

We can assume, without significant loss of generality, that G acts effectively on open subsets of M, meaning that the only group element that fixes *every* point in any open $U \subset M$ is the identity element. This implies, [**76**], that the prolonged action is locally free on a dense open subset $\mathcal{V}^n \subset \mathrm{J}^n$ for $n \gg 0$ sufficiently large. In all known examples, the prolonged action is, in fact, free on such a $\mathcal{V}^n$ although there is, frustratingly, no general proof of this property. The points $z^{(n)} \in \mathcal{V}^n$ are known as *regular jets*.

The normalization construction based on a choice of local cross-section $\mathcal{K}^n \subset \mathcal{V}^n$ to the prolonged group orbits can be used to produce a (locally defined) n^{th} *order equivariant* *moving frame* $\rho^{(n)}\colon \mathrm{J}^n \to G$ in a neighborhood of any regular jet. Once the moving frame is established, the induced invariantization process will map general differential functions $F(x,u^{(n)})$ to differential invariants $I = \iota(F)$. The *fundamental differential invariants* are obtained by invariantization of the coordinate functions:

$$H^i = \iota(x^i), \qquad I^\alpha_J = \iota(u^\alpha_J), \qquad \alpha = 1,\ldots,q, \qquad \#J \geq 0. \tag{3.1}$$

These naturally split into two classes: The $r = \dim G$ combinations defining the cross-section will be constant, and are known as the *phantom differential invariants*. The remainder, called the *basic differential invariants*, form a complete system of functionally independent differential invariants. Indeed, if $I(x,u^{(n)}) = I(\ldots x^i \ldots u^\alpha_J \ldots)$ is any differential invariant, then the Replacement Rule (2.5) allows one to immediately rewrite $I = I(\ldots H^i \ldots I^\alpha_J \ldots)$ in terms of the fundamental differential invariants. The moving frame also produces p independent invariant differential operators by invariantizing the usual total derivative operators, $\mathcal{D}_1 = \iota(D_1), \ldots, \mathcal{D}_p = \iota(D_p)$, which can be iteratively applied to lower order differential invariants to generate the higher order differential invariants; see below for full details.

Example 3.1. The paradigmatic example is the action of the orientation-preserving Euclidean group $\mathrm{SE}(2)$ on plane curves $C \subset M = \mathbb{R}^2$. The group transformation $g \in \mathrm{SE}(2)$ maps the point $z = (x,u)$ to the point $w = (y,v) = g \cdot z$, given by

$$y = x\cos\phi - u\sin\phi + a, \qquad v = x\sin\phi + u\cos\phi + b, \tag{3.2}$$

where $g = (\phi, a, b) \in \mathrm{SE}(2)$ are the group parameters. The prolonged group transformations are obtained by successively applying the implicit differentiation operator

$$D_y = (\cos\phi - u_x \sin\phi)^{-1}\, D_x \tag{3.3}$$

to v, producing

$$v_y = \frac{\sin\phi + u_x\cos\phi}{\cos\phi - u_x\sin\phi}\,, \qquad v_{yy} = \frac{u_{xx}}{(\cos\phi - u_x\sin\phi)^3}\,, \qquad \ldots \tag{3.4}$$

Observe that the prolonged action is locally free on the first order jet space J^1. (To simplify the exposition, we gloss over the remaining discrete ambiguity caused by a 180° rotation; see [**77**] for a more precise development.) The classical moving frame is based on the cross-section

$$\mathcal{K}^1 = \{\, x = u = u_x = 0 \,\}. \tag{3.5}$$

Solving the corresponding normalization equations $y = v = v_y = 0$ for the group parameters produces the right moving frame

$$\phi = -\tan^{-1} u_x\,, \qquad a = -\,\frac{x + u u_x}{\sqrt{1+u_x^2}}\,, \qquad b = \frac{x u_x - u}{\sqrt{1+u_x^2}}\,. \tag{3.6}$$

The classical left-equivariant Frenet frame, [**39**], is obtained by inverting the Euclidean group element given by (3.6). Invariantization of the coordinate functions, which is done by substituting the moving frame formulae (3.6) into the prolonged group transformations (3.4), produces the fundamental normalized differential invariants:

$$\begin{aligned} &\iota(x) = H = 0, && \iota(u) = I_0 = 0, && \iota(u_x) = I_1 = 0, \\ &\iota(u_{xx}) = I_2 = \kappa, && \iota(u_{xxx}) = I_3 = \kappa_s, && \iota(u_{xxxx}) = I_4 = \kappa_{ss} + 3\kappa^3, \end{aligned} \tag{3.7}$$

and so on. The first three are the *phantom invariants*. The lowest order basic differential invariant is the Euclidean curvature $I_2 = \kappa = (1+u_x^2)^{-3/2}u_{xx}$. The corresponding invariant differential operator is the arc length derivative,

$$\mathcal{D} = D_s = \frac{1}{\sqrt{1+u_x^2}}\, D_x \tag{3.8}$$

which is obtained by invariantizing (3.3). Using the general recursion formulae, that relate the normalized and differentiated differential invariants, to be presented in detail below, we can readily prove that the curvature and its successive derivatives with respect to arc length, $\kappa, \kappa_s, \kappa_{ss}, \ldots$, form a complete system of differential invariants.

4. Equivalence and Signatures.

A motivating application of the moving frame method is to solve problems of equivalence and symmetry of submanifolds under group actions. Given a group action of G on M, two submanifolds $S, \overline{S} \subset M$ are said to be *equivalent* if $\overline{S} = g \cdot S$ for some $g \in G$. A *symmetry* of a submanifold is a self-equivalence, that is a group transformation $g \in G$ that maps S to itself: $S = g \cdot S$. The solution to the equivalence and symmetry problems for submanifolds is based on the functional interrelationships among the fundamental differential invariants restricted to the submanifold.

Suppose we have constructed an n^{th} order moving frame $\rho^{(n)} \colon \mathrm{J}^n \to G$ defined on an open subset of jet space. A submanifold S is called *regular* if its n-jet $\mathrm{j}_n S$ lies in the domain of definition of the moving frame. For any $k \geq n$, we use $J^{(k)} = I^{(k)} \mid \mathrm{j}_k S$, where $I^{(k)} = (\ldots H^i \ldots I_J^\alpha \ldots)$, $\#J \leq k$, to denote the k^{th} order *restricted differential invariants*.

Definition 4.1. The k^{th} order *signature* $\mathcal{S}^{(k)} = \mathcal{S}^{(k)}(S)$ is the set parametrized by the restricted differential invariants $J^{(k)} \colon \mathrm{j}_k S \to \mathbb{R}^{n_k}$, where $n_k = p + q\binom{p+k}{k} = \dim \mathrm{J}^k$.

The submanifold S is called *fully regular* if $J^{(k)}$ has constant rank $0 \leq t_k \leq p = \dim S$ for all $k \geq n$. In this case, $\mathcal{S}^{(k)}$ forms a submanifold of dimension t_k — perhaps with self-intersections. In the fully regular case,

$$t_n < t_{n+1} < t_{n+2} < \cdots < t_s = t_{s+1} = \cdots = t \leq p, \tag{4.1}$$

where t is the *differential invariant rank* and s the *differential invariant order* of S.

Theorem 4.2. *Two fully regular p-dimensional submanifolds $S, \overline{S} \subset M$ are* (*locally*) *equivalent if and only if they have the same differential invariant order s and their signature manifolds of order $s+1$ are identical*: $\mathcal{S}^{(s+1)}(\overline{S}) = \mathcal{S}^{(s+1)}(S)$.

Since symmetries are merely self-equivalences, the signature also determines the symmetry group of the submanifold.

Theorem 4.3. *If $S \subset M$ is a fully regular p-dimensional submanifold of differential invariant rank t, then its symmetry group G_S is an $(r-t)$–dimensional subgroup of G that acts locally freely on S.*

A submanifold with maximal differential invariant rank $t = p$, and hence only a discrete symmetry group, is called *nonsingular*. The number of symmetries of a nonsingular submanifold is determined by its *index*, which is defined as the number of points in S map to a single generic point of its signature:

$$\operatorname{ind} S = \min \left\{ \# \, (J^{(s+1)})^{-1}\{\zeta\} \;\middle|\; \zeta \in \mathcal{S}^{(s+1)} \right\}. \tag{4.2}$$

Theorem 4.4. *If S is a nonsingular submanifold, then its symmetry group is a discrete subgroup of cardinality* $\operatorname{ind} S$.

At the other extreme, a rank 0 or *maximally symmetric* submanifold, [**82**], has all constant differential invariants, and so its signature degenerates to a single point.

Theorem 4.5. *A regular p-dimensional submanifold S has differential invariant rank* 0 *if and only if its symmetry group is a p-dimensional subgroup $H \subset G$ and hence S is an open submanifold of an H–orbit: $S \subset H \cdot z_0$.*

Remark: "Totally singular" submanifolds may have even larger, non-free symmetry groups, but these are not covered by the preceding results. See [**76**] for details, including Lie algebraic characterizations.

Remark: See [**72**] for some counterexamples when one tries to relax the regularity assumptions in the above results.

Example 4.6. The *Euclidean signature* for a curve $C \subset M = \mathbb{R}^2$ is the planar curve $\mathcal{S}(C) = \{ (\kappa, \kappa_s) \}$ parametrized by the curvature invariant κ and its first derivative with respect to arc length. Two fully regular planar curves are equivalent under an oriented rigid motion if and only if they have the same signature curve. The maximally symmetric curves have constant Euclidean curvature, and so their signature curve degenerates to a single point. These are the circles and straight lines, and, in accordance with Theorem 4.5, each is the orbit of its one-parameter symmetry subgroup of SE(2). The number of Euclidean symmetries of a nonsingular curve is equal to its index — the number of times the Euclidean signature is retraced as we go around the curve.

In Figure 1 we display some signature curves computed from the left ventricle of a gray-scale digital MRI scan of a canine heart. The boundary of the ventricle has been automatically segmented through use of the conformally Riemannian snake flow proposed in [**51**, **96**]. The ventricle boundary curve is then smoothed with the Euclidean-invariant curve shortening flow (see the final section for details) and the Euclidean signatures of the resulting curves computed. As the progressively smoothed curves approach circularity, their signatures exhibit less variation in curvature and wind more and more tightly around a single point, which is the signature of a circle of area equal to the area inside the evolving curve. Despite the rather extensive smoothing involved, except for an overall shrinking as the contour approaches circularity, the basic qualitative features of the different signature curves appear to be remarkably robust.

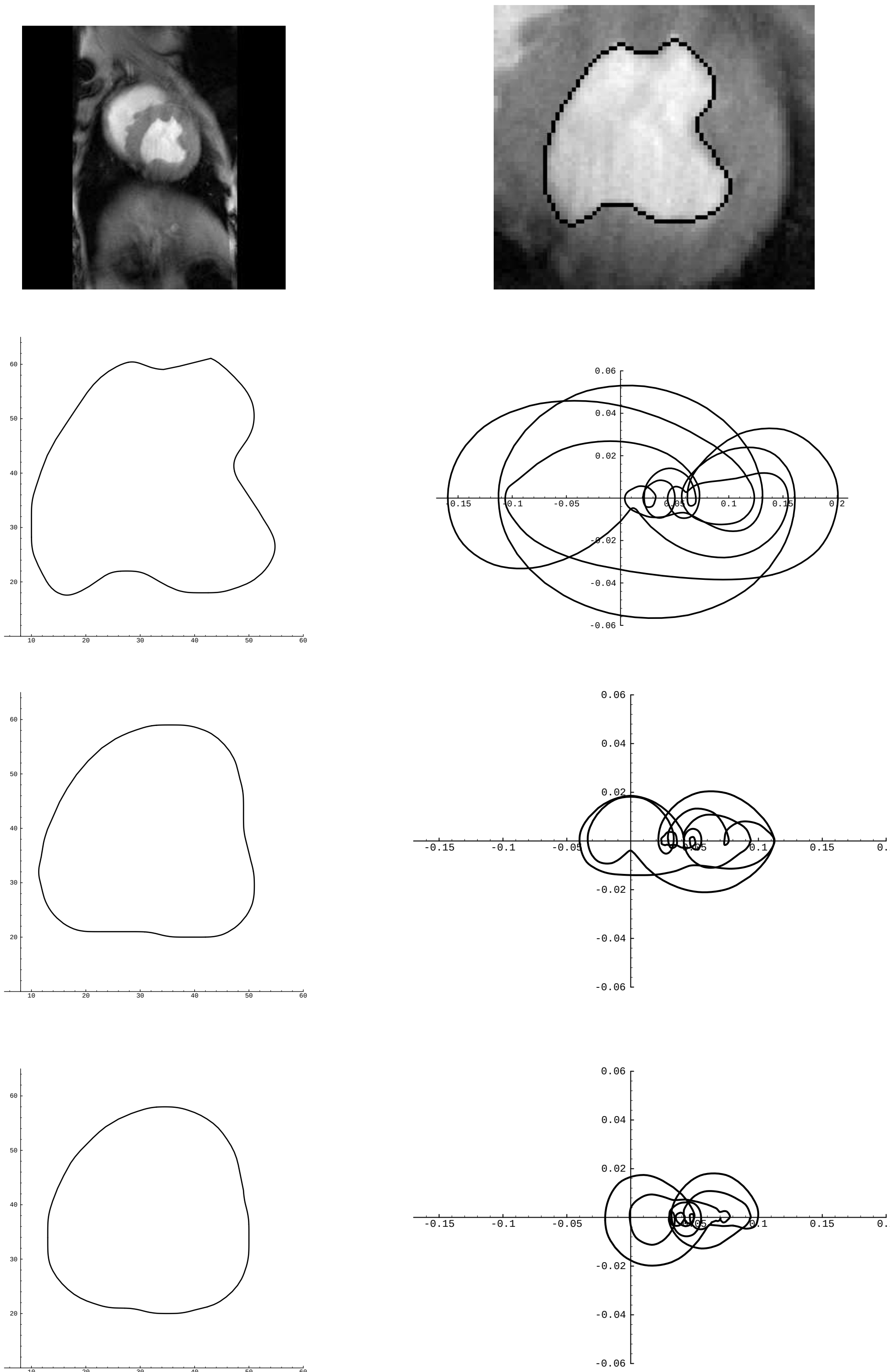

Figure 1. Signature of a Canine Ventricle.

5. Joint Invariants and Joint Differential Invariants.

As always, the starting point the the action of a Lie group G on a manifold M. Consider the *joint action*

$$g \cdot (z_0, \ldots, z_n) = (g \cdot z_0, \ldots, g \cdot z_n), \qquad g \in G, \quad z_0, \ldots, z_n \in M. \tag{5.1}$$

on the $(n+1)$-fold Cartesian product $M^{\times (n+1)} = M \times \cdots \times M$. An invariant $I(z_0, \ldots, z_n)$ of (5.1) is an $(n+1)$-*point joint invariant* of the original transformation group. In most cases of interest (although not in general), if G acts effectively on M, then, for $n \gg 0$ sufficiently large, the product action is free and regular on an open subset of $M^{\times (n+1)}$, cf. [**77**]. Consequently, the equivariant moving frame method can be applied to such joint actions, and thereby establish complete classifications of joint invariants and, via prolongation to Cartesian products of jet spaces, joint differential invariants.

Example 5.1. Consider the Euclidean group SE(2) acting on curves $C \subset M = \mathbb{R}^2$. For the Cartesian product action on $M^{\times n}$, we take the simplest cross-section $\mathcal{K} = \left\{ x_0 = u_0 = x_1 = 0, u_1 > 0 \right\}$ leads to the normalization equations

$$\begin{gathered} y_0 = x_0 \cos\phi - u_0 \sin\phi + a = 0, \qquad\qquad v_0 = x_0 \sin\phi + u_0 \cos\phi + b = 0, \\ y_1 = x_1 \cos\phi - u_1 \sin\phi + a = 0. \end{gathered} \tag{5.2}$$

Solving, we obtain a right moving frame

$$\phi = \tan^{-1}\left(\frac{x_1 - x_0}{u_1 - u_0}\right), \qquad a = -x_0 \cos\phi + u_0 \sin\phi, \qquad b = -x_0 \sin\phi - u_0 \cos\phi, \tag{5.3}$$

along with the fundamental interpoint distance invariant

$$I = \iota(u_1) = \| \, z_1 - z_0 \, \|. \tag{5.4}$$

Substituting (5.3) into the prolongation formulae (3.4) leads to the the normalized first and second order joint differential invariants

$$J_k = \iota\left(\frac{du_k}{dx}\right) = -\,\frac{(z_1 - z_0) \cdot \dot z_k}{(z_1 - z_0) \wedge \dot z_k}\,, \qquad K_k = \iota\left(\frac{d^2 u_k}{dx^2}\right) = -\,\frac{\| \, z_1 - z_0 \, \|^3 \, (\dot z_k \wedge \ddot z_k)}{\left[\, (z_1 - z_0) \wedge \dot z_0 \, \right]^3}\,, \tag{5.5}$$

where the dots indicate derivatives of $z_k(t_k)$ with respect to the curve parameter t_k.

Theorem 5.2. *If $n \geq 2$, then every n-point joint Euclidean differential invariant is a function of the interpoint distances $\| \, z_i - z_j \, \|$ and their iterated derivatives with respect to the invariant differential operators*

$$\mathcal{D}_k = \iota(D_{t_k}) = -\,\frac{\| \, z_1 - z_0 \, \|}{(z_1 - z_0) \wedge \dot z_k}\, D_{t_k}.$$

Consequently, to create a Euclidean signature based entirely on joint invariants, we can take four points z_0, z_1, z_2, z_3 on our curve $C \subset \mathbb{R}^2$. As illustrated in Figure 2, there are six different interpoint distance invariants

$$\begin{aligned} a &= \| \, z_1 - z_0 \, \|, \qquad & b &= \| \, z_2 - z_0 \, \|, \qquad & c &= \| \, z_3 - z_0 \, \|, \\ d &= \| \, z_2 - z_1 \, \|, \qquad & e &= \| \, z_3 - z_1 \, \|, \qquad & f &= \| \, z_3 - z_2 \, \|, \end{aligned} \tag{5.6}$$

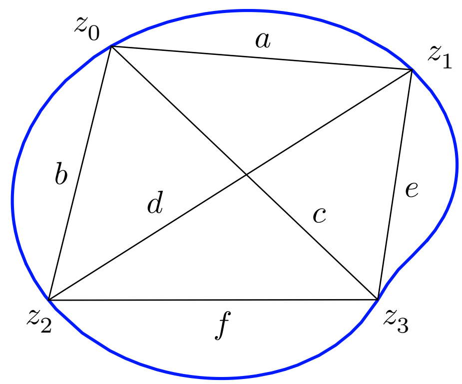

Figure 2. Four-Point Euclidean Curve Invariants.

which parametrize the joint signature $\widehat{\mathcal{S}} = \widehat{\mathcal{S}}(C)$ that uniquely characterizes the curve C up to Euclidean motion. Since this signature avoids any differentiation, it is insensitive to noisy image data. There are two local syzygies

$$\Phi_1(a,b,c,d,e,f) = 0, \qquad \Phi_2(a,b,c,d,e,f) = 0, \tag{5.7}$$

among the the six interpoint distances. One of these is the universal *Cayley–Menger syzygy*

$$\det \begin{vmatrix} 2a^2 & a^2+b^2-d^2 & a^2+c^2-e^2 \\ a^2+b^2-d^2 & 2b^2 & b^2+c^2-f^2 \\ a^2+c^2-e^2 & b^2+c^2-f^2 & 2c^2 \end{vmatrix} = 0, \tag{5.8}$$

which is valid for all possible planar configurations of the four points, cf. [**8**]. The second syzygy in (5.7) is curve-dependent and serves to effectively characterize the joint invariant signature.

A variety of additional examples, including curves and surfaces in two and three-dimensional space under the Euclidean, equi-affine, affine and projective groups, are investigated in detail in [**77**].

6. Invariant Numerical Approximations.

In modern numerical analysis, the development of numerical schemes that incorporate additional structure enjoyed by the problem being approximated, e.g., symmetries, conservation laws, symplectic structure, etc., is now known as *geometric numerical integration*, [**18**, **29**, **40**, **65**]. In practical applications of invariant theory to computer vision, group-invariant numerical schemes to approximate differential invariants have been applied to the problem of symmetry-based object recognition, [**10**, **20**, **19**]. In this section., I discuss the use of moving frame methods to construct symmetry-preserving numerical approximations.

The first step is to construct a suitable manifold that incorporates both the differential equation under consideration and its numerical approximations. Currently, only the case of ordinary differential equations, involving $p = 1$ independent variables, is completely understood, and so we restrict ourselves to this context.

Finite difference approximations to the derivatives of a function $u = f(x)$ rely on its values $u_0 = f(x_0), \ldots, u_n = f(x_n)$ at several distinct points $z_i = (x_i, u_i) = (x_i, f(x_i))$ on the graph. Thus, discrete approximations to jet coordinates on J^n are functions $F(z_0, \ldots, z_n)$ defined on the $(n+1)$-fold Cartesian product space $M^{\times(n+1)}$. As the points $z_0, \ldots, z_n$ coalesce, the approximation $F(z_0, \ldots, z_n)$ will not be well-defined unless we specify the "direction" of convergence. Thus, strictly speaking, F is not defined on all of $M^{\times(n+1)}$, but, rather, on the "off-diagonal" part

$$M^{\diamond(n+1)} = \left\{ \, (z_0, \ldots, z_n) \;\middle|\; z_i \neq z_j \text{ for all } i \neq j \, \right\} \subset M^{\times(n+1)}.$$

As two or more points come together, the limiting value of $F(z_0, \ldots, z_n)$ will be governed by the derivatives (or jet) of the appropriate order governing the direction of convergence. This motivates our construction of the n^{th} *order multi-space* $M^{(n)}$.

Definition 6.1. An $(n+1)$-*pointed curve* $\mathbf{C} = (z_0, \ldots, z_n; C)$ consists of a smooth curve C and $n+1$ not necessarily distinct points $z_0, \ldots, z_n \in C$ thereon. Two $(n+1)$-pointed curves $\mathbf{C} = (z_0, \ldots, z_n; C)$, $\widetilde{\mathbf{C}} = (\widetilde{z}_0, \ldots, \widetilde{z}_n; \widetilde{C})$, have n^{th} *order multi-contact* if and only if $z_i = \widetilde{z}_i$, and $\mathrm{j}_{\#i-1} C|_{z_i} = \mathrm{j}_{\#i-1} \widetilde{C}|_{z_i}$, where $\#i = \#\{\, j \mid z_j = z_i \,\}$, for each $i = 0, \ldots, n$.

Definition 6.2. The n^{th} order *multi-space*, denoted $M^{(n)}$ is the set of equivalence classes of $(n+1)$-pointed curves in M under the equivalence relation of n^{th} order multi-contact. The equivalence class of an $(n+1)$-pointed curves $\mathbf{C}$ is called its n^{th} order *multi-jet*, and denoted $\mathbf{j}_n \mathbf{C} \in M^{(n)}$.

We can identify the subset of multi-jets of multi-pointed curves having distinct points with the off-diagonal Cartesian product space $M^{\diamond(n+1)} \subset \mathrm{J}^n$. On the other hand, the multi-space equivalence relation reduces to the ordinary jet space equivalence relation on the set of coincident multi-pointed curves, and in this way $\mathrm{J}^n \subset M^{(n)}$. Intermediate cases, when some but not all points coincide, correspond to "off-diagonal" Cartesian products of jet spaces

$$\mathrm{J}^{k_1} \diamond \cdots \diamond \mathrm{J}^{k_i} \equiv \left\{ \, (z_0^{(k_1)}, \ldots, z_i^{(k_i)}) \in \mathrm{J}^{k_1} \times \cdots \times \mathrm{J}^{k_i} \;\middle|\; \pi(z_\nu^{(k_\nu)}) \text{ are distinct} \, \right\}, \tag{6.1}$$

where $\sum k_\nu = n$ and $\pi \colon \mathrm{J}^k \to M$ is the usual jet space projection.

Theorem 6.3. *If M is a smooth m-dimensional manifold, then its n^{th} order multi-space $M^{(n)}$ is a smooth manifold of dimension $(n+1)m$, which contains the off-diagonal part $M^{\diamond(n+1)}$ of the Cartesian product space as an open, dense submanifold, and the n^{th} order jet space J^n as a smooth submanifold.*

Just as local coordinates on J^n are provided by the coefficients of Taylor polynomials, local coordinates on $M^{(n)}$ are provided by the coefficients of interpolating polynomials, which are the classical divided differences of numerical interpolation theory, [**78**].

Definition 6.4. Given an $(n+1)$-pointed graph $\mathbf{C} = (z_0, \ldots, z_n; C)$, its divided differences are defined by $[\, z_j \,]_C = f(x_j)$, and

$$[\, z_0 z_1 \ldots z_{k-1} z_k \,]_C = \lim_{z \to z_k} \frac{[\, z_0 z_1 z_2 \ldots z_{k-2} z \,]_C - [\, z_0 z_1 z_2 \ldots z_{k-2} z_{k-1} \,]_C}{x - x_{k-1}}. \tag{6.2}$$

When taking the limit, the point $z = (x, f(x))$ must lie on the curve C, and take limiting values $x \to x_k$ and $f(x) \to f(x_k)$.

It is not hard to show that two $(n+1)$-pointed graphs $\mathbf{C}, \widetilde{\mathbf{C}}$ have n^{th} order multi-contact if and only if they have the same divided differences: $[\, z_0 z_1 \ldots z_k \,]_C = [\, z_0 z_1 \ldots z_k \,]_{\widetilde{C}}$ for all $k = 0, \ldots, n$. Therefore, the required local coordinates on multi-space $M^{(n)}$ consist of the independent variables along with all the divided differences

$$x_0, \ldots, x_n, \quad u^{(0)} = u_0 = [\, z_0 \,]_C, \quad u^{(1)} = [\, z_0 z_1 \,]_C, \quad \ldots \quad u^{(n)} = n!\, [\, z_0 z_1 \ldots z_n \,]_C. \tag{6.3}$$

The $n!$ factor is included so that $u^{(n)}$ agrees with the usual derivative coordinate when restricted to J^n.

In general, implementation of a finite difference numerical solution scheme for a system of ordinary differential equations

$$\Delta_1(x, u, u^{(1)}, \ldots, u^{(n)}) = \ \cdots \ = \Delta_k(x, u, u^{(1)}, \ldots, u^{(n)}) = 0, \tag{6.4}$$

requires suitable discrete approximations to each of its defining differential functions Δ_ν. This requires extending the differential functions from the jet space to the associated multi-space, in accordance with the following definition.

Definition 6.5. An $(n+1)$*-point numerical approximation of order* k to a differential function $\Delta \colon \mathrm{J}^n \to \mathbb{R}$ is an function $F \colon M^{(n)} \to \mathbb{R}$ that, when restricted to the jet space, agrees with Δ to order k.

Now let us consider an r-dimensional Lie group G which acts smoothly on M. Since G evidently maps multi-pointed curves to multi-pointed curves while preserving the multi-contact equivalence relation, it induces an action on the multi-space $M^{(n)}$ that will be called the n^{th} *multi-prolongation* of G and denoted by $G^{(n)}$. On the jet subset $\mathrm{J}^n \subset M^{(n)}$ the multi-prolonged action reduces to the usual jet space prolongation. On the other hand, on the off-diagonal part $M^{\diamond(n+1)} \subset M^{(n)}$ the action coincides with the $(n+1)$-fold Cartesian product action of G on $M^{\times(n+1)}$.

We define a *multi-invariant* to be a function $K \colon M^{(n)} \to \mathbb{R}$ on multi-space which is invariant under the multi-prolonged action of $G^{(n)}$. The restriction of a multi-invariant K to jet space will be a differential invariant, $I = K \mid \mathrm{J}^n$, while restriction to $M^{\diamond(n+1)}$ will define a joint invariant $J = K \mid M^{\diamond(n+1)}$. Restriction to intermediate multi-jet subspaces (6.1) will produce joint differential invariants. Smoothness of K will imply that the joint invariant J is an *invariant* n^{th} *order numerical approximation to the differential invariant* I. Moreover, every invariant finite difference numerical approximation arises in this manner. Thus, the theory of multi-invariants *is* the theory of invariant numerical approximations!

Assuming regularity and freeness of the multi-prolonged action on an open subset of $M^{(n)}$, we can apply the equivariant moving frame construction. The resulting *multi-frame* $\rho^{(n)} \colon M^{(n)} \to G$ will lead us immediately to the required multi-invariants and hence a general, systematic construction for invariant numerical approximations to differential invariants through its induced invariantization procedure. The basic multi-invariants are

$$(H_i, K_i) = I_i = \iota(z_i) = \big(\, \iota(x_i), \iota(u_i) \,\big), \qquad\qquad i = 1, \ldots, n, \tag{6.5}$$

and their divided differences

$$I^{(k)} = \iota(\,[\,z_0 z_1 \ldots z_k\,]\,) = [\,I_0 \ldots I_k\,] = \frac{[\,I_0 \ldots I_{k-2} I_k\,] - [\,I_0 \ldots I_{k-2} I_{k-1}\,]}{H_k - H_{k-1}}\,. \tag{6.6}$$

Example 6.6. For the planar Euclidean action (3.2), the multi-prolonged action is locally free on $M^{(n)}$ for $n \geq 1$. We can thereby determine a first order multi-frame and use it to completely classify Euclidean multi-invariants. The first order transformation formulae are

$$\begin{aligned} y_0 &= x_0 \cos\phi - u_0 \sin\phi + a, & v_0 &= x_0 \sin\phi + u_0 \cos\phi + b, \\ y_1 &= x_1 \cos\phi - u_1 \sin\phi + a, & v^{(1)} &= \frac{\sin\phi + u^{(1)} \cos\phi}{\cos\phi - u^{(1)} \sin\phi}\,, \end{aligned} \tag{6.7}$$

where $u^{(1)} = [\,z_0 z_1\,] = (u_1 - u_0)/(x_1 - x_0)$. Normalization based on the cross-section $y_0 = v_0 = v_1 = 0$ results in the right moving frame

$$\begin{aligned} a &= -x_0 \cos\phi + u_0 \sin\phi = -\frac{x_0 + u^{(1)}\,u_0}{\sqrt{1 + (u^{(1)})^2}}\,, \\ b &= -x_0 \sin\phi - u_0 \cos\phi = \frac{x_0\,u^{(1)} - u_0}{\sqrt{1 + (u^{(1)})^2}}\,, \end{aligned} \qquad \tan\phi = -u^{(1)}\,. \tag{6.8}$$

Substituting the moving frame formulae (6.8) into the lifted divided differences produces a complete system of (oriented) Euclidean multi-invariants. These are easily computed by beginning with the fundamental joint invariants

$$\begin{aligned} H_k &= \iota(x_k) = \frac{(x_k - x_0) + u^{(1)}\,(u_k - u_0)}{\sqrt{1 + (u^{(1)})^2}} = (x_k - x_0)\,\frac{1 + [\,z_0 z_1\,]\,[\,z_0 z_k\,]}{\sqrt{1 + [\,z_0 z_1\,]^2}}\,, \\ K_k &= \iota(u_k) = \frac{(u_k - u_0) - u^{(1)}\,(x_k - x_0)}{\sqrt{1 + (u^{(1)})^2}} = (x_k - x_0)\,\frac{[\,z_0 z_k\,] - [\,z_0 z_1\,]}{\sqrt{1 + [\,z_0 z_1\,]^2}}\,. \end{aligned}$$

The higher order multi-invariants are obtained by forming divided difference quotients

$$[\,I_0 I_k\,] = \frac{K_k - K_0}{H_k - H_0} = \frac{K_k}{H_k} = \frac{(x_k - x_1)[\,z_0 z_1 z_k\,]}{1 + [\,z_0 z_k\,]\,[\,z_0 z_1\,]}\,,$$

where, in particular, $I^{(1)} = [\,I_0 I_1\,] = 0$. The second order multi-invariant

$$\begin{aligned} I^{(2)} = 2\,[\,I_0 I_1 I_2\,] = 2\,\frac{[\,I_0 I_2\,] - [\,I_0 I_1\,]}{H_2 - H_1} &= \frac{2\,[\,z_0 z_1 z_2\,]\sqrt{1 + [\,z_0 z_1\,]^2}}{\big(1 + [\,z_0 z_1\,]\,[\,z_1 z_2\,]\big)\big(1 + [\,z_0 z_1\,]\,[\,z_0 z_2\,]\big)} \\ &= \frac{u^{(2)}\sqrt{1 + (u^{(1)})^2}}{\left[\,1 + (u^{(1)})^2 + \frac{1}{2} u^{(1)} u^{(2)} (x_2 - x_0)\,\right]\left[\,1 + (u^{(1)})^2 + \frac{1}{2} u^{(1)} u^{(2)} (x_2 - x_1)\,\right]} \end{aligned}$$

provides a Euclidean–invariant numerical approximation to the Euclidean curvature:

$$\lim_{z_1, z_2 \to z_0} I^{(2)} = \kappa = \frac{u^{(2)}}{(1 + (u^{(1)})^2)^{3/2}}\,.$$

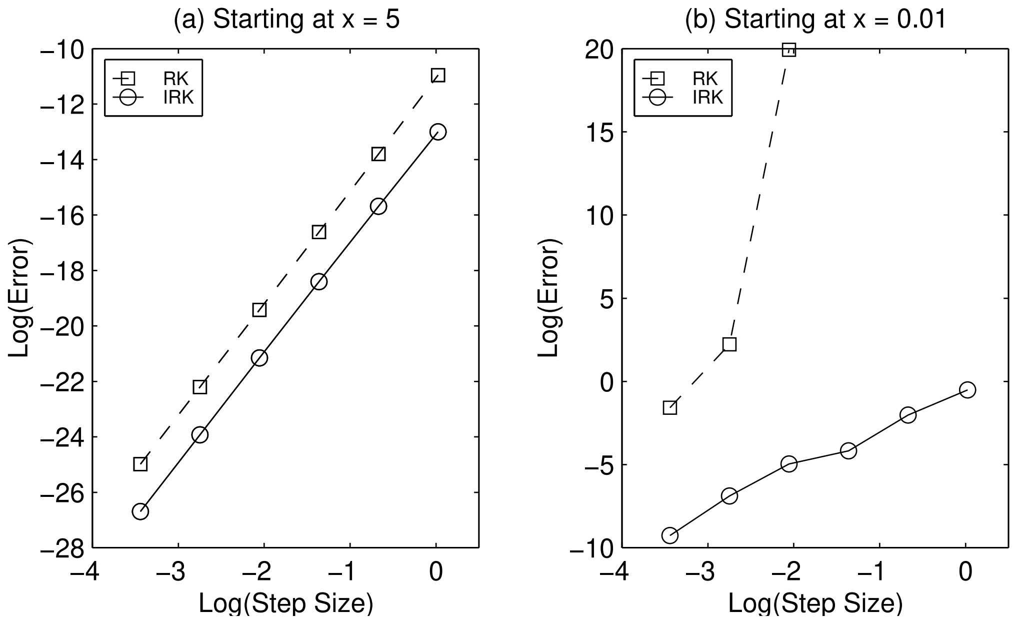

Figure 3. Invariantized Runge–Kutta Schemes for Ames' Equation.

Similarly, the third order multi-invariant

$$I^{(3)} = 6\,[\,I_0 I_1 I_2 I_3\,] = 6\,\frac{[\,I_0 I_1 I_3\,] - [\,I_0 I_1 I_2\,]}{H_3 - H_2}$$

will form a Euclidean–invariant approximation for the normalized differential invariant $\kappa_s = \iota(u_{xxx})$, the derivative of curvature with respect to arc length, [**20**, **31**]. In [**26**], my undergraduate student Derek Dalle makes detailed comparisons between the various divided difference approximations to differential invariants, and shows a number of advantages of such moving frame-based approximations.

Given a symmetry group of an ordinary differential equation, we can construct a moving frame on the associated multispace and apply the induced invariantization procedure to standard numerical schemes, e.g., Runge–Kutta methods, to systematically derive invariantized schemes that respect the symmetries. As emphasized by Pilwon Kim, [**52**, **53**, **54**], the key to the success of the invariantized numerical scheme lies in the intelligent choice of cross-section for the moving frame. Let us look at one simple illustrative example taken from [**52**].

Example 6.7. *Ames' equation*

$$u_{xx} = -\frac{u_x}{x} - e^u \tag{6.9}$$

is a well-studied stiff ordinary differential equation that arises in a wide range of fields, including kinetics and heat transfer, vortex motion of incompressible fluids, and the mass distribution of gaseous interstellar material under influence of its own gravitational fields, [**3**]. The infinitesimal generators

$$\mathbf{v}_1 = -x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial u}, \qquad\qquad \mathbf{v}_2 = -\frac{1}{2}\,x\log x\frac{\partial}{\partial x} + (1+\log x)\frac{\partial}{\partial u},$$

induce one-parameter symmetry groups mapping (x,u) to

$$(e^{\varepsilon_1}x, u + 2\,\varepsilon_1), \qquad\qquad (x^{e^{-\varepsilon_2/2}}, u + 2(1 - e^{-\varepsilon_2/2})\log x + \varepsilon_2),$$

respectively. Individually, neither group leads to a significant improvement in the integration scheme, but a suitable combination induces a moving frame that maps every point (x,u) to the cross-section $\{\, u = 0\,\}$. Figure 3 compares the Runge–Kutta and the invariantized Runge–Kutta schemes starting at $x = 5$. Even in this region, the invariantized scheme outperforms the standard scheme. A more dramatic effect appears when they are applied around $x = 0$, where the ordinary Runge–Kutta breaks down, while the invariantized Runge–Kutta method successfully avoids the stiffness of the equation in that regime.

Extensions to partial differential equations are under development. In [**53**], Kim develops an invariantized Crank-Nicolson scheme for Burgers' equation that avoids problems with numerical oscillations near sharp transition regions. In [**95**], the authors develop invariant schemes for nonlinear partial differential equations of use in image processing, including the Hamilton–Jacobi equation.

7. The Invariant Bicomplex.

Let us return to the case of prolonged group actions on jet space and develop some further machinery required in the more advanced applications of moving frames to differential invariants, differential equations, and the calculus of variations. The full power of the equivariant construction becomes evident once we incorporate the contact structure and induced variational bicomplex on the infinite order jet bundle $\mathrm{J}^\infty = \mathrm{J}^\infty(M,p)$, which we now review, [**4, 74**].

Separating the local coordinates $(x,u) = (x^1,\ldots,x^p,u^1,\ldots,u^q)$ on M into independent and dependent variables naturally splits† the differential one-forms on J^∞ into *horizontal forms*, spanned by $dx^1,\ldots,dx^p$, and *vertical forms*, spanned by the *basic contact one-forms*

$$\theta^\alpha_J = du^\alpha_J - \sum_{i=1}^{p} u^\alpha_{J,i}\,dx^i, \qquad \alpha = 1,\ldots,q, \qquad \#J \geq 0. \tag{7.1}$$

Let π_H and π_V denote the projections mapping one-forms on J^∞ to their horizontal and vertical (contact) components, respectively. We accordingly decompose the differential

† The splitting, which depends on the choice of local coordinates, only works at infinite order, which is the reason we work on J^∞.

$d = \pi_H \circ d + \pi_V \circ d = d_H + d_V$, which results in the *variational bicomplex* on J^∞. If $F(x, u^{(n)})$ is any differential function, its horizontal differential is

$$d_H F = \sum_{i=1}^{p} (D_i F)\, dx^i, \tag{7.2}$$

in which $D_i = D_{x^i}$ denote the usual total derivatives with respect to the independent variables. Thus, $d_H F$ can be identified with the "total gradient" of F. Similarly, its vertical differential is

$$d_V F = \sum_{\alpha,J} \frac{\partial F}{\partial u_J^\alpha}\, \theta_J^\alpha = \sum_{\alpha,J} \frac{\partial F}{\partial u_J^\alpha}\, D_J \theta^\alpha = \mathrm{D}_F(\theta), \tag{7.3}$$

in which the total derivatives act as Lie derivatives on the contact forms $\theta = (\theta^1, \ldots, \theta^q)^T$, and D_F denotes the *formal linearization operator* or *Fréchet derivative* of the differential function F. Thus, the vertical differential $d_V F$ can be identified† with the (first) *variation*, hence the name "variational bicomplex".

Let $\pi_n \colon J^\infty \to J^n$ be the natural jet space projections. Choosing a cross-section $\mathcal{K}^n \subset \mathcal{V}^n \subset J^n$, we extend the induced n^{th} order moving frame $\rho^{(n)}$ to the infinite jet bundle by setting $\rho(x, u^{(\infty)}) = \rho^{(n)}(x, u^{(n)})$ whenever $(x, u^{(n)}) = \pi_n(x, u^{(\infty)})$ lies in the domain of definition of $\rho^{(n)}$. We will employ our moving frame to *invariantize* the variational bicomplex. As before, the invariantization of a differential form is the unique invariant differential form that agrees with its progenitor on the cross-section. In particular, the invariantization process does not affect invariant differential forms. In practice, one determines the invariantization by first transforming the differential form by the prolonged group action and then substituting the moving frame formulae for the group parameters.

As in (3.1), the fundamental differential invariants are obtained by invariantizing the jet coordinates: $H^i = \iota(x^i)$, $I_J^\alpha = \iota(u_J^\alpha)$. Let

$$\varpi^i = \iota(dx^i) = \omega^i + \eta^i, \qquad \text{where} \qquad \omega^i = \pi_H(\varpi^i), \quad \eta^i = \pi_V(\varpi^i), \tag{7.4}$$

denote the *invariantized horizontal one-forms*. Their horizontal components $\omega^1, \ldots, \omega^p$ prescribe, in the language of [**74**], a contact-invariant coframe for the prolonged group action, while the contact forms $\eta^1, \ldots, \eta^p$ are required to make $\varpi^1, \ldots, \varpi^p$ fully G-invariant. Finally, the *invariantized basis contact forms* are denoted by

$$\vartheta_J^\alpha = \iota(\theta_J^\alpha), \qquad \alpha = 1, \ldots, q, \qquad \#J \geq 0. \tag{7.5}$$

Invariantization of more general differential forms relies on the fact that it preserves the exterior algebra structure, and so

$$\iota(\Omega + \Psi) = \iota(\Omega) + \iota(\Psi), \qquad \iota(\Omega \wedge \Psi) = \iota(\Omega) \wedge \iota(\Psi), \tag{7.6}$$

for any differential forms (or functions) Ω, Ψ on J^∞.

† This becomes clearer when you rewrite $\theta_J^\alpha = \delta u_J^\alpha$.

As in the ordinary bicomplex construction, the decomposition of invariant one-forms on J^∞ into invariant horizontal and invariant contact components induces a decomposition of the differential. However, now $d = d_{\mathcal{H}} + d_{\mathcal{V}} + d_{\mathcal{W}}$ splits into three constituents, where $d_{\mathcal{H}}$ adds an invariant horizontal form, $d_{\mathcal{V}}$ adds a invariant contact form, while $d_{\mathcal{W}}$ replaces an invariant horizontal one-form with a combination of wedge products of two invariant contact forms. They satisfy the "quasi-tricomplex" identities

$$\begin{aligned} &d_{\mathcal{H}}^2 = 0, \qquad && d_{\mathcal{H}}\, d_{\mathcal{V}} + d_{\mathcal{V}}\, d_{\mathcal{H}} = 0, \\ &d_{\mathcal{W}}^2 = 0, \qquad && d_{\mathcal{V}}\, d_{\mathcal{W}} + d_{\mathcal{W}}\, d_{\mathcal{V}} = 0, \end{aligned} \qquad d_{\mathcal{V}}^2 + d_{\mathcal{H}}\, d_{\mathcal{W}} + d_{\mathcal{W}}\, d_{\mathcal{H}} = 0. \tag{7.7}$$

Fortunately, the third, anomalous component $d_{\mathcal{W}}$ plays no role (to date) in the applications; in particular, $d_{\mathcal{W}} F = 0$ for any differential function F. Even better, if the group acts projectably, $d_{\mathcal{W}} \equiv 0$. The corresponding dual invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$ are then defined so that

$$d_{\mathcal{H}}\, F = \sum_{i=1}^{p} (\mathcal{D}_i F)\, \varpi^i, \qquad d_{\mathcal{H}}\, \Omega = \sum_{i=1}^{p} \varpi^i \wedge \mathcal{D}_i\, \Omega, \tag{7.8}$$

for any differential function F and, more generally, differential form Ω, on which the $\mathcal{D}_i$ act via Lie differentiation. Keep in mind that, in general, the invariant differential operators do not commute; see (7.17) below.

The most important fact underlying the moving frame construction is that, while it does preserve algebraic structure, the invariantization map ι does *not* respect the differential. The *recurrence formulae*, [**31**, **57**], which we now review, provide the missing "correction terms", i.e., $d\iota(\Omega) - \iota(d\Omega)$. Remarkably, they can be explicitly and algorithmically constructed using merely linear differential algebra — without knowing the explicit formulas for either the differential invariants or invariant differential forms, the invariant differential operators, or even the moving frame!

Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be a basis for the infinitesimal generators of our transformation group. For conciseness, we will retain the same notation for the corresponding prolonged vector fields on J^∞ which, in local coordinates, take the form

$$\mathbf{v}_\kappa = \sum_{i=1}^{p} \xi^i_\kappa(x,u)\, \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{j=\#J\geq 0} \varphi^\alpha_{J,\kappa}(x,u^{(j)})\, \frac{\partial}{\partial u^\alpha_J}\,, \qquad \kappa = 1, \ldots, r. \tag{7.9}$$

The coefficients $\varphi^\alpha_{J,\kappa} = \mathbf{v}_\kappa(u^\alpha_J)$ can be successively constructed by Lie's recursive prolongation formula, [**73**, **74**]:

$$\varphi^\alpha_{Ji,\kappa} = D_i \varphi^\alpha_{J,\kappa} - \sum_{j=1}^{p} u^\alpha_{Jj}\, D_i \xi^j_\kappa. \tag{7.10}$$

With this in hand, we can formulate the *universal recurrence formula*.

Theorem 7.1. *If Ω is any differential function or form on J^∞, then*

$$d\,\iota(\Omega) = \iota(d\,\Omega) + \sum_{\kappa=1}^{r} \nu^\kappa \wedge \iota\,[\mathbf{v}_\kappa(\Omega)], \tag{7.11}$$

where $\nu^1, \ldots, \nu^r$ *are the* invariantized Maurer–Cartan forms *dual to the infinitesimal generators* $\mathbf{v}_1, \ldots, \mathbf{v}_r$, *while* $\mathbf{v}_\kappa(\Omega)$ *denotes the corresponding Lie derivative of* Ω.

In general, the invariantized Maurer–Cartan forms are obtained by pulling back the dual Maurer–Cartan forms $\mu^1, \ldots, \mu^r$ on G via the moving frame map: $\nu^\kappa = \rho^* \mu^\kappa$. The full details, [**57**], are, fortunately, not required thanks to the following marvelous result that allows us to compute them directly without reference to their underlying definition:

Proposition 7.2. *Let* $\mathcal{K} = \{ Z_1(x, u^{(n)}) = c_1, \ldots, Z_r(x, u^{(n)}) = c_r \}$ *be the cross-section defining our moving frame, so that* $c_\lambda = \iota(Z_\lambda)$ *are the phantom differential invariants. Then the corresponding* phantom recurrence formulae

$$0 = d\,\iota(Z_\lambda) = \iota(dZ_\lambda) + \sum_{\kappa=1}^{r} \nu^\kappa \wedge \iota\,[\mathbf{v}_\kappa(Z_\lambda)], \qquad\qquad \lambda = 1, \ldots, r, \tag{7.12}$$

can be uniquely solved for the invariantized Maurer–Cartan forms:

$$\nu^\kappa = \sum_{i=1}^{p} R_i^\kappa \,\varpi^i \;+\; \sum_{\alpha, J} S_\alpha^{\kappa, J}\, \vartheta_J^\alpha, \tag{7.13}$$

where $R_i^\kappa, S_\alpha^{\kappa,J}$ *are certain differential invariants.*

The R_i^κ are called the *Maurer–Cartan invariants*, [**44**, **79**]. In the case of curves, $p = 1$, they are the entries of the Frenet–Serret matrix $\mathcal{D}\rho^{(n)}(x, u^{(n)}) \cdot \rho^{(n)}(x, u^{(n)})^{-1}$, cf. [**39**].

Substituting (7.13) into the universal formula (7.11) produces a complete system of explicit recurrence relations for all the differentiated invariants and invariant differential forms. In particular, taking Ω to be any one of the individual jet coordinate functions x^i, u_J^α, results in the recurrence formulae for the fundamental differential invariants (3.1):

$$\mathcal{D}_i H^j = \delta_i^j + \sum_{\kappa=1}^{r} R_i^\kappa \,\iota(\xi_\kappa^i), \qquad\qquad \mathcal{D}_i I_J^\alpha = I_{Ji}^\alpha + \sum_{\kappa=1}^{r} R_i^\kappa \,\iota(\varphi_{J,\kappa}^\alpha), \tag{7.14}$$

where δ_i^j is the usual Kronecker delta, and $\xi_\kappa^i, \varphi_{J,\kappa}^\alpha$ are the coefficients of the prolonged infinitesimal generators (7.9). Owing to the functional independence of the non-phantom differential invariants, these formulae, in fact, serve to completely prescribe the structure of the non-commutative differential invariant algebra engendered by G, [**31**, **43**, **79**].

Similarly, the recurrence formulae (7.11) for the invariant horizontal forms are

$$\begin{aligned} d\varpi^i = d[\iota(dx^i)] &= \iota(d^2 x^i) + \sum_{\kappa=1}^{r} \nu^\kappa \wedge \iota\,[\mathbf{v}_\kappa(dx^i)] \\ &= \sum_{\kappa=1}^{r} \sum_{k=1}^{p} \iota\left(D_k \xi_\kappa^i\right) \nu^\kappa \wedge \varpi^k + \sum_{\kappa=1}^{r} \sum_{\alpha=1}^{q} \iota\left(\frac{\partial \xi_\kappa^i}{\partial u^\alpha}\right) \nu^\kappa \wedge \vartheta^\alpha. \end{aligned} \tag{7.15}$$

The terms in (7.15) involving wedge products of two horizontal forms are

$$d_{\mathcal{H}}\, \varpi^i = - \sum_{j<k} Y_{jk}^i \,\varpi^j \wedge \varpi^k,$$

where

$$Y^i_{jk} = -\,Y^i_{kj} = \sum_{\kappa=1}^{r} \left[\, R^\kappa_k \,\iota(D_j \xi^i_\kappa) - R^\kappa_j \,\iota(D_k \xi^i_\kappa) \,\right] \tag{7.16}$$

are called the *commutator invariants*, since they prescribe the commutators of the invariant differential operators:

$$[\,\mathcal{D}_j, \mathcal{D}_k\,] = \sum_{i=1}^{p} Y^i_{jk}\, \mathcal{D}_i. \tag{7.17}$$

The terms in (7.15) involving wedge products of a horizontal and a contact form yield

$$d_{\mathcal{V}}\, \varpi^i = \sum_{\kappa=1}^{r} \left[\, \sum_{\alpha=1}^{q} \iota\left(\frac{\partial \xi^i_\kappa}{\partial u^\alpha}\right) R^\kappa_j\, \varpi^i \wedge \vartheta^\alpha - \sum_{k=1}^{p} \iota(D_k \xi^i_\kappa)\, S^{\kappa,J}_\alpha\, \varpi^k \wedge \vartheta^\alpha_J \right]. \tag{7.18}$$

Finally, the remaining terms, involving wedge products of two contact forms, provide the formulas for the anomalous third component of the differential:

$$d_{\mathcal{W}}\, \varpi^i = \sum_{\kappa=1}^{r} \sum_{\alpha=1}^{q} \iota\left(\frac{\partial \xi^i_\kappa}{\partial u^\alpha}\right) S^{\kappa,J}_\alpha\, \vartheta^\alpha_J \wedge \vartheta^\alpha. \tag{7.19}$$

In a similar fashion, we derive the recurrence formulae (7.11) for the differentiated invariant contact forms: In particular, the horizontal components

$$\mathcal{D}_i \vartheta^\alpha_J = \vartheta^\alpha_{Ji} + \sum_{\kappa=1}^{r} R^\kappa_i\, \iota\big(\, \mathbf{v}_\kappa(\theta^\alpha_J)\,\big). \tag{7.20}$$

can be inductively solved to express the higher order invariantized contact forms as certain invariant derivatives of those of order 0:

$$\vartheta^\alpha_J = \mathcal{E}^\alpha_J(\vartheta) = \sum_{\beta=1}^{q} \mathcal{E}^\alpha_{J,\beta}(\vartheta^\beta), \tag{7.21}$$

in which $\vartheta = (\vartheta^1, \ldots, \vartheta^q)^T$ denotes the column vector containing the order zero invariantized contact forms, while $\mathcal{E}^\alpha_J = (\mathcal{E}^\alpha_{J,1}, \ldots, \mathcal{E}^\alpha_{J,q})$ is a row vector of invariant differential operators, i.e., each $\mathcal{E}_{J,\alpha} = \sum A^K_{J,\alpha} \mathcal{D}^K$ for certain differential invariants $A^K_{J,\alpha}$.

Combining these formulae allows us to express the invariant vertical derivative or *invariant variation* of any differential invariant K in the form

$$d_{\mathcal{V}}\, K = \mathcal{A}_K(\vartheta), \tag{7.22}$$

in which $\mathcal{A}_K$ is a row vector of invariant differential operators. Formula (7.22) can be viewed as the invariant version of the vertical differentiation formula (7.3), and so will refer to $\mathcal{A}_K$ as the *invariant linearization operator* associated with the differential invariant K. Similarly, we derive the recurrence formulae for the vertical differentials of the invariant horizontal forms:

$$d_{\mathcal{V}}\, \varpi^i = \sum_{j=1}^{p} \sum_{\alpha=1}^{q} \mathcal{B}^i_{j\alpha}(\vartheta^\alpha) \wedge \varpi^j = \sum_{j=1}^{p} \mathcal{B}^i_j(\vartheta) \wedge \varpi^j \tag{7.23}$$

in which $\mathcal{B}^i_j = (\mathcal{B}^i_{j1}, \ldots, \mathcal{B}^i_{jq})$ is a family of p^2 row-vector-valued invariant differential operators, known, collectively, as the *invariant Hamiltonian operator complex*, stemming from its role in the calculus of variations, cf. [**57**, **88**].

Example 7.3. Let us return to the Euclidean group acting on plane curves initiated in Example 3.1. The basic invariant horizontal one-form $\varpi = \iota(dx)$ is obtained by first transforming dx by a general group element:

$$dx \;\longmapsto\; dy = (\cos\phi - u_x \sin\phi)\, dx + (\sin\phi)\theta, \tag{7.24}$$

where

$$\theta = du - u_x\, dx, \qquad \theta_x = du_x - u_{xx}\, dx, \qquad \ldots, \tag{7.25}$$

are the ordinary basis contact forms. Substituting the moving frame formulae (3.6) for the group parameters into (7.24) yields the basic invariant horizontal one-form

$$\varpi = \iota(dx) = \frac{dx + u_x\, du}{\sqrt{1+u_x^2}} = \sqrt{1+u_x^2}\; dx + \frac{u_x}{\sqrt{1+u_x^2}}\, \theta. \tag{7.26}$$

Its (non-invariant) horizontal component is the contact-invariant arc length form

$$\omega = \pi_H(\varpi) = ds = \sqrt{1+u_x^2}\; dx,$$

and so the corresponding invariant differential operator is the usual arc length derivative $\mathcal{D} = D_s$. In the same manner we obtain the basis invariant contact forms

$$\vartheta = \iota(\theta) = \frac{\theta}{\sqrt{1+u_x^2}}\,, \qquad \vartheta_1 = \iota(\theta_x) = \frac{(1+u_x^2)\,\theta_x - u_x u_{xx}\theta}{(1+u_x^2)^2}\,, \qquad \ldots\,. \tag{7.27}$$

To construct the recurrence formulae for the differentiated functions and forms, we begin with the prolonged infinitesimal generators of SE(2):

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_u,$$
$$\mathbf{v}_3 = -u\,\partial_x + x\,\partial_u + (1+u_x^2)\,\partial_{u_x} + 3\,u_x u_{xx}\,\partial_{u_{xx}} + (4\,u_x u_{xxx} + 3\,u_{xx}^2)\,\partial_{u_{xxx}} + \;\cdots\;.$$

The pulled back dual Maurer–Cartan forms ν^1, ν^2, ν^3 are found by applying the universal recurrence formulae (7.11) to the phantom invariants:

$$\begin{aligned}
0 = dH &= \iota(dx) + \iota(\mathbf{v}_1(x))\,\nu^1 + \iota(\mathbf{v}_2(x))\,\nu^2 + \iota(\mathbf{v}_3(x))\,\nu^3 = \varpi + \nu^1,\\
0 = dI_0 &= \iota(du) + \iota(\mathbf{v}_1(u))\,\nu^1 + \iota(\mathbf{v}_2(u))\,\nu^2 + \iota(\mathbf{v}_3(u))\,\nu^3 = \vartheta + \nu^2,\\
0 = dI_1 &= \iota(du_x) + \iota(\mathbf{v}_1(u_x))\,\nu^1 + \iota(\mathbf{v}_2(u_x))\,\nu^2 + \iota(\mathbf{v}_3(u_x))\,\nu^3 = \kappa\,\varpi + \vartheta_1 + \nu^3,
\end{aligned}$$

since $du = u_x\, dx + \theta$, $du_x = u_{xx}\, dx + \theta_x$. Therefore,

$$\nu^1 = -\varpi, \qquad \nu^2 = -\,\vartheta, \qquad \nu^3 = -\,\kappa\,\varpi - \vartheta_1. \tag{7.28}$$

We are now ready to substitute the non-phantom invariants into (7.11):

$$\begin{aligned}
d\kappa = d\iota(u_{xx}) &= \iota(du_{xx}) + \iota(\mathbf{v}_1(u_{xx}))\,\nu^1 + \iota(\mathbf{v}_2(u_{xx}))\,\nu^2 + \iota(\mathbf{v}_3(u_{xx}))\,\nu^3\\
&= \iota(u_{xxx}\, dx + \theta_{xx}) - \iota(3\,u_x u_{xx})\,(\kappa\,\varpi + \vartheta_1) = I_3\varpi + \vartheta_2,\\
dI_3 = d\iota(u_{xxx}) &= \iota(du_{xxx}) + \iota(\mathbf{v}_1(u_{xxx}))\,\nu^1 + \iota(\mathbf{v}_2(u_{xxx}))\,\nu^2 + \iota(\mathbf{v}_3(u_{xxx}))\,\nu^3\\
&= \iota(u_{xxxx}\, dx + \theta_{xxx}) - \iota(4\,u_x u_{xxx} + 3u_{xx}^2)\,(\kappa\,\varpi + \vartheta_1) = (I_4 - 3\,\kappa^3)\varpi + \vartheta_3 - 3\,\kappa^2\vartheta_1,
\end{aligned}$$

and so on. Breaking these formulas into their horizontal and vertical components yields

$$\begin{aligned} I_3 &= \mathcal{D}\kappa = \kappa_s, & d_{\mathcal{V}}\,\kappa &= \vartheta_2, \\ I_4 &= \mathcal{D}I_3 + 3\kappa^3 = \kappa_{ss} + 3\kappa^3, & d_{\mathcal{V}}\, I_3 &= d_{\mathcal{V}}\,\kappa_s = \vartheta_3 - 3\kappa^2\,\vartheta_1. \end{aligned} \tag{7.29}$$

To proceed further, we compute the differentials of the invariant contact forms, again using (7.11, 28):

$$\begin{aligned} d\vartheta &= \iota(d\theta) + \nu^1 \wedge \iota(\mathbf{v}_1(\theta)) + \nu^2 \wedge \iota(\mathbf{v}_2(\theta)) + \nu^3 \wedge \iota(\mathbf{v}_3(\theta)) \\ &= \iota(dx \wedge \theta_x) - (\kappa\,\varpi + \vartheta_1) \wedge \iota(u_x\,\theta) = \varpi \wedge \vartheta_1, \\ d\vartheta_1 &= \iota(d\theta_x) + \nu^1 \wedge \iota(\mathbf{v}_1(\theta_x)) + \nu^2 \wedge \iota(\mathbf{v}_2(\theta_x)) + \nu^3 \wedge \iota(\mathbf{v}_3(\theta_x)) \\ &= \iota(dx \wedge \theta_{xx}) - (\kappa\,\varpi + \vartheta_1) \wedge \iota(2\,u_x\theta_x + u_{xx}\theta) = \varpi \wedge (\vartheta_2 - \kappa^2\,\vartheta) - \kappa\,\vartheta_1 \wedge \vartheta, \\ d\vartheta_2 &= \iota(d\theta_{xx}) + \nu^1 \wedge \iota(\mathbf{v}_1(\theta_{xx})) + \nu^2 \wedge \iota(\mathbf{v}_2(\theta_{xx})) + \nu^3 \wedge \iota(\mathbf{v}_3(\theta_{xx})) \\ &= \iota(dx \wedge \theta_{xxx}) - (\kappa\,\varpi + \vartheta_1) \wedge \iota(3\,u_x\theta_{xx} + 3\,u_{xx}\theta_x + u_{xxx}\theta) \\ &= \varpi \wedge (\vartheta_3 - 3\kappa^2\,\vartheta_1 - \kappa\,\kappa_s\vartheta) - \kappa_s\vartheta_1 \wedge \vartheta, \end{aligned}$$

and so on. Concentrating on the terms involving the invariant horizontal form and comparing with (7.8), we deduce

$$\begin{gathered} \vartheta_1 = \mathcal{D}\vartheta, \qquad \vartheta_2 = \mathcal{D}\vartheta_1 + \kappa^2\,\vartheta = (\mathcal{D}^2 + \kappa^2)\,\vartheta, \\ \vartheta_3 = \mathcal{D}\vartheta_2 + 3\kappa^2\vartheta_1 + \kappa\,\kappa_s\vartheta = (\mathcal{D}^3 + 4\kappa^2\mathcal{D} + 3\kappa\,\kappa_s)\,\vartheta. \end{gathered}$$

Substituting back into (7.29), we find

$$d_{\mathcal{V}}\,\kappa = (\mathcal{D}^2 + \kappa^2)\,\vartheta, \qquad d_{\mathcal{V}}\,\kappa_s = (\mathcal{D}^3 + \kappa^2\mathcal{D} + 3\kappa\,\kappa_s)\,\vartheta.$$

Thus, the invariant linearization operators for the curvature and its arc length derivative are

$$\mathcal{A}_\kappa = \mathcal{D}^2 + \kappa^2, \qquad \mathcal{A}_{\kappa_s} = \mathcal{D}^3 + \kappa^2\mathcal{D} + 3\kappa\,\kappa_s. \tag{7.30}$$

Finally, applying (7.11) and (7.28) to the invariant arc length form $\varpi = \iota(dx)$ yields

$$\begin{aligned} d\varpi &= \iota(d^2x) + \nu^1 \wedge \iota(\mathbf{v}_1(dx)) + \nu^2 \wedge \iota(\mathbf{v}_2(dx)) + \nu^3 \wedge \iota(\mathbf{v}_3(dx)) \\ &= (\kappa\,\varpi + \vartheta_1) \wedge \iota(u_x\,dx + \theta) = \kappa\,\varpi \wedge \vartheta + \vartheta_1 \wedge \vartheta. \end{aligned}$$

Therefore,

$$d_{\mathcal{V}}\,\varpi = -\,\kappa\,\vartheta \wedge \varpi, \qquad \text{and so} \qquad \mathcal{B} = -\,\kappa \tag{7.31}$$

is the invariant Hamiltonian operator.

8. Generating Differential Invariants.

Let us now apply the recurrence formulae to study the structure of the differential invariant algebra associated with the prolonged group action. A set of differential invariants $\mathcal{I} = \{I_1, \ldots, I_k\}$ is said to be *generating* if, locally, every differential invariant can be expressed as a function of the generators and their iterated invariant derivatives $\mathcal{D}_J I_\nu$. Let

$$\mathcal{I}^{(n)} = \{H^1, \ldots, H^p\} \,\cup\, \{\, I_J^\alpha \mid \alpha = 1, \ldots, q,\ \#J \le n \,\} \tag{8.1}$$

denote the entire set of fundamental differential invariants (3.1) of order $\leq n$. In particular, assuming we choose a cross-section that projects to a cross-section on M, then $\mathcal{I}^{(0)} = \{ H^1, \ldots, H^p, I^1, \ldots I^q \}$ are the ordinary invariants for the action of G on M. If, as in the examples treated here, G acts transitively on M, the normalized order 0 invariants are all constant, and hence are superfluous in any generating systems.

The first result is a direct consequence of the recurrence formulae (7.14) for the fundamental differential invariants and the fact that the Maurer–Cartan invariants, being solutions to the phantom recurrence relations, have order bounded by that of the moving frame.

Theorem 8.1. *If the moving frame has order n, then the set of normalized differential invariants $\mathcal{I}^{(n+1)}$ of order $\leq n+1$ forms a generating set.*

Almost all applications rely on a cross-section $\mathcal{K}^n \subset \mathrm{J}^n$ of *minimal order*, which means that its projections $\mathcal{K}^k = \pi^n_k(\mathcal{K}^n) \subset \mathrm{J}^k$ form cross-sections for all $0 \leq k < n$. In this case, one can significantly reduce the set of required generators, [**43**, **79**]:

Theorem 8.2. *If $\mathcal{K}^n = \{ Z_1(x, u^{(n)}) = c_1, \ldots, Z_r(x, u^{(n)}) = c_r \}$ is a minimal order cross-section, then $\mathcal{I}^{(0)} \cup \mathcal{Z}^{(1)}$, where $\mathcal{Z}^{(1)} = \{ \iota(\mathcal{D}_i(Z_j)) \mid 1 \leq i \leq p,\ 1 \leq j \leq r \}$, form a generating set of differential invariants.*

The result is false in general if the cross-section is not minimal, [**79**]. An alternative interesting generating system was found in [**44**]; again, the proof is entirely based on the recurrence formulae.

Theorem 8.3. *Let $\mathcal{R} = \{ R^i_a \mid 1 \leq i \leq p,\ 1 \leq a \leq r \}$ be the Maurer–Cartan invariants. Then $\mathcal{I}^{(0)} \cup \mathcal{R}$ form a generating system.*

In both cases, the $\mathcal{I}^{(0)}$ constituent can be omitted if G acts transitively on M. The preceding generating sets are rarely minimal. For curves, where $p = 1$, under mild restrictions on the group action (specifically transitivity and no pseudo-stabilization under prolongation), there are exactly $m-1$ independent generating differential invariants, and any other differential invariant is a function of the generating invariants and their successive derivatives with respect to the G-invariant arc length element. Thus, for instance, the differential invariants of a space curve $C \subset \mathbb{R}^3$ under the standard action of the Euclidean group $\mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$ are generated by $m - 1 = 2$ differential invariants, namely its curvature and torsion.

For higher dimensional submanifolds, the minimal number of generating differential invariants cannot be fixed a priori, but depends the particularities of the group action and, in fact, can be arbitrarily large, even for surfaces in three-dimensional space, [**79**]. Even in very well-studied, classical situations, there are interesting subtleties that have not been noted before, [**47**, **81**].

Example 8.4. Consider the standard action of the special Euclidean group SE(3) on surfaces $S \subset \mathbb{R}^3$. The classical moving frame construction, [**39**; Chapter 10], or its equivariant reformulation, [**57**; Example 9.9], relies on the cross-section

$$x = y = u = u_x = u_y = u_{xy} = 0, \qquad u_{xx} \neq u_{yy}. \tag{8.2}$$

The two basic differential invariants are the principal curvatures

$$\kappa_1 = \iota(u_{xx}), \qquad\qquad \kappa_2 = \iota(u_{yy}), \tag{8.3}$$

or, equivalently, the *mean curvature* and *Gauss curvature*

$$H = \tfrac{1}{2}(\kappa_1 + \kappa_2), \qquad\qquad K = \kappa_1 \kappa_2. \tag{8.4}$$

The surface admits a classical moving frame provided we are at a non-umbilic point, where $\kappa_1 \neq \kappa_2$. (At a non-degenerate umbilic, one could, in principle, employ a higher order moving frame.) The corresponding invariant horizontal coframe $\varpi^1 = \iota(dx)$, $\varpi^2 = \iota(dy)$, can be identified with the diagonalizing Frenet frame on the surface, [**39**]. We let $\mathcal{D}_1, \mathcal{D}_2$ denote the dual invariant differential operators.

Let $I_{jk} = \iota(u_{jk})$ denote the higher order normalized differential invariants, so $I_{20} = \kappa_1$, $I_{11} = 0$, $I_{02} = \kappa_2$. The third order recurrence relations are readily found:

$$I_{30} = \mathcal{D}_1\kappa_1 = \kappa_{1,1}, \quad I_{21} = \mathcal{D}_2\kappa_1 = \kappa_{1,2}, \quad I_{12} = \mathcal{D}_1\kappa_2 = \kappa_{2,1}, \quad I_{03} = \mathcal{D}_2\kappa_2 = \kappa_{2,2}. \tag{8.5}$$

The two fourth order recurrence relations for

$$I_{22} = \mathcal{D}_2 I_{21} + \frac{I_{30}I_{12} - 2\,I_{12}^2}{\kappa_1 - \kappa_2} + \kappa_1\kappa_2^2 = \mathcal{D}_1 I_{12} - \frac{I_{21}I_{03} - 2\,I_{21}^2}{\kappa_1 - \kappa_2} + \kappa_1^2\kappa_2$$

imply the celebrated *Codazzi syzygy*

$$\kappa_{1,22} - \kappa_{2,11} + \frac{\kappa_{1,1}\kappa_{2,1} + \kappa_{1,2}\kappa_{2,2} - 2\,\kappa_{2,1}^2 - 2\,\kappa_{1,2}^2}{\kappa_1 - \kappa_2} - \kappa_1\kappa_2(\kappa_1 - \kappa_2) = 0. \tag{8.6}$$

The well-known fact that the principal curvatures κ_1, κ_2, or, equivalently, the Gauss and mean curvatures H, K, form a generating system follows from Theorem 8.1 combined with (8.5). Remarkably, as we now show, neither is a minimal generating set!

Applying the moving frame machinery, the recurrence relations for the invariant horizontal forms are found to be

$$\begin{aligned} d_{\mathcal{H}}\,\varpi^1 &= Y_2\,\varpi^1 \wedge \varpi^2, \\ d_{\mathcal{H}}\,\varpi^2 &= Y_1\,\varpi^1 \wedge \varpi^2, \end{aligned} \qquad \text{where} \qquad Y_1 = \frac{\kappa_{2,1}}{\kappa_1 - \kappa_2}\,, \qquad Y_2 = \frac{\kappa_{1,2}}{\kappa_2 - \kappa_1}\,, \tag{8.7}$$

are the commutator invariants. The invariant differential operators therefore satisfy the commutation relation

$$\big[\,\mathcal{D}_1, \mathcal{D}_2\,\big] = \mathcal{D}_1\,\mathcal{D}_2 - \mathcal{D}_2\,\mathcal{D}_1 = Y_2\,\mathcal{D}_1 - Y_1\,\mathcal{D}_2. \tag{8.8}$$

An easy computation shows that the Codazzi syzygy (8.6) can be written compactly as

$$K = \kappa_1\kappa_2 = -\,(\mathcal{D}_1 + Y_1)Y_1 - (\mathcal{D}_2 + Y_2)Y_2. \tag{8.9}$$

which is the key identity employed by Guggenheimer, [**39**], for a short proof of Gauss' Theorema Egregium.

Let us now show how, for suitably nondegenerate surfaces, we can write the Gauss curvature K as a universal rational combination of the invariant derivatives of the mean curvature H. In view of the Codazzi formula (8.9), it suffices to write the commutator invariants Y_1, Y_2 in terms of the mean curvature. To this end, we note that the commutator identity (8.8) can be applied to any differential invariant. In particular,

$$\mathcal{D}_1 \mathcal{D}_2 H - \mathcal{D}_2 \mathcal{D}_1 H = Y_2 \, \mathcal{D}_1 H - Y_1 \, \mathcal{D}_2 H, \tag{8.10}$$

and, furthermore, for $j = 1$ or 2,

$$\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_j H - \mathcal{D}_2 \mathcal{D}_1 \mathcal{D}_j H = Y_2 \, \mathcal{D}_1 \mathcal{D}_j H - Y_1 \, \mathcal{D}_2 \mathcal{D}_j H. \tag{8.11}$$

Provided the nondegeneracy condition

$$(\mathcal{D}_1 H)(\mathcal{D}_2 \mathcal{D}_j H) \neq (\mathcal{D}_2 H)(\mathcal{D}_1 \mathcal{D}_j H), \qquad \text{for } j = 1 \text{ or } 2, \tag{8.12}$$

holds, we can solve (8.10–11) to write the commutator invariants Y_1, Y_2 as rational functions of invariant derivatives of H. Plugging these expressions into the right hand side of the Codazzi identity (8.9) produces an explicit formula for the Gauss curvature as a rational function of the invariant derivatives, of order ≤ 4, of the mean curvature, valid for all surfaces satisfying the nondegeneracy condition (8.12).

In [**81**] it was also proved that, for suitably generic surfaces in $\mathbb{R}^3$, the algebra of equi-affine differential invariants is generated by the third order Pick invariant alone through invariant differentiation. In [**47**] it was proved that the algebras of conformal and projective differential invariants are also both generated by a single differential invariant.

9. Invariant Variational Problems.

As first recognized by Sophus Lie, [**59**], every invariant variational problem can be written in terms of the differential invariants of the symmetry group. The associated Euler-Lagrange equations automatically inherit the symmetry group of the variational problem, and so can also be written in terms of the differential invariants, [**73**]. The formula for directly constructing the differential invariant form of the Euler–Lagrange equations from that of the variational problem was only known in a handful of particular cases, [**4**, **38**], until, applying the invariant variational bicomplex machinery, the general version was established in [**57**]. Recent applications to the equilibrium configurations of flexible Möbius bands can be found in [**93**].

Let us begin by recalling how variational problems $\mathcal{L}[u] = \int L(x, u^{(n)}) \, d\mathbf{x}$ appear in the variational bicomplex, [**4**]. The integrand or *Lagrangian form*

$$\lambda = L(x, u^{(n)}) \, d\mathbf{x} = L(x, u^{(n)}) \, dx^1 \wedge \cdots \wedge dx^p, \tag{9.1}$$

is a differential form on J^∞ of type $(p, 0)$, meaning that it involves p horizontal forms and no contact forms. Classically, to compute the associated Euler–Lagrange equations, one begins with the first variation, followed by an integration by parts. According to (7.3), we identify the first variation with the vertical differential $d_V \lambda = d_V L \wedge d\mathbf{x}$ of the Lagrangian form, which is a form of type $(p, 1)$. Integration by parts can be viewed as quotienting out by the image of the horizontal differential, so $\omega \equiv \widetilde{\omega}$ whenever $\omega - \widetilde{\omega} = d_H \psi$ for some differential form ψ. The induced equivalence classes are represented by *source forms*

$$\omega = \sum_{\alpha=1}^{q} \Delta_\alpha(x, u^{(n)})\, \theta^\alpha \wedge d\mathbf{x}, \tag{9.2}$$

whose vanishing defines a system of differential equations: $\Delta_\alpha(x, u^{(n)}) = 0$. In the case of a variational problem, $\Delta_\alpha = E_\alpha(L) = 0$ are the classical Euler–Lagrange equations.

The Lagrangian of a G-invariant variational problem can be written in the invariant form

$$\lambda = \widetilde{L}(I^{(n)})\, \omega^1 \wedge \ \cdots \ \wedge \omega^p,$$

where $\omega^1, \ldots, \omega^p$ denote the contact invariant coframe induced by the moving frame, (7.4), while $\widetilde{L}(I^{(n)})$ is a function of the generating differential invariants $I = (I^1, \ldots, I^l)$ and their invariant derivatives $\mathcal{D}_J I^\kappa$ up to some finite order $\#J \leq k$. Since they differ by contact forms (which vanish when evaluated on submanifold jets), we do not affect anything by replacing the ω^i by their fully invariant counterparts ϖ^i, and so will use the fully *invariant Lagrangian form*

$$\widetilde{\lambda} = \widetilde{L}(I^{(n)})\, \varpi^1 \wedge \ \cdots \ \wedge \varpi^p \tag{9.3}$$

in our subsequent computations. To find the invariant form of the Euler–Lagrange equations, we first compute the invariant variation $d_{\mathcal{V}} \widetilde{\lambda}$, followed by an invariant integration by parts. Two new complications arise: first, whereas the ordinary vertical derivative does not affect the basis horizontal forms dx^i, formula (7.18) shows that this is not true for the invariant vertical derivatives of the invariant horizontal forms ϖ^i. Secondly, invariant integration by parts, which amounts to working modulo the image of the invariant horizontal differential $d_{\mathcal{H}}$, also introduces new terms owing to (7.16). As a result, the invariant Euler–Lagrange equation expressions are considerably more complicated.

For simplicity, let's just work out the case of curves, so we have only $p = 1$ independent variable, and $q \geq 1$ dependent variables. (The higher dimensional case has some extra twists; see [**57**] for details.) Consider an invariant Lagrangian form $\widetilde{\lambda} = \widetilde{L}(I^{(n)})\, \varpi$ depending on the generating differential invariants $I = (I^1, \ldots, I^l)$, their invariant derivatives $I^\alpha_{,i} = \mathcal{D}^i I^\alpha$, and the fully G-invariant arc length form $\varpi = \iota(dx)$. Its first variation is computed as follows:

$$d_{\mathcal{V}} \widetilde{\lambda} = d_{\mathcal{V}} (\widetilde{L}\, \varpi) = d_{\mathcal{V}} \widetilde{L} \wedge \varpi + \widetilde{L}\, d_{\mathcal{V}} \varpi = \sum_{i,\alpha} \frac{\partial \widetilde{L}}{\partial I^\alpha_{,i}}\, d_{\mathcal{V}} I^\alpha_{,i} \wedge \varpi + \widetilde{L}\, d_{\mathcal{V}} \varpi. \tag{9.4}$$

We then invariantly integrate by parts by applying the basic identity

$$F\, d_{\mathcal{V}} (\mathcal{D}H) \wedge \varpi \equiv - \mathcal{D}F\, d_{\mathcal{V}} H \wedge \varpi - F(\mathcal{D}H)\, d_{\mathcal{V}} \varpi, \tag{9.5}$$

where we work modulo the image of $d_{\mathcal{H}}$. We eventually arrive at the formula

$$d_{\mathcal{V}} \widetilde{\lambda} \equiv \mathcal{E}(\widetilde{L})\, d_{\mathcal{V}} I \wedge \varpi - \mathcal{H}(\widetilde{L})\, d_{\mathcal{V}} \varpi, \tag{9.6}$$

where $\mathcal{E}(\widetilde{L})$, the *invariantized Eulerian* of $\widetilde{L}$, has components

$$\mathcal{E}_\alpha(\widetilde{L}) = \sum_{i=0}^{\infty} (-\mathcal{D})^i \, \frac{\partial \widetilde{L}}{\partial I^\alpha_{,i}} \; , \qquad\qquad \alpha = 1, \ldots, l, \tag{9.7}$$

while

$$\mathcal{H}(\widetilde{L}) = \sum_{\alpha=1}^{m} \sum_{i>j} I^\alpha_{,i-j} (-\mathcal{D})^j \, \frac{\partial \widetilde{L}}{\partial I^\alpha_{,i}} \, - \widetilde{L} \tag{9.8}$$

is known as the *invariantized Hamiltonian*, being the invariant counterpart of the usual Hamiltonian associated with a higher order Lagrangian $L(x, u^{(n)})$, cf. [**4**, **88**].

In the second phase of the computation, we use the recurrence formulae (7.22, 23) to compute the vertical differentials

$$d_{\mathcal{V}} \, I = \mathcal{A}(\vartheta), \qquad\qquad d_{\mathcal{V}} \, \varpi = \mathcal{B}(\vartheta) \wedge \varpi, \tag{9.9}$$

of the differential invariants $I = (I^1, \ldots, I^l)$ and the invariant horizontal (arc length) form in terms of invariant derivatives of the zero$^{\text{th}}$ order invariant contact forms $\vartheta = (\vartheta^1, \ldots, \vartheta^q)$. Substituting (9.9) into (9.6) and performing one last integration by parts, we arrive at the key formula

$$d_{\mathcal{V}} \, \widetilde{\lambda} \equiv \mathcal{E}(\widetilde{L}) \, \mathcal{A}(\vartheta) \wedge \varpi - \mathcal{H}(\widetilde{L}) \, \mathcal{B}(\vartheta) \wedge \varpi \equiv \big[\, \mathcal{A}^* \mathcal{E}(\widetilde{L}) - \mathcal{B}^* \mathcal{H}(\widetilde{L}) \, \big] \vartheta \wedge \varpi,$$

where * denotes the *formal invariant adjoint* of an invariant differential operator, so if

$$\mathcal{P} = \sum_n P_k \, \mathcal{D}^k, \qquad \text{then} \qquad \mathcal{P}^* = \sum_k \, (-\mathcal{D})^k \cdot P_k.$$

We conclude that the Euler-Lagrange equations for our invariant variational problem are equivalent to the invariant system of differential equations

$$\mathcal{A}^* \mathcal{E}(\widetilde{L}) - \mathcal{B}^* \mathcal{H}(\widetilde{L}) = 0. \tag{9.10}$$

Example 9.1. Any Euclidean-invariant variational problem corresponds to an invariant Lagrangian $\widetilde{\lambda} = \widetilde{L}(\kappa, \kappa_s, \kappa_{ss}, \ldots) \, \varpi$ depending on the arc length derivatives of the curvature, and the invariant arc length form (7.26). According to (7.30, 31), $\mathcal{A} = \mathcal{D}^2 + \kappa^2 = \mathcal{A}^*$, while $\mathcal{B} = -\kappa = \mathcal{B}^*$. The invariant Euler-Lagrange formula (9.10) reduces to the known formula

$$(\mathcal{D}^2 + \kappa^2) \, \mathcal{E}(\widetilde{L}) + \kappa \, \mathcal{H}(\widetilde{L}) = 0 \tag{9.11}$$

for the Euclidean-invariant Euler-Lagrange equation, [**4**, **38**].

Additional, more intricate examples can be found in [**80**].

10. Invariant Curve Flows.

Finally, let us discuss some recent applications of the invariant variational bicomplex construction to invariant curve flows. (Extensions to higher dimensional invariant submanifold flows can be found in [**80**].) Setting $p = 1$, let us single out the $m = 1 + q$ invariant one-forms

$$\varpi, \; \vartheta^1, \ldots, \vartheta^q \tag{10.1}$$

consisting of the invariant arc length form $\varpi = \iota(dx)$ and the order 0 invariant contact forms $\vartheta^\alpha = \iota(\theta^\alpha)$. Let $C \subset M$ be a curve. Evaluating the coefficients of (10.1) on the curve jet $(x, u^{(n)}) = \mathrm{j}_n C|_z$ produces a G-equivariant coframe, i.e., a basis for the cotangent space $T^*M|_z$ at $z = (x, u) \in C$. Let $\mathbf{t}, \mathbf{n}_1, \ldots, \mathbf{n}_q$, denote the corresponding dual G–equivariant *frame* on C, with $\mathbf{t}$ tangent, while $\mathbf{n}_1, \ldots, \mathbf{n}_q$ form a basis for the complementary G–invariant *normal bundle* $N \to C$ induced by the moving frame.

In general, let

$$\mathbf{V} = \mathbf{V}_T + \mathbf{V}_N = I\,\mathbf{t} + \sum_{\alpha=1}^{q} J^\alpha\,\mathbf{n}_\alpha \tag{10.2}$$

be a G-equivariant section of $TM \to C$, where $\mathbf{V}_T, \mathbf{V}_N$ denote, respectively, its tangential and normal components, while $I, J^1, \ldots, J^q$ are differential invariants. We will, somewhat imprecisely, refer to $\mathbf{V}$ as a *vector field*, even though it depends on the underlying curve jet. Any $\mathbf{V}$ generates a G-invariant curve flow:

$$\frac{\partial C}{\partial t} = \mathbf{V}|_{C(t)}. \tag{10.3}$$

The tangential component $\mathbf{V}_T$ only affects the curve's internal parametrization, and hence can be ignored as far as the external curve geometry goes. For example, if we set $\mathbf{V}_T = 0$, the resulting vector field $\mathbf{V}_N$ is said to generate a *normal flow*, since each point on the curve moves in the G-invariant normal direction.

Example 10.1. The most well-studied are the Euclidean-invariant plane curve flows. The dual frame vectors to the invariant one-forms (7.26, 27) are the usual Euclidean frame vectors† — the unit tangent and unit normal:

$$\mathbf{t} = \frac{1}{\sqrt{1+u_x^2}}\left(\frac{\partial}{\partial x} + u_x\,\frac{\partial}{\partial u}\right), \qquad \mathbf{n} = \frac{1}{\sqrt{1+u_x^2}}\left(-u_x\,\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right). \tag{10.4}$$

A Euclidean-invariant normal flow is generated by a vector field of the form $\mathbf{V} = \mathbf{V}_N = J\,\mathbf{n}$, in which $J(\kappa, \kappa_s, \ldots)$ is any differential invariant. Particular cases include:

- $\mathbf{V} = \mathbf{n}$: the geometric optics or grassfire flow, [**9**, **89**];
- $\mathbf{V} = \kappa\,\mathbf{n}$: the celebrated curve shortening flow, [**33**, **35**], also used to great effect in image processing, [**86**, **89**];
- $\mathbf{V} = \kappa^{1/3}\,\mathbf{n}$: the induced flow is equivalent, modulo reparametrization, to the equi-affine invariant curve shortening flow, also used in image processing, [**5**, **86**, **89**];
- $\mathbf{V} = \kappa_s\,\mathbf{n}$: this flow induces the modified Korteweg–deVries equation for the curvature evolution, and is the simplest example of a soliton equation arising in a geometric curve flow, [**25**, **34**, **64**];
- $\mathbf{V} = \kappa_{ss}\,\mathbf{n}$: this flow models thermal grooving of metals, [**15**].

A key question is how the differential invariants evolve under an invariant curve flow.

† For simplicity, we are assuming the curve is represented as the graph of a function $u = u(x)$; generalizing the formulas to arbitrarily parametrized curves is straightforward, [**80**].

Theorem 10.2. *Let $\mathbf{V}_N = \sum J^\alpha \, \mathbf{n}_\alpha$ generate an invariant normal curve flow. If K is any differential invariant, then*

$$\frac{\partial K}{\partial t} = \mathbf{V}(K) = \mathcal{A}_K(J), \tag{10.5}$$

where $\mathcal{A}_K$ is the corresponding invariant linearization operator .

Example 10.3. For any of the Euclidean invariant normal plane curve flows $C_t = J\,\mathbf{n}$ listed in Example 10.1, we have, according to (7.30),

$$\frac{\partial \kappa}{\partial t} = (\mathcal{D}^2 + \kappa^2)\,J, \qquad\qquad \frac{\partial \kappa_s}{\partial t} = (\mathcal{D}^3 + \kappa^2\mathcal{D} + 3\,\kappa\,\kappa_s)\,J. \tag{10.6}$$

For instance, for the grassfire flow $J = 1$, and so

$$\frac{\partial \kappa}{\partial t} = \kappa^2, \qquad\qquad \frac{\partial \kappa_s}{\partial t} = 3\,\kappa\,\kappa_s. \tag{10.7}$$

The first equation immediately implies finite time blow-up at a caustic for a convex initial curve segment, where $\kappa > 0$. For the curve shortening flow, $J = \kappa$, and

$$\frac{\partial \kappa}{\partial t} = \kappa_{ss} + \kappa^3, \qquad\qquad \frac{\partial \kappa_s}{\partial t} = \kappa_{sss} + 4\,\kappa^2\kappa_s, \tag{10.8}$$

thereby recovering formulas used in Gage and Hamilton's analysis, [**33**]; see also [**68**]. Finally, for the modified Korteweg-deVries flow, $J = \kappa_s$,

$$\frac{\partial \kappa}{\partial t} = \kappa_{sss} + \kappa^2\kappa_s, \qquad\qquad \frac{\partial \kappa_s}{\partial t} = \kappa_{ssss} + \kappa^2\kappa_{ss} + 3\,\kappa\,\kappa_s^2. \tag{10.9}$$

Warning: Normal flows *do not preserve arc length*, and so the arc length parameter s will vary in time. Or, to phrase it another way, time differentiation ∂_t and arc length differentiation $\mathcal{D} = D_s$ *do not commute* — as can easily be seen in the preceding examples. Thus, one must be very careful *not* to interpret the resulting evolutions (10.7–9) as partial differential equations in the usual sense. Rather, one should regard the differential invariants $\kappa, \kappa_s, \kappa_{ss}, \ldots$ as satisfying an infinite dimensional dynamical system of coupled ordinary differential equations.

A second important class are the invariant curve flows that preserve arc length, which requires $\big[\,\mathbf{V}, \mathcal{D}\,\big] = 0$, or, equivalently that the Lie derivative $\mathbf{V}(\varpi) \equiv 0$ is a contact form. Applying the Cartan formula and (7.23) to the latter characterization, we conclude that arc length preservation under (10.2) requires

$$\mathcal{D}I = \mathcal{B}(J) = \sum_{\alpha=1}^{q} \mathcal{B}_\alpha(J^\alpha), \tag{10.10}$$

where $\mathcal{D}$ is the arc length derivative, while $\mathcal{B} = (\mathcal{B}_1, \ldots, \mathcal{B}_q)$ is the *invariant Hamiltonian operator* (7.23).

Theorem 10.4. *Under an arc-length preserving flow,*

$$\kappa_t = \mathcal{R}_\kappa(J) \qquad \text{where} \qquad \mathcal{R}_\kappa = \mathcal{A}_\kappa - \kappa_s\, \mathcal{D}^{-1}\mathcal{B}. \tag{10.11}$$

More generally, the time evolution of $\kappa_n = \mathcal{D}^n\kappa$ is given by arc length differentiation: $\partial\kappa_n/\partial t = \mathcal{D}^n\mathcal{R}_\kappa(J)$.

Here, the arc length and time derivatives commute, and hence the arc-length preserving flow (10.11) is an ordinary evolution equation — albeit possibly with nonlocal terms. Moreover, when (10.11) is a local evolution equation, it often turns out to be integrable, with $\mathcal{R}_\kappa$ the associated recursion operator, [**73**]. However, as yet, there is no general explanation for this phenomenon.

Example 10.5. For the Euclidean action on plane curves, the condition (10.10) that a curve flow generated by the vector field $\mathbf{V} = I\,\mathbf{t} + J\,\mathbf{n}$ preserve arc length is that

$$\mathcal{D}I = -\,\kappa\, J. \tag{10.12}$$

Most of the curve flows listed in Example 10.1 have *non-local* arc length preserving counterparts owing to the non-invertibility of the arc length derivative operator on $\kappa\, J$. An exception is the modified Korteweg-deVries flow, where $J = \kappa_s$, and so $I = -\frac{1}{2}\kappa^2$. For such flows, the evolution of the curvature is given by (10.11), where

$$\mathcal{R}_\kappa = \mathcal{A}_\kappa - \kappa_s\mathcal{D}^{-1}\mathcal{B} = \mathcal{D}^2 + \kappa^2 + \kappa_s\mathcal{D}^{-1}\cdot\kappa = D_s^2 + \kappa^2 + \kappa_s\, D_s^{-1}\cdot\kappa \tag{10.13}$$

is the modified Korteweg-deVries recursion operator, [**73**]. In particular, when $J = \kappa_s$, (10.11) is the modified Korteweg-deVries equation

$$\kappa_t = \mathcal{R}_\kappa(\kappa_s) = \kappa_{sss} + \tfrac{3}{2}\kappa^2\kappa_s.$$

Example 10.6. In the case of space curves $C \subset \mathbb{R}^3$, under the usual action of the Euclidean group $G = \mathrm{SE}(3)$, the coordinate cross-section

$$\mathcal{K}^2 = \{\, x = u = v = u_x = v_x = v_{xx} = 0 \,\}$$

produces the classical moving frame, [**39, 57**]. There are two generating differential invariants: the curvature $\kappa = \iota(u_{xx})$ and the torsion $\tau = \iota(v_{xxx}/u_{xx})$. According to [**57**], the relevant moving frame formulae are

$$d_{\mathcal{V}}\,\kappa = \mathcal{A}_\kappa(\vartheta), \qquad d_{\mathcal{V}}\,\tau = \mathcal{A}_\tau(\vartheta), \qquad d_{\mathcal{V}}\,\varpi = \mathcal{B}(\vartheta)\wedge\varpi,$$

where $\vartheta = (\vartheta_1, \vartheta_2)^T$ are the order 0 invariant contact forms, while

$$\begin{aligned}
&\mathcal{A}_\kappa = \big(\, \mathcal{D}^2 + (\kappa^2 - \tau^2),\ -2\,\tau\,\mathcal{D} - \tau_s \,\big), \\
&\mathcal{A}_\tau = \left(\ \frac{2\,\tau}{\kappa}\,\mathcal{D}^2 + \frac{3\,\kappa\,\tau_s - 2\kappa_s\tau}{\kappa^2}\,\mathcal{D} + \frac{\kappa\,\tau_{ss} - \kappa_s\tau_s + 2\,\kappa^3\tau}{\kappa^2}\,, \right. \qquad\qquad \mathcal{B} = \big(-\,\kappa,\ 0\,\big). \\
&\qquad\qquad \left. \frac{1}{\kappa}\,\mathcal{D}^3 - \frac{\kappa_s}{\kappa^2}\,\mathcal{D}^2 + \frac{\kappa^2 - \tau^2}{\kappa}\,\mathcal{D} + \frac{\kappa_s\tau^2 - 2\kappa\,\tau\,\tau_s}{\kappa^2}\ \right),
\end{aligned}$$

Thus, under an arc length preserving flow with normal component $\mathbf{V}_N = J\,\mathbf{n}_1 + K\,\mathbf{n}_2$, the curvature and torsion evolve according to

$$\begin{pmatrix} \kappa_t \\ \tau_t \end{pmatrix} = \mathcal{R} \begin{pmatrix} J \\ K \end{pmatrix}, \qquad \text{where} \qquad \mathcal{R} = \begin{pmatrix} \mathcal{R}_\kappa \\ \mathcal{R}_\tau \end{pmatrix} = \begin{pmatrix} \mathcal{A}_\kappa \\ \mathcal{A}_\tau \end{pmatrix} - \begin{pmatrix} \kappa_s \mathcal{D}^{-1}\kappa & 0 \\ \tau_s \mathcal{D}^{-1}\kappa & 0 \end{pmatrix}$$

is the recursion operator for the integrable vortex filament flow, which corresponds to the choice $J = \kappa_s$, $K = \tau_s$. The latter flow can be mapped to the nonlinear Schrödinger equation via the Hasimoto transformation, [**42**, **58**].

Further developments, including applications to image processing and object recognition, can be found in Kenney's thesis, [**50**].

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