Differential Invariant Algebras of Lie Pseudo–Groups

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Abstract. The aim of this paper is to describe, in as much detail as possible and constructively, the structure of the algebra of differential invariants of a Lie pseudo-group acting on the submanifolds of an analytic manifold. Under the assumption of local freeness of a suitably high order prolongation of the pseudo-group action, we develop computational algorithms for locating a finite generating set of differential invariants, a complete system of recurrence relations for the differentiated invariants, and a finite system of generating differential syzygies among the generating differential invariants. In particular, if the pseudo-group acts transitively on the base manifold, then the algebra of differential invariants is shown to form a rational differential algebra with non-commuting derivations.

The essential features of the differential invariant algebra are prescribed by a pair of commutative algebraic modules: the usual symbol module associated with the infinitesimal determining system of the pseudo-group, and a new "prolonged symbol module" constructed from the symbols of the annihilators of the prolonged pseudo-group generators. Modulo low order complications, the generating differential invariants and differential syzygies are in one-to-one correspondence with the algebraic generators and syzygies of an invariantized version of the prolonged symbol module. Our algorithms and proofs are all constructive, and rely on combining the moving frame approach developed in earlier papers with Gröbner basis algorithms from commutative algebra.

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1. Introduction.

In this, the third paper in our series, [**44**, **45**], developing the method of moving frames for pseudo-groups, our aim is to establish the basic theoretical results underlying our earlier constructions and algorithms. Applications of these results and techniques can be found in [**9**, **10**, **38**, **54**]. The reader is advised to consult these papers before delving deeply into the detailed constructions and proofs presented here.

Consider an analytic Lie pseudo-group $\mathcal{G}$ acting on a manifold M. The induced action of $\mathcal{G}$ on submanifolds $S \subset M$ of a fixed dimension p has natural prolongations to the submanifold jet spaces $\mathrm{J}^n(M,p)$, $0 \leq n \leq \infty$, [**41**, **45**]. By a *differential invariant*, we mean a locally defined[†] invariant function $I\colon \mathrm{J}^n(M,p) \to \mathbb{R}$. Our principal object of study is the algebra of differential invariants, denoted by $\mathcal{I}(\mathcal{G})$. Thus, in the geometric language of differential equations, we are dealing with pseudo-groups of *point transformations*. Our methods extend, with minimal effort, to pseudo-groups of contact transformations, [**41**].

Remark: Since differential invariants are, in general, only locally defined, a more technically precise development would recast everything in the language of sheaves, [**56**, **32**]. However, since the experts can readily translate our constructions into sheaf-theoretic language, we will refrain from employing this additional level of abstraction, and, instead, work locally on suitable open subsets of the indicated manifolds and bundles.

A theorem first formulated by Lie in the finite-dimensional Lie group case, [**33**; Theorem 42, p. 760], and then extended by Tresse to infinite-dimensional pseudo-groups, [**55**], states that, under suitable hypotheses, the differential invariant algebra $\mathcal{I}(\mathcal{G})$ is finitely generated. This means that there exists a finite system of differential invariants $I_1, \ldots, I_\ell$, and exactly p invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$ that preserve $\mathcal{I}(\mathcal{G})$, such that *every* differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives, namely $\mathcal{D}_J I_\kappa = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_k} I_\kappa$ for $k = \#J \geq 0$. In general, the invariant differential operators need not commute, and so the order of differentiation is important. Moreover, except in the case of curves, $p = 1$, the differentiated invariants are typically *not* functionally independent, but are subject to certain functional relations or *syzygies* $H(\ \ldots\ \mathcal{D}_J I_\kappa\ \ldots\ ) \equiv 0$.

A rigorous version of the Lie–Tresse Theorem, based on the machinery of Spencer cohomology, was established by Kumpera, [**32**]; see also [**31**] for a generalization to pseudo-group actions on differential equations (submanifolds of jet space), and [**39**] for an approach based on Weil algebras. None of these references provide constructive algorithms for pinpointing a system of generating differential invariants, nor methods for classifying the recurrence and commutator formulae, nor do they investigate the finiteness of the generating differential syzygies. All of these are, in fact, direct consequences of our moving frame algorithms. In the present paper, we establish a constructive algorithm for producing a (non-minimal) generating set of differential invariants for any eventually locally free

[†] Our notational conventions for functions, maps, etc., allows the domain of I to be a proper open subset of its indicated source space: $\operatorname{dom} I \subset \mathrm{J}^n(M,p)$.

pseudo-group action, cf. [**45**], and, in addition, establish, constructively, the existence of a finite number of generating differential syzygies.

As in the finite-dimensional theory, [**19**], (local) freeness of the prolonged pseudo-group action, as formalized by Definition 5.1, underlies the construction of a moving frame, and hence the construction of differential invariants and invariant differential forms. (Extending our methods to the non-free case remains a challenge.) Freeness serves to bound the possible dimensions of the pseudo-group jet bundles, and thus can be regarded as a more transparent geometric version of the Spencer cohomological growth requirements in Kumpera's approach, [**32**]. Indeed, many "large" pseudo-groups, such as volume–preserving diffeomorphisms, or canonical diffeomorphisms on a symplectic manifold, do not possess any local differential invariants. Since freeness is the essential ingredient for our constructions, the first order of business is to establish its persistence under prolongation. Specifically, we use algebraic techniques to prove that a pseudo-group that acts locally freely on a jet space of order $n \geq 1$ necessarily acts locally freely on all higher order jet spaces.

Many of the structural properties of systems of differential equations, both linear and nonlinear, are based on the algebraic structure of their symbols, [**5, 23, 52**]. At each point, the symbols of an involutive system generate a submodule of the module of vector-valued polynomials, and hence can be analyzed by modern computational algebra — in particular the method of Gröbner bases, [**1, 6, 12, 15**]. For linear systems, the symbol polynomials are intrinsically realized as elements of the dual space to the space spanned by the jets of their solutions, and we will exploit this duality throughout. By definition, the Lie pseudo-group transformations are the solutions to a formally integrable system of nonlinear partial differential equations on the diffeomorphism jet bundle, known as the *determining equations* of the pseudo-group. Our methods rely on its infinitesimal generators, which are subject to a linear, involutive system of partial differential equations, known as the *linear determining equations*. These can be obtained by linearizing the nonlinear determining equations at the identity pseudo-group element. In particular, Lie's algorithm for determining the symmetry (pseudo-)group of a system of differential equations, [**40**], leads directly to the symmetry group's linear determining equations. In this manner, the basic properties of the pseudo-group $\mathcal{G}$ are prescribed by the symbol module of its linear determining equations.

Remark: For the historically inclined reader, it is worth noting that the modern theory of Gröbner bases has its foundations in early research on the integrability of systems of partial differential equations. Intimations can be found in [**14, 49, 50**], culminating in the mostly unrecognized work of Gjunter, cf. [**22**] (which summarizes his earlier papers from 1910–1913), in which he anticipates Gröbner basis methods and the Buchberger algorithm, [**6**]. Gjunter's work was rediscovered by Renschuch and his students and collaborators, and an English summary has recently appeared, [**48**]. Vice versa, recent developments in the method of involutive bases for algebraic modules, [**21, 23, 52**], have been directly inspired by the older computational approaches to involutive partial differential equations.

The space of annihilators $\mathcal{L}$ of the prolonged pseudo-group generators plays a key role in our constructions. We realize $\mathcal{L}$ as a subspace of a certain polynomial module. The symbols of the annihilators serve to define the *prolonged symbol submodule* associated with the pseudo-group action on submanifolds; it is related to the usual symbol module by a simple, explicit, linear map.. Specifying a complementary subspace to the annihilator at a point serves to fix a moving frame through the process of normalization, [**45**]. The moving frame engenders an invariantization process that maps differential functions to differential invariants, differential forms to invariant differential forms, and so on. The fundamental differential invariants are obtained by invariantizing the basic submanifold jet coordinates, while the associated invariant differential operators are dual to the invariantized horizontal coordinate one-forms.

The key consequence of the moving frame construction is the all-important *recurrence formulae*, [**19**, **45**], that relate the fundamental differential invariants to their invariant derivatives. The recurrence formuale serve to completely specify the entire structure of the associated differential invariant algebra. Remarkably, both the structure equations for the pseudo-group and the recurrence formulae can be explicitly generated, using only linear differentiable algebra, from solely the formulae for the prolonged infinitesimal generators of the pseudo-group and the choice of moving frame cross-section — the formulas for the differential invariants, the invariant differential operators, the Maurer–Cartan forms, and even the pseudo-group transformations themselves are *not* required! In particular, if the pseudo-group acts transitively on the base manifold, then the recurrence formulae, differential invariant syzygies, and commutation relations for the invariant derivations can all be expressed as rational combinations of the generating invariants. This, perhaps surprisingly, proves that the algebra of differential invariants of any eventually locally freely acting, transitive pseudo-group has the structure of a rational differential algebra with non-commuting derivations, [**24**, **26**, **58**]; see Theorem 8.4 for a more general formulation of this result. Modulo low order embellishments, which can arise when $\mathcal{G}$ does not act freely on lower order jet spaces, the commutative algebraic structure of the invariantized prolonged symbol module encodes the basic structural features of the differential invariant algebra $\mathcal{I}(\mathcal{G})$. Specifically, the finite generating system of (higher order) differential invariants is in one-to-one correspondence with the Gröbner basis generators of the prolonged symbol module. Moreover, the algebraic syzygies of the prolonged symbol module correspond to the (higher order) differential syzygies of $\mathcal{I}(\mathcal{G})$, which thereby produces a finite collection of generating differential syzygies that are quasilinear in the highest order differential invariants. In this manner, standard methods from computational commutative algebra, e.g., Gröbner bases, [**1**, **12**], yield constructive algorithms for extracting the full differential algebraic structure of the differential invariant algebra $\mathcal{I}(\mathcal{G})$.

2. Algebraic Preliminaries.

Let $\mathbb{R}[t,T]$ denote the algebra of real polynomials in the variables $t = (t_1, \ldots, t_m)$, $T = (T^1, \ldots, T^m)$. The subspace

$$\mathcal{T} = \left\{ \eta(t,T) = \sum_{a=1}^{m} \eta_a(t)\, T^a \right\} \simeq \mathbb{R}[t] \otimes \mathbb{R}^m \subset \mathbb{R}[t,T] \tag{2.1}$$

of homogeneous linear polynomials in the T's forms a free module over the polynomial algebra $\mathbb{R}[t]$, isomorphic to the module of vector-valued polynomials $\boldsymbol{\eta}\colon \mathbb{R}^m \to \mathbb{R}^m$. We grade $\mathcal{T} = \bigoplus_{n\geq 0} \mathcal{T}^n$, where $\mathcal{T}^n$ consists of the homogeneous polynomials of *degree* n in t. We set $\mathcal{T}^{\leq n} = \bigoplus_{k=0}^{n} \mathcal{T}^k$ to be the space of polynomials of degree $\leq n$, and shall also use the notation $\mathcal{T}^{\geq n} = \bigoplus_{k=n}^{\infty} \mathcal{T}^k$. In order to unambiguously specify Gröbner bases of submodules, we shall fix a convenient degree compatible term ordering, e.g., degree lexicographic, [**1**, **12**], on the monomials in $\mathcal{T}$ from the outset.

Given a subspace $\mathcal{I} \subset \mathcal{T}$, we set $\mathcal{I}^n = \mathcal{I} \cap \mathcal{T}^n$, $\mathcal{I}^{\leq n} = \mathcal{I} \cap \mathcal{T}^{\leq n}$, and $\mathcal{I}^{\geq n} = \mathcal{I} \cap \mathcal{T}^{\geq n}$. The subspace is *graded* if $\mathcal{I} = \bigoplus_{n\geq 0} \mathcal{I}^n$ is the sum of its homogeneous constituents. A subspace $\mathcal{I} \subset \mathcal{T}$ is a *submodule* if the product $\lambda(t)\,\eta(t,T) \in \mathcal{I}$ whenever $\eta(t,T) \in \mathcal{I}$ and $\lambda(t) \in \mathbb{R}[t]$. In this paper, all submodules (but not all subspaces) are graded. A subspace $\mathcal{I} \subset \mathcal{T}$ spanned by monomials

$$t_A T^b = t_{a_1} \cdots\, t_{a_n} T^b, \qquad \text{where} \qquad 1 \leq a_1, \ldots, a_n, b \leq m,$$

is called a *monomial subspace*, and is automatically graded. In particular, a *monomial submodule* is a submodule that is spanned by monomials.

A polynomial $0 \neq \eta \in \mathcal{T}^{\leq n}$ has *degree* $n = \deg \eta$ and *highest order terms* $\mathbf{H}(\eta) \in \mathcal{T}^n$ provided $\eta = \mathbf{H}(\eta) + \lambda$, where $\mathbf{H}(\eta) \neq 0$ and $\lambda \in \mathcal{T}^{\leq n-1}$ is of lower degree. By convention, only the zero polynomial has zero highest order term. Observe that the map $\mathbf{H}\colon \mathcal{T} \to \mathcal{T}$ is not, in general, linear. Indeed, $\mathbf{H}(\eta + \lambda) = \mathbf{H}(\eta)$ if $\deg \eta > \deg \lambda$, while if $\deg \eta = \deg \lambda$ then $\mathbf{H}(\eta + \lambda) = \mathbf{H}(\eta) + \mathbf{H}(\lambda)$ if and only if either $\mathbf{H}(\eta) + \mathbf{H}(\lambda) \neq 0$ or $\eta + \lambda = 0$.

3. The Symbol Module.

We will review the construction of the symbol module associated with a linear system of partial differential equations. Although the constructions work for arbitrary numbers of independent and dependent variables, in our applications the system in question consists of (the involutive completion of) the linear determining equations for infinitesimal generators of a Lie pseudo-group acting on an m-dimensional manifold M, and so we only deal with the case when there are the same number, namely m, of independent and dependent variables.

For simplicity, we work in the analytic category throughout, although, modulo the usual technical complications, e.g., existence theorems for systems of partial differential equations, all our constructions retain their validity in the smooth (C^∞) category, [**5**, **41**]. We will use $z = (z^1, \ldots, z^m)$ to denote local coordinates on the m-dimensional analytic manifold M. Let $\mathcal{X}(M)$ denote the space of locally defined analytic vector fields

$$\mathbf{v} = \sum_{b=1}^{m} \zeta^b(z)\,\frac{\partial}{\partial z^b}\,, \tag{3.1}$$

i.e., analytic local sections of the tangent bundle TM. For $0 \leq n \leq \infty$, let $\mathrm{J}^n TM$ denote the associated n^{th} order jet bundle, whose fiber coordinates

$$\zeta^b_A = \frac{\partial^{\#A}\zeta^b}{\partial z^A} = \frac{\partial^k \zeta^b}{\partial z^{a_1}\cdots\partial z^{a_k}}\,, \qquad \begin{matrix} b = 1, \ldots, m, & A = (a_1, \ldots, a_k), \\ 1 \leq a_\nu \leq m, & 0 \leq k = \#A \leq n, \end{matrix} \tag{3.2}$$

represent partial derivatives of the vector field coefficients with respect to the base coordinates on M. For $n < \infty$, let $(\mathrm{J}^n TM)^*$ denote the dual bundle; further, $(\mathrm{J}^\infty TM)^* =$

$\lim_{n \to \infty} (\mathrm{J}^n TM)^*$ is the direct limit under the dual projections $(\pi_n^{n+1})^* : (\mathrm{J}^n TM)^* \to (\mathrm{J}^{n+1} TM)^*$. In local coordinates, a section of $(\mathrm{J}^\infty TM)^*$ represents a linear differential polynomial, and thus defines a homogeneous, linear partial differential equation on the space of vector fields

$$L\big(\mathrm{j}_n \mathbf{v}|_z\big) = L(z, \zeta^{(n)}) = \sum_{b=1}^{m} \sum_{\#A \le n} h_b^A(z)\, \zeta_A^b = 0. \tag{3.3}$$

By indicating the (necessarily finite) order $n < \infty$ in (3.3), we assume, by convention, that when $L \not\equiv 0$, at least one of the highest order coefficients is not identically zero: $h_b^A(z) \not\equiv 0$ for some A, b with $\#A = n$.

Recalling (2.1), we can locally identify

$$(\mathrm{J}^\infty TM)^* \simeq M \times \mathcal{T} \qquad \text{via the pairing} \qquad \langle\, \mathrm{j}_\infty \mathbf{v}|_z \,; t_A T^b \,\rangle = \zeta_A^b. \tag{3.4}$$

A linear differential polynomial (3.3) is thereby identified with the analytically parametrized polynomial

$$\eta(z; t, T) = \sum_{b=1}^{m} \sum_{\#A \le n} h_b^A(z)\, t_A T^b, \tag{3.5}$$

whereby

$$L(z, \zeta^{(n)}) = \big\langle\, \mathrm{j}_\infty \mathbf{v}|_z \,;\, \eta(z; t, T) \,\big\rangle\,. \tag{3.6}$$

Definition 3.1. The *symbol* $\mathbf{\Sigma}(L)$ of a non-zero order n linear differential polynomial (3.3) consists of the highest order terms of its defining polynomial (3.5):

$$\mathbf{\Sigma}[L(z, \zeta^{(n)})] = \mathbf{H}[\eta(z; t, T)] = \sum_{b=1}^{m} \sum_{\#A = n} h_b^A(z)\, t_A T^b. \tag{3.7}$$

Suppose $\mathcal{G}$ is a Lie pseudo-group acting analytically on M. The blanket technical hypotheses of regularity and tameness will be assumed throughout, and we refer the reader to [**44**] for complete details. For $0 \le n \le \infty$, let $\mathcal{G}^{(n)} \subset \mathrm{J}^n(M, M)$ denote the subbundle (or, more precisely, subgroupoid, [**34, 44**]) consisting of all n-jets of pseudo-group diffeomorphisms. Let $\mathfrak{g} \subset \mathcal{X}(M)$ be the space spanned by the infinitesimal generators of $\mathcal{G}$. Let $\mathrm{J}^n \mathfrak{g} \subset \mathrm{J}^n TM$ denote the subbundle spanned by their n^{th} order jets. In view of our regularity assumptions, the inverse limit bundle $\mathrm{J}^\infty \mathfrak{g}$ is prescribed by the *linear determining system*

$$L^{(\infty)}(z, \zeta^{(\infty)}) = L^{(\infty)}(\, \ldots\ z^a\ \ldots\ \zeta_A^b\ \ldots\,) = 0, \tag{3.8}$$

which is a formally integrable and locally solvable system of homogeneous linear partial differential equations, [**44**]. Formal integrability requires that the linear determining system be closed under application of the usual total derivative operators

$$\mathbb{D}_{z^a} = \frac{\partial}{\partial z^a} + \sum_{c=1}^{m} \sum_{\#B \ge 0} \zeta_{B,a}^c \,\frac{\partial}{\partial \zeta_B^c}\,, \qquad\qquad a = 1, \ldots, m. \tag{3.9}$$

For computational reasons, one often replaces formal integrability, which cannot in general be verified algorithmically, by the slightly more restrictive assumption of involutivity; see [**23**, **52**] for details. In applications to symmetry groups of differential equations, (3.8) represents the formally integrable (or involutive) completion, under total differentiation, of the usual determining equations obtained by Lie's infinitesimal algorithm, [**40**].

Let

$$\mathcal{L} = (\mathrm{J}^\infty \mathfrak{g})^\perp \subset (\mathrm{J}^\infty TM)^* \tag{3.10}$$

denote the *annihilator subbundle*† of the infinitesimal generator jet bundle. Observe that each equation in the determining system (3.8) can be represented by a parametrized polynomial, as in (3.6), which cumulatively span $\mathcal{L}$. Let

$$\mathcal{I} = \mathbf{H}(\mathcal{L}) \subset (\mathrm{J}^\infty TM)^* \tag{3.11}$$

be spanned by the highest order terms of the annihilating polynomials at each $z \in M$. We will make the further regularity assumption that $\mathcal{I}$ forms a subbundle, known as the *symbol subbundle* for the linear determining system (3.8), which means that, when truncated at any sufficiently large, finite order, $\mathcal{I}^{\leq n}$ forms a subbundle of $(\mathrm{J}^n TM)^*$. Note that

$$\mathcal{I} = \textstyle\bigoplus_{n \geq 0} \mathcal{I}^n, \qquad \text{where} \qquad \mathcal{I}^n \simeq (\mathrm{J}^n \mathfrak{g})^\perp / (\mathrm{J}^{n-1} TM)^*$$

are its homogeneous components.

Remark: Points at which $\mathcal{I}$ fails to be a subbundle are *singular points* for the linear determining system, and are not well understood. Even for linear ordinary differential equations, the distinction between regular and irregular singular points, [**29**], highlights the inherent complications.

On the symbol level, total differentiation, (3.9), corresponds to multiplication:

$$\mathbf{H}(\mathbb{D}_{z^a} L) = t_a\, \mathbf{H}(L), \qquad a = 1, \ldots, m. \tag{3.12}$$

Thus, formal integrability implies that, at each point $z \in M$, the fiber $\mathcal{I}|_z \subset \mathcal{T}$ forms a graded submodule, known as the *symbol module* of the pseudo-group at the point z. On the other hand, the annihilator $\mathcal{L}|_z \subset \mathcal{T}$ is typically *not* a submodule. A notable exception is when the linear determining system consists of partial differential equations that have constant coefficients in some coordinate system.

Let $\mathcal{M}|_z \subset \mathcal{T}$ denote the monomial module generated by the leading (with respect to the specified term ordering) monomials of the polynomials in the symbol module $\mathcal{I}|_z$. We can assume, possibly by restricting to an open subset and employing δ–regular coordinates, [**23**, **52**], that $\mathcal{M}|_z = \mathcal{M}$ does not depend upon z. Let

$$\mathcal{C} = \mathrm{Span}\{\, t_B T^c \notin \mathcal{M} \,\} \subset \mathcal{T} \tag{3.13}$$

† Our regularity assumptions ensure that, for n sufficiently large, $\mathrm{J}^n\mathfrak{g} \subset \mathrm{J}^n TM$ forms a subbundle, and, consequently, so does its annihilator $(\mathrm{J}^n\mathfrak{g})^\perp \subset (\mathrm{J}^n TM)^*$.

be the *complementary monomial subspace* spanned by all monomials *not* in the monomial module $\mathcal{M}$. Applying standard Gaussian Elimination, we are able to construct a linear basis for the space of symbol polynomials $\mathcal{I}|_z$ of the form

$$t_A T^b + \sum_{t_B T^c \in \mathcal{C}^n} h^B_c(z)\, t_B T^c \qquad \text{for all} \qquad t_A T^b \in \mathcal{M}^n, \qquad n \geq 0,$$

with analytic coefficients $h^B_c(z)$. A similar statement holds for the subspace $\mathcal{L}|_z$, where the sum now runs over all monomials in $\mathcal{C}^{\leq n}$. Therefore, $\mathcal{C}$ forms a fixed complement to the symbol module $\mathcal{I}|_z$ as well as the annihilating subspace $\mathcal{L}|_z$:

$$\mathcal{T} = \mathcal{C} \oplus \mathcal{M} = \mathcal{C} \oplus \mathcal{I}|_z = \mathcal{C} \oplus \mathcal{L}|_z \qquad \text{at each} \qquad z \in M. \tag{3.14}$$

Reinterpreting this decomposition in terms of the linear determining system, we conclude that, locally, the differential equations can be rewritten in solved triangular form, [**47**]:

$$\zeta^b_A \;=\; - \sum_{t_B T^c \in \mathcal{C}^{\leq n}} h^B_c(z)\, \zeta^c_B \qquad \text{for all} \qquad t_A T^b \in \mathcal{M}^n, \qquad n \geq 0. \tag{3.15}$$

The *parametric derivatives* ζ^c_B, indexed by the complementary monomials $t_B T^c \in \mathcal{C}$, thus serve to uniquely parametrize the infinitesimal generator jets of the pseudo-group $\mathcal{G}$. In view of (3.10), we can identify the complementary subspace

$$\mathcal{C} \simeq \mathcal{T} \,/\, (\mathrm{J}^\infty \mathfrak{g}|_z)^\perp \simeq (\mathrm{J}^\infty \mathfrak{g}|_z)^* \tag{3.16}$$

as the corresponding dual vector space. In particular,

$$\dim \mathcal{C}^{\leq n} = \dim \mathrm{J}^n \mathfrak{g}|_z = \dim \mathcal{G}^{(n)}|_z = r_n \tag{3.17}$$

is the same as the fiber dimension of the n^{th} order pseudo-group jet bundle, i.e., the minimal number of independent parameters required to locally represent the n^{th} order pseudo-group jets.

Let

$$\widetilde{H}(n) = \text{codim } \mathcal{I}^{\leq n}|_z = \text{codim } \mathcal{M}^{\leq n} = \dim \mathcal{C}^{\leq n} = r_n$$

denote the (affine) Hilbert function of the symbol module. According to [**12**; p. 453] (as adapted to modules) when n is sufficiently large, the Hilbert function coincides with a polynomial:

$$\widetilde{H}(n) = H(n) \qquad \text{for all} \qquad n \geq n_0. \tag{3.18}$$

The *Hilbert polynomial* of the symbol module is necessarily the form

$$H(n) = \sum_{i=0}^{d} b_i \binom{n}{d-i} = b\, \frac{n^d}{d!} + \mathrm{O}(n^{d-1}), \tag{3.19}$$

for certain integer coefficients $b = b_0, b_1, \ldots, b_d \in \mathbb{Z}$. The integer $0 \leq d \leq m$ is the *dimension* of the symbol module. Unless $\mathcal{I} = \mathcal{T}$, in which case $\widetilde{H}(n) \equiv 0$ and the pseudo-group is discrete, the leading coefficient is positive, $b = b_0 > 0$, and is known as the submodule's *degree*, [**12**; p. 465].

In the pseudo-group context, the dimension d and degree b of the symbol module are interpreted as follows. Informally, the general solution to the determining equations — that is, the general pseudo-group generator, and hence the general pseudo-group transformation — can be written in terms of b arbitrary functions $f_1, \ldots, f_b$, each depending on d variables†. In particular, the system is of finite type — and hence $\mathcal{G}$ is, in fact, a b-dimensional Lie group action — if and only if the symbol module has dimension $d = 0$. This interpretation can be made precise if we assume that the system satisfies the hypotheses of the Cartan–Kähler Existence Theorem, [**5**, **41**]. Since this result will not be used, the proof is omitted. See Seiler, [**51**, **52**], for additional details.

Theorem 3.2. *If the linear determining equations for the pseudo-group form a regular system in the sense of the Cartan–Kähler Theorem, then the last nonzero Cartan character is $c_d = b$.*

The smallest integer n_0 for which (3.18) holds is called the *index of regularity* of the symbol module, [**12**; p. 449]. It appears to be related to the maximal order of the integrability constraints for the pseudo-group, but this remains to be completely clarified.

4. Prolongation Symbols.

Our primary object of study is the induced action of the pseudo-group $\mathcal{G}$ on the submanifolds of M of a fixed dimension. Let $\mathrm{J}^n(M,p)$, for $0 \le n \le \infty$, denote the bundle of n-jets of p-dimensional submanifolds $S \subset M$, [**41**, **45**]. We use $\widetilde{\pi}^k_n \colon \mathrm{J}^k(M,p) \to \mathrm{J}^n(M,p)$ for $k \ge n$ to indicate the standard projections. Splitting the coordinates on M into independent and dependent variables:

$$(z^1, \ldots, z^m) = (x^1, \ldots, x^p, u^1, \ldots, u^q), \qquad \text{where} \qquad p + q = m = \dim M, \tag{4.1}$$

fixes a system of local coordinates on the submanifold jet bundle $\mathrm{J}^n(M,p)$, written

$$z^{(n)} = (x, u^{(n)}) = (\ \ldots\ x^i\ \ldots\ u^\alpha_J \ldots), \quad i = 1, \ldots, p, \quad \alpha = 1, \ldots, q, \quad 0 \le \#J \le n. \tag{4.2}$$

The induced fiber coordinates

$$u^\alpha_J = \frac{\partial^{\#J} u^\alpha}{\partial x^J} = \frac{\partial^k u^\alpha}{\partial x^{j_1} \cdots \partial x^{j_k}}\,, \qquad \begin{aligned} &\alpha = 1, \ldots, q, \qquad && J = (j_1, \ldots, j_k), \\ &1 \le j_\nu \le p, && 0 \le k = \#J \le n, \end{aligned} \tag{4.3}$$

ranging over all *unordered multi-indices* J of order $\#J \le n$, represent partial derivatives of the dependent variables with respect to the independent variables.

A real-valued function‡ $F \colon \mathrm{J}^n(M,p) \to \mathbb{R}$ is known as a *differential function*. We will not distinguish between F and its compositions $F \circ \widetilde{\pi}^k_n \colon \mathrm{J}^k(M,p) \to \mathbb{R}$ for $n \le k \le \infty$. The

† The precise meaning of this remark was the principal subject of the famous Cartan–Einstein correspondence, [**8**].

‡ As noted above, our notational conventions allow the domain of F to be an open subset of the jet bundle.

order of F is the lowest order jet space on which it is well defined, or, equivalently, the highest order jet coordinate(s) u^α_J that it explicitly depends on.

Given a pseudo-group $\mathcal{G}$ acting on M, its action on p-dimensional submanifolds $S \subset M$ induces an action on the submanifold jet bundle $\mathrm{J}^n(M,p)$, known as the n^{th} *prolonged action*, and denoted by $\mathcal{G}^{(n)}$. In many applications, $\mathcal{G}$ represents the symmetry pseudo-group of a system of differential equations

$$\mathcal{S}_\Delta = \{ \Delta(x, u^{(n)}) = 0 \} \subset \mathrm{J}^n(M,p)$$

defined by the vanishing of one or more differential functions $\Delta\colon \mathrm{J}^n(M,p) \to \mathbb{R}^l$, and the submanifolds $S = \{ u = f(x) \} \subset M$ of interest are the graphs of candidate solutions. A *differential invariant* is a differential function $I\colon \mathrm{J}^n(M,p) \to \mathbb{R}$ that is invariant under the prolonged pseudo-group action: $I(g^{(n)} \cdot z^{(n)}) = I(z^{(n)})$ for all submanifold jets $z^{(n)}$ and all prolonged pseudo-group transformations $g^{(n)}$ close to the identity such that both $z^{(n)}$ and $g^{(n)} \cdot z^{(n)}$ lie in the domain of I.

Given an analytic vector field

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^a(z) \frac{\partial}{\partial z^a} = \sum_{i=1}^{p} \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x,u) \frac{\partial}{\partial u^\alpha} \in \mathcal{X}(M), \tag{4.4}$$

let

$$\mathbf{v}^{(\infty)} = \sum_{i=1}^{p} \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{k=\#J \geq 0} \widehat{\varphi}^{\,\alpha}_J(x, u^{(k)}) \frac{\partial}{\partial u^\alpha_J} \in \mathcal{X}(\mathrm{J}^\infty(M,p)) \tag{4.5}$$

denote its infinite prolongation. The hats on the prolonged vector field coefficients $\widehat{\varphi}^{\,\alpha}_J$ serve to distinguish them from the partial derivatives φ^α_B of the vector field coefficients with respect to the independent and dependent variables, as in (3.2). They are explicitly prescribed by the well-known prolongation formula, [**40**; eq. (2.39)], which we review. For each $i = 1, \ldots, p$, let

$$\mathrm{D}_{x^i} = \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{\#J \geq 0} u^\alpha_{J,i} \frac{\partial}{\partial u^\alpha_J} \tag{4.6}$$

denote the *total derivative* on the submanifold jet bundle with respect to the independent variable x^i. Then,

$$\widehat{\varphi}^{\,\alpha}_J = D_J\, Q^\alpha + \sum_{i=1}^{p} u^\alpha_{J,i}\, \xi^i, \qquad \alpha = 1, \ldots, q, \qquad 0 \leq \#J, \tag{4.7}$$

where

$$Q^\alpha(x, u^{(1)}) = \varphi^\alpha(x,u) - \sum_{i=1}^{p} u^\alpha_i\, \xi^i(x,u), \qquad \alpha = 1, \ldots, q, \tag{4.8}$$

are the components of the *characteristic* of the vector field $\mathbf{v}$, and

$$D_J = \mathrm{D}_{x^{j_1}} \cdots \mathrm{D}_{x^{j_k}}, \qquad J = (j_1, \ldots, j_k), \qquad 1 \leq j_\nu \leq p,$$

denotes the corresponding iterated total derivative. Expanding the prolongation formula (4.7) in full, we conclude that each prolonged vector field coefficient

$$\widehat{\varphi}\,^{\alpha}_{J} = \Phi^{\alpha}_{J}(u^{(k)};\zeta^{(k)}) = \Phi^{\alpha}_{J}(\ \ldots\ u^{\beta}_{K}\ \ldots\ ;\ \ldots\ \xi^{i}_{A}\ \ldots\ \varphi^{\beta}_{A}\ \ldots\ ), \qquad \#K, \#A \leq k = \#J, \tag{4.9}$$

is a particular linear combination of the vector field jet coordinates, i.e., the *partial* derivatives $(\ \ldots\ \zeta^{b}_{A}\ \ldots\ ) = (\ \ldots\ \xi^{i}_{A}\ \ldots\ \varphi^{\beta}_{A}\ \ldots\ )$ of the vector field coefficients with respect to both the independent and dependent variables, of orders $1 \leq \#A \leq k = \#J$, of the jet coordinates u^{β}_{K} for $1 \leq \#K \leq k$. Therefore, the n^{th} order prolongation of vector fields factors through the n^{th} order vector field jet bundle. Explicitly, given $z^{(n)} \in \mathrm{J}^{n}(M,p)|_{z}$, we let

$$\mathbf{p}^{(n)} = \mathbf{p}^{(n)}_{z^{(n)}} : \mathrm{J}^{n}TM|_{z} \ \longrightarrow\ T\mathrm{J}^{n}|_{z^{(n)}}, \qquad \mathbf{p}^{(n)}\big(\mathrm{j}_{n}\mathbf{v}|_{z}\big) = \mathbf{v}^{(n)}|_{z^{(n)}}, \tag{4.10}$$

denote the associated *prolongation map* that takes the n–jet of a vector field at the base point $z = \widetilde{\pi}^{n}_{0}(z^{(n)}) \in M$ to the tangent vector $\mathbf{v}^{(n)} \in T\mathrm{J}^{n}|_{z^{(n)}}$ prescribed by the prolongation formula (4.5). We will usually abbreviate the infinite order prolongation map as

$$\mathbf{p} = \mathbf{p}^{(\infty)} : \mathrm{J}^{\infty}TM|_{z} \ \longrightarrow\ T\mathrm{J}^{\infty}|_{z^{(\infty)}}. \tag{4.11}$$

Remark: To extend these results to pseudo-groups of contact transformations — which, by Bäcklund's Theorem, [**3**], only generalize point transformations in the case of codimension 1 submanifolds, i.e., for $q = m - p = 1$ dependent variable — the only significant difference (modulo some minor low order technicalities) in the formalism is that the characteristics of the infinitesimal generators can be more general functions of the first order jet coordinates, cf. [**41**; (4.60)].

We will now develop a symbol algebra for the prolonged infinitesimal generators of a pseudo-group action. We introduce variables $s = (s_1, \ldots, s_p)$, $S = (S^1, \ldots, S^q)$, and let

$$\widehat{\mathcal{S}} = \left\{ \widehat{\sigma}(s,S) = \sum_{\alpha=1}^{q} \widehat{\sigma}_{\alpha}(s)\, S^{\alpha} \right\} \simeq \mathbb{R}[s] \otimes \mathbb{R}^{q} \subset \mathbb{R}[s,S], \tag{4.12}$$

be the $\mathbb{R}[s]$ module consisting of polynomials that are linear in S. Further, define

$$\mathcal{S} = \mathbb{R}^{p} \oplus \widehat{\mathcal{S}} = \bigoplus_{n=-1}^{\infty} \mathcal{S}^{\,n}, \tag{4.13}$$

whose non-negative summands $\mathcal{S}^{\,n} = \widehat{\mathcal{S}}^{\,n}$, for $n \geq 0$, contain all polynomials $\widehat{\sigma}(s,S) \in \widehat{\mathcal{S}}$ that are homogeneous of degree n in s, while, by convention,

$$\mathcal{S}^{-1} = \left\{ c_1 \tilde{s}_1 + \cdots + c_p \tilde{s}_p \;\middle|\; c_i \in \mathbb{R} \right\} \simeq \mathbb{R}^{p},$$

where $\tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_p) \in \mathbb{R}^{p}$ are extra variables, not to be confused with the polynomial variables $s = (s_1, \ldots, s_p)$. Thus, an element $\sigma \in \mathcal{S}$ takes the form

$$\sigma(\tilde{s}, s, S) = c \cdot \tilde{s} + \widehat{\sigma}(s,S) = \sum_{i=1}^{p} c_i \tilde{s}_i + \sum_{\alpha=1}^{q} \widehat{\sigma}_{\alpha}(s)\, S^{\alpha}, \tag{4.14}$$

where $c = (c_1, \ldots, c_p) \in \mathbb{R}^p$ and $\widehat{\sigma}(s,S) \in \widehat{\mathcal{S}}$. We endow $\mathcal{S}$ with the structure of an $\mathbb{R}[s]$ module by taking the usual module structure of $\widehat{\mathcal{S}}$ and then setting

$$\tau(s)\tilde{s}_i = \tau(0)\tilde{s}_i \qquad \text{for any polynomial} \qquad \tau(s) \in \mathbb{R}[s].$$

We define the *highest order term map* $\mathbf{H}\colon \mathcal{S} \to \widehat{\mathcal{S}}$ so that

$$\mathbf{H}[\sigma(\tilde{s},s,S)] = \mathbf{H}\big[\,\widehat{\sigma}(s,S)\,\big], \qquad \text{where} \qquad \sigma(\tilde{s},s,S) = c\cdot\tilde{s} + \widehat{\sigma}(s,S). \tag{4.15}$$

Thus, our convention is that all elements of $\mathcal{S}^{-1}$ have zero highest order term.

For each $n < \infty$, consider the cotangent bundle $T^*\mathrm{J}^n(M,p)$ of the n^{th} order submanifold jet bundle. In local coordinates, we can identify their direct limit

$$T^*\mathrm{J}^\infty(M,p) = \lim_{n\to\infty} T^*\mathrm{J}^n(M,p) \simeq \mathrm{J}^\infty(M,p)\times\mathcal{S}$$

by adopting the explicit pairing

$$\begin{aligned} &\langle\, \mathbf{V}\,;\tilde{s}_i\,\rangle = \xi^i, \qquad &&\langle\, \mathbf{V}\,;S^\alpha\,\rangle = Q^\alpha = \varphi^\alpha - \sum_{i=1}^p u_i^\alpha\,\xi^i,\\ &\langle\, \mathbf{V}\,;s_J S^\alpha\,\rangle = \widehat{\varphi}{}_J^\alpha, \qquad &&\text{for} \quad n = \#J \geq 1, \end{aligned} \tag{4.16}$$

whenever

$$\mathbf{V} = \sum_{i=1}^p \xi^i\,\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \sum_{k=\#J\geq 0} \widehat{\varphi}{}_J^\alpha\,\frac{\partial}{\partial u_J^\alpha} \;\in\; T\mathrm{J}^\infty|_{z^{(\infty)}}$$

is any tangent vector (not necessarily a prolonged vector field) at a point $z^{(\infty)} = (x, u^{(\infty)})$. (The reason for the appearance of the characteristic Q^α, cf. (4.8), at order 0 in (4.16) will become evident in formula (4.25) below.) Every one-form on $\mathrm{J}^\infty(M,p)$, i.e., analytic section of $T^*\mathrm{J}^\infty(M,p)$, is thereby represented, locally, by a parametrized polynomial

$$\sigma(x,u^{(k)};\tilde{s},s,S) \;=\; \sum_{i=1}^p h_i(x,u^{(k)})\,\tilde{s}_i + \sum_{\alpha=1}^q \sum_{\#J\leq n} h_\alpha^J(x,u^{(k)})s_J S^\alpha, \tag{4.17}$$

depending linearly on the variables $(\tilde{s},S) \in \mathbb{R}^m$, polynomially on the variables $s \in \mathbb{R}^p$, and analytically on the jet coordinates $(x,u^{(k)})$ of some finite order $k < \infty$. (And note that the first set of summands does not depend on s.)

In what follows, let us fix a submanifold jet $z^{(\infty)} \in \mathrm{J}^\infty(M,p)$ at the point $\widetilde{\pi}_0^\infty(z^{(\infty)}) = z \in M$. Keeping in mind our identification of the dual spaces to those appearing in (4.10), we let

$$\mathbf{p}^* = (\mathbf{p}^{(\infty)})^*\colon\; \mathcal{S} \;\longrightarrow\; \mathcal{T} \tag{4.18}$$

be the *dual prolongation map*, which is defined so that

$$\langle\, \mathrm{j}_\infty\mathbf{v}\,;\mathbf{p}^*(\sigma)\,\rangle = \langle\,\mathbf{p}(\mathrm{j}_\infty\mathbf{v})\,;\sigma\,\rangle = \langle\,\mathbf{v}^{(\infty)}\,;\sigma\,\rangle \quad \text{for all} \quad \mathrm{j}_\infty\mathbf{v} \in \mathrm{J}^\infty TM|_z, \quad \sigma \in \mathcal{S}. \tag{4.19}$$

In general, $\mathbf{p}^*$ is *not* a module morphism. However, on the symbol level it essentially is, as we now demonstrate.

Consider the particular linear polynomials

$$\begin{aligned} \beta_i(t) &= t_i + \sum_{\alpha=1}^{q} u_i^\alpha\, t_{p+\alpha}, && i = 1, \ldots, p, \\ B^\alpha(T) &= T^{p+\alpha} - \sum_{i=1}^{p} u_i^\alpha\, T^i, && \alpha = 1, \ldots, q, \end{aligned} \tag{4.20}$$

where $u_i^\alpha = \partial u^\alpha / \partial x^i$ are the first order jet coordinates of our point $z^{(\infty)}$. Note that $B^\alpha(T)$ is the symbol of Q^α, the α^{th} component of the characteristic of $\mathbf{v}$. Furthermore, $\beta_i(t)$ represents the symbol† of the i^{th} total derivative operator (4.6), meaning that

$$\boldsymbol{\Sigma}(\mathrm{D}_{x^i} L) = \beta_i(t)\, \boldsymbol{\Sigma}(L), \tag{4.21}$$

for any linear differential polynomial

$$L(x, u^{(n)}; \zeta^{(n)}) = \sum_{b=1}^{m} \sum_{\#A \le n} h_b^A(x, u^{(n)})\, \zeta_A^b.$$

Keep in mind that total differentiation in (4.21) acts on both the submanifold jet variables $(x, u^{(n)})$ and the vector field jets ζ_A^b.

For fixed first order jet coordinates u_i^α, the functions (4.20) serve to define a linear map

$$\boldsymbol{\beta} : \mathbb{R}^{2m} \longrightarrow \mathbb{R}^m, \qquad \text{given by} \qquad s_i = \beta_i(t), \qquad S^\alpha = B^\alpha(T). \tag{4.22}$$

Since $\boldsymbol{\beta}$ has maximal rank, the pull-back map

$$\boldsymbol{\beta}^* \left[\, \widehat{\sigma}(s_1, \ldots, s_p, S^1, \ldots, S^q) \,\right] = \widehat{\sigma}\big(\, \beta_1(t), \ldots, \beta_p(t), B^1(T), \ldots, B^q(T) \,\big) \tag{4.23}$$

defines an injection $\boldsymbol{\beta}^* : \widehat{\mathcal{S}} \to \mathcal{T}$. The key lemma is a direct consequence of the prolongation formula (4.7) combined with (4.21).

Lemma 4.1. *The symbols of the prolonged vector field coefficients are*

$$\begin{aligned} &\boldsymbol{\Sigma}(\xi^i) = T^i, \\ &\boldsymbol{\Sigma}(\varphi^\alpha) = T^{\alpha+p}, && i = 1, \ldots, p, \\ &\boldsymbol{\Sigma}(Q^\alpha) = \boldsymbol{\beta}^*(S^\alpha) = B^\alpha(T), && \alpha = 1, \ldots, q, \\ &\boldsymbol{\Sigma}(\widehat{\varphi}_J^{\,\alpha}) = \boldsymbol{\beta}^*(s_J S^\alpha) = \boldsymbol{\beta}^*(s_{j_1} \cdots s_{j_k} S^\alpha) = \beta_{j_1}(t) \,\cdots\, \beta_{j_k}(t)\, B^\alpha(T), && k = \#J \ge 1. \end{aligned} \tag{4.24}$$

In general, given a polynomial $\widehat{\sigma}(s, S) \in \widehat{\mathcal{S}} \subset \mathcal{S}$, the highest order terms of its pull-back under the prolongation map can be found by pulling back its highest order terms under the linear map (4.22):

$$\mathbf{H}\big[\, \mathbf{p}^*(\widehat{\sigma}) \,\big] = \boldsymbol{\beta}^* \big[\, \mathbf{H}(\widehat{\sigma}) \,\big]. \tag{4.25}$$

---

† We extend the Definition 3.1 of the symbol map to linear differential polynomials whose coefficients also depend on submanifold jet coordinates in the obvious manner.

Remark: The reason for our original definition of the pairing (4.16) was to ensure the general validity of (4.25), including at order 0.

Now given a Lie pseudo-group $\mathcal{G}$ acting on M, let

$$\mathfrak{g}^{(\infty)}|_{z^{(\infty)}} = \mathbf{p}(\mathrm{J}^\infty \mathfrak{g}|_z) \subset T\mathrm{J}^\infty|_{z^{(\infty)}}$$

denote the subspace spanned by its prolonged infinitesimal generators at $z^{(\infty)} \in \mathrm{J}^\infty(M,p)$. Pulling back the annihilator (3.10) by the dual prolongation map produces the *prolonged annihilator subbundle*

$$\mathcal{Z} = (\mathfrak{g}^{(\infty)})^\perp = (\mathbf{p}^*)^{-1}\mathcal{L} \subset \mathcal{S}, \tag{4.26}$$

containing those polynomials (4.14) that annihilate all prolonged infinitesimal generators $\mathbf{v}^{(\infty)} \in \mathfrak{g}^{(\infty)}$. In other words, the prolonged vector field coefficients are subject to the linear constraints

$$\langle\, \mathbf{v}^{(\infty)}\,; \sigma \,\rangle = 0 \quad \text{for all} \quad \mathbf{v} \in \mathfrak{g} \qquad \text{if and only if} \qquad \sigma \in \mathcal{Z} \tag{4.27}$$

is a section of the prolonged annihilator subbundle. In particular, a prolonged pseudo-group acts locally transitively near $z^{(\infty)} \in \mathrm{J}^\infty(M,p)$, and hence has no differential invariants, if and only if $\mathcal{Z} = \{0\}$ is trivial, and so there are no constraints.

Further, define the subspace

$$\mathcal{U} = \mathbf{H}(\mathcal{Z}) \subset \mathcal{S} \tag{4.28}$$

to be spanned by the highest order terms (symbols) of the prolonged annihilators. In general (and, perhaps, surprisingly), $\mathcal{U}$ *is not* a submodule, although it inherits considerable algebraic structure that we intend to exploit.

Definition 4.2. The *prolonged symbol submodule* is defined† as the inverse image of the symbol module (3.11) under the polynomial pull-back morphism (4.23):

$$\mathcal{J} = (\boldsymbol{\beta}^*)^{-1}(\mathcal{I}) = \left\{\, \widehat{\sigma}(s,S) \;\middle|\; \boldsymbol{\beta}^*(\widehat{\sigma})(t,T) = \widehat{\sigma}(\beta(t), B(T)) \in \mathcal{I} \,\right\} \subset \widehat{\mathcal{S}}\,. \tag{4.29}$$

In view of (4.25) and (4.26),

$$\mathcal{U} \subset \mathcal{J}. \tag{4.30}$$

However, these two subspaces *are not* necessarily equal, as can be seen in the following example.

Example 4.3. Consider the Lie pseudo-group $\mathcal{G}$ acting on $M = \mathbb{R}^3$ consisting of all local diffeomorphisms of the form

$$X = a(x), \qquad Y = a'(x)\,y + b(x), \qquad U = u + \frac{a''(x)\,y + b'(x)}{a'(x)}, \tag{4.31}$$

---

† To streamline the notation, we have suppressed the dependence of $\mathcal{J}$ on the first order jets $z^{(1)} \in \mathrm{J}^1(M,p)$.

where $a(x) \in \mathcal{D}(\mathbb{R})$ is an arbitrary local analytic diffeomorphism of $\mathbb{R}$, while $b(x)$ is an arbitrary analytic function. Its infinitesimal generators consist of all vector fields of the form

$$\mathbf{v} = \xi\,\partial_x + \eta\,\partial_y + \varphi\,\partial_u = f(x)\,\partial_x + [\,f'(x)\,y + g(x)\,]\,\partial_y + [\,f''(x)\,y + g'(x)\,]\,\partial_u, \qquad (4.32)$$

where $f(x), g(x)$ are arbitrary analytic functions of a single variable. The linearized determining equations characterizing the infinitesimal generators are obtained by prolonging the first order system

$$\xi_y = \xi_u = 0, \qquad \eta_x = \varphi, \qquad \eta_y = \xi_x, \qquad \eta_u = 0, \qquad \varphi_u = 0. \qquad (4.33)$$

In particular, the additional second order determining equations are

$$\begin{gathered} \xi_{xx} = \varphi_y, \qquad \xi_{xy} = \xi_{yy} = \xi_{xu} = \xi_{yu} = \xi_{uu} = 0, \qquad \eta_{xx} = \varphi_x, \qquad \eta_{xy} = \varphi_y, \\ \eta_{yy} = \eta_{xu} = \eta_{yu} = \eta_{uu} = 0, \qquad \varphi_{yy} = \varphi_{xu} = \varphi_{yu} = \varphi_{uu} = 0. \end{gathered} \qquad (4.34)$$

Let $t_1, t_2, t_3, T^1, T^2, T^3$ be the polynomial variables corresponding to $x, y, u, \xi, \eta, \varphi$, respectively. To second order, then, the annihilator $\mathcal{L}|_z$ at any $z = (x, y, u) \in M$ is spanned by

$$\begin{gathered} t_2 T^1, \quad t_3 T^1, \quad t_1 T^2 - T^3, \quad t_2 T^2 - t_1 T^1, \quad t_3 T^2, \quad t_3 T^3, \\ t_1^2 T^1 - t_2 T^3, \quad t_1 t_2 T^1, \quad t_2^2 T^1, \quad t_1 t_3 T^1, \quad t_2 t_3 T^1, \quad t_3^2 T^1, \quad t_1^2 T^2 - t_1 T^3, \\ t_1 t_2 T^2 - t_2 T^3, \quad t_2^2 T^2, \quad t_1 t_3 T^2, \quad t_2 t_3 T^2, \quad t_3^2 T^2, \quad t_2^2 T^3, \quad t_1 t_3 T^3, \quad t_2 t_3 T^3, \quad t_3^2 T^3. \end{gathered}$$

Adopting the term orderings $t_1 < t_2 < t_3$, $T^1 < T^2 < T^3$, the symbol module $\mathcal{I}|_z$ is generated by the Gröbner basis elements

$$t_2 T^1, \quad t_3 T^1, \quad t_1^2 T^1, \quad t_1 T^2, \quad t_2 T^2 - t_1 T^1, \quad t_3 T^2, \quad t_3 T^3, \quad t_2^2 T^3. \qquad (4.35)$$

The complementary monomials

$$T^1, \quad T^2, \quad T^3, \quad t_1 T^1, \quad t_1 T^3, \quad t_2 T^3, \quad t_1^2 T^3, \quad t_1 t_2 T^3, \quad t_1^3 T^3, \quad t_1^2 t_2 T^3, \quad \ldots,$$

serve to index the free derivatives

$$\xi, \quad \eta, \quad \varphi, \quad \xi_x, \quad \varphi_x, \quad \varphi_y, \quad \varphi_{xx}, \quad \varphi_{xy}, \quad \varphi_{xxx}, \quad \varphi_{xxy}, \quad \ldots,$$

in the infinitesimal determining equations (4.33–34).

We are interested in the action of this pseudo-group on functions $u = f(x, y)$, and hence work on the surface jet bundles $\mathrm{J}^n(M, 2)$. The prolonged infinitesimal generators are, up to order 3,

$$\begin{aligned} \mathbf{v}^{(3)} = {} & \xi\,\partial_x + \eta\,\partial_y + \varphi\,\partial_u + \big(\,\varphi_x - u_x\,\xi_x - u_y\,\varphi\,\big)\,\partial_{u_x} + \big(\,\varphi_y - u_y\,\xi_x\,\big)\,\partial_{u_y} \\ & + \big(\,\varphi_{xx} - u_x\,\varphi_y - u_y\,\varphi_x - 2\,u_{xx}\,\xi_x - 2\,u_{xy}\,\varphi\,\big)\,\partial_{u_{xx}} \\ & + \big(\,\varphi_{xy} - u_y\,\varphi_y - 2\,u_{xy}\,\xi_x - u_{yy}\,\varphi\,\big)\,\partial_{u_{xy}} - 2\,u_{yy}\,\xi_x \partial_{u_{yy}} \\ & + \big(\,\varphi_{xxx} - u_x\,\varphi_{xy} - u_y\,\varphi_{xx} - 3\,u_{xx}\,\varphi_y - 3\,u_{xy}\,\varphi_x - 3\,u_{xxx}\,\xi_x - 3\,u_{xxy}\,\varphi\,\big)\,\partial_{u_{xxx}} \\ & + \big(\,\varphi_{xxy} - u_y\,\varphi_{xy} - 3\,u_{xy}\,\varphi_y - u_{yy}\,\varphi_x - 3\,u_{xxy}\,\xi_x - 2\,u_{xyy}\,\varphi\,\big)\,\partial_{u_{xxy}} \\ & + \big(-2\,u_{yy}\,\varphi_y - 3\,u_{xyy}\,\xi_x - u_{yyy}\,\varphi\,\big)\,\partial_{u_{xyy}} - 3\,u_{yyy}\,\xi_x \partial_{u_{yyy}}, \end{aligned} \qquad (4.36)$$

where we have used the linearized determining equations (4.33–34) to condense the expressions.

We employ s_1, s_2, S to designate the polynomial variables in $\widehat{\mathcal{S}}$ representing x, y, φ, respectively, so that, for any prolonged infinitesimal generator $\mathbf{v}^{(\infty)}$,

$$\langle\, \mathbf{v}^{(\infty)}\,;s_1^j s_2^k S\,\rangle = \begin{cases} \widehat{\varphi}_{jk} = D_x^j D_y^k(\varphi - \xi\, u_x - \eta\, u_y) + \xi\, u_{j+1,k} + \eta\, u_{j,k+1}, & j+k>0, \\ \varphi - \xi\, u_x - \eta\, u_y, & j+k=0, \end{cases}$$

with $u_{jk} = \partial^{j+k}u/\partial x^j \partial y^k$. Continuing to prolong the infinitesimal generators (4.36), we discover that, at each surface jet $z^{(\infty)} = (x, u^{(\infty)}) \in \mathrm{J}^\infty(M,2)$, the prolonged annihilator $\mathcal{Z}$ is spanned by the polynomials

$$\big(\, u_{yy} s_2^3 - \tfrac{3}{2} u_{yyy} s_2^2 \,\big)\, S, \quad \big(\, u_{yy} s_1 s_2^2 - \big(\, \tfrac{3}{2} u_{xyy} + u_y u_{yy}\,\big) s_2^2 + 2\, u_{yy}^2 s_2 + u_{yy} u_{yyy} \,\big)\, S, \quad \ldots\, .$$

The subspace $\mathcal{U}$ is spanned by their highest order terms and so, provided $u_{yy} \neq 0$, is the submodule generated by $s_2^3 S$ and $s_1 s_2^2 S$.

On the other hand, the linear polynomial map (4.22) is defined by

$$s_1 = \beta_1(t_1,t_2,t_3) = t_1 + u_x\, t_3, \qquad s_2 = \beta_2(t_1,t_2,t_3) = t_2 + u_y\, t_3,$$
$$S = B(T^1,T^2,T^3) = T^3 - u_x\, T^1 - u_y\, T^2.$$

Thus, according to (4.29), the prolonged symbol submodule $\mathcal{J} = (\boldsymbol{\beta}^*)^{-1}(\mathcal{I})$ contains all polynomials $\sigma(s_1,s_2,S) = \sigma_1(s_1,s_2)\, S$ such that

$$\boldsymbol{\beta}^*\big[\, \sigma_1(s_1,s_2)\, S\,\big] = \sigma_1(t_1 + u_x\, t_3, t_2 + u_y\, t_3)\, (T^3 - u_x\, T^1 - u_y\, T^2) \in \mathcal{I}|_z.$$

It is not hard to see that $\mathcal{J}$ is generated by the single monomial $s_2^2 S$, which does *not* appear in $\mathcal{U}$, and hence the subspace $\mathcal{U} \subsetneq \mathcal{J}$ is a strict subset of the prolonged symbol submodule.

5. Freeness of Prolonged Pseudo-group Actions.

We are now in a position to study the local freeness of the prolonged pseudo-group action. Let us begin with the infinitesimal version of the basic definition, which is inspired by the well-accepted concept of freeness for a finite-dimensional group action. Details can be found in [**45**].

Definition 5.1. A pseudo-group acts *locally freely* at a submanifold jet $z^{(n)} \in \mathrm{J}^n(M,p)|_z$ whenever the prolongation map $\mathbf{p}^{(n)}\colon \mathrm{J}^n\mathfrak{g}|_z \xrightarrow{\ \sim\ } \mathfrak{g}^{(n)}|_{z^{(n)}}$ is a linear isomorphism.

In the analytic category, local freeness at $z^{(n)}$ implies local freeness in a dense open subset of $\mathrm{J}^n(M,p)$. Definition 5.1 implies that *all* pseudo-group actions are locally free at order 0, and so the condition has nontrivial implications only at the jet level.

Warning: According to the standard definition, [**19**], any locally free action of a finite-dimensional Lie group satisfies the local freeness condition of Definition 5.1, but not necessarily conversely. For instance, the four-dimensional group with infinitesimal generators $\partial_x, \partial_u, x\partial_u, x^2\partial_u$ acting on $M = \mathbb{R}^2$ has locally free action on $\mathrm{J}^n(M,p)$ for all $n \geq 0$ according to Definition 5.1; whereas, in the usual Lie group terminology, the action is only locally free when $n \geq 2$. In this paper, even when dealing with finite-dimensional Lie group actions, we will consistently employ the more general notion of freeness adopted in Definition 5.1.

Freeness imposes a requirement that the bundle of pseudo-group jets $\mathcal{G}^{(n)}$ (or, equivalently, the infinitesimal generator jet bundle $\mathrm{J}^n\mathfrak{g}$) not be too large. Specifically, local freeness at a point of $z^{(n)} \in \mathrm{J}^n(M,p)|_z$ requires the fiber dimensions be bounded by

$$r_n = \dim \mathcal{G}^{(n)}|_z = \dim \mathrm{J}^n\mathfrak{g}|_z \leq \dim \mathrm{J}^n(M,p)|_z = q\binom{p+n}{p}. \tag{5.1}$$

According to Section 3, for sufficiently large $n \gg 0$,

$$r_n = \dim \mathrm{J}^n\mathfrak{g}|_z = \dim \mathcal{C}^{\leq n} = H(n)$$

is characterized by the symbol module's Hilbert polynomial. Thus, by comparing (3.19) and (5.1), we deduce:

Proposition 5.2. *If the symbol module $\mathcal{I}$ has either dimension $d > p$, or else $d = p$ and degree $b > q$, then the pseudo-group cannot act locally freely on $\mathrm{J}^n(M,p)$ for $n \gg 0$ sufficiently large.*

Remark: Theorem 5.4 below will allow us to replace $n \gg 0$ by simply $n > 0$.

In other words, if the pseudo-group is to act locally freely, as required for our moving frame constructions to be valid, the rate of growth of the dimensions of its jet subgroups cannot be too rapid. Intuitively, Proposition 5.2 says that, since we are acting on p-dimensional submanifolds of an $m = p+q$ dimensional space, which are thus parametrized by q functions of p variables, a locally freely acting pseudo-group can itself depend upon arbitrary functions of at most p variables, and at most q functions of exactly p variables. In Kumpera's approach, [**32, 31**], assumptions on the dimensions of $\mathcal{G}^{(n)}$ are encoded in terms of Spencer cohomology groups. Here, the required growth rate assumptions are simply stated in constructive algebraic terms — specifically the dimension and degree of the symbol module. Further details on the interconnections between the algebraic and cohomological approaches can be found in Seiler's forthcoming monograph, [**52**].

Proposition 5.2 merely provides a preliminary dimension bound required for local freeness. In order to fully characterize a locally free action, we must understand the structure of its prolonged infinitesimal generators (4.5) in more detail.

Lemma 5.3. *The pseudo-group $\mathcal{G}$ acts locally freely at $z^{(n)} \in \mathrm{J}^n(M,p)$ if and only if*

$$\mathbf{p}^*(\mathcal{S}^{\leq n}) + \mathcal{L}^{\leq n}|_z = \mathcal{T}^{\leq n}. \tag{5.2}$$

Proof: Note first that $(\mathbf{p}^{(n)})^* = \mathbf{p}^* \,|\, \mathcal{S}^{\leq n}$. In view of (3.10), the condition required for local freeness in Definition 5.1 can be restated as

$$\{0\} = \ker \mathbf{p}^{(n)} \cap \mathrm{J}^n \mathfrak{g}|_z = \bigl(\operatorname{rng}\,(\mathbf{p}^{(n)})^*\bigr)^\perp \cap \bigl(\mathcal{L}^{\leq n}|_z\bigr)^\perp = \bigl(\mathbf{p}^*(\mathcal{S}^{\leq n}) + \mathcal{L}^{\leq n}|_z\bigr)^\perp,$$

from which (5.2) immediately follows. *Q.E.D.*

We are now in a position to prove a key result that guarantees local freeness of higher order prolonged pseudo-group actions.

Theorem 5.4. *If the pseudo-group $\mathcal{G}$ acts locally freely at $z^{(n)} \in \mathrm{J}^n(M,p)$ for some $n > 0$, then, for all $k \geq n$, it also acts locally freely at any jet $z^{(k)} \in \mathrm{J}^k(M,p)$ such that $\widetilde{\pi}^k_n(z^{(k)}) = z^{(n)}$.*

Proof: The proof is by induction on the order $k \geq n \geq 1$. According to Lemma 5.3, local freeness at order k implies that

$$\mathbf{p}^*(\mathcal{S}^{\leq k}) + \mathcal{L}^{\leq k}|_z = \mathcal{T}^{\leq k}. \tag{5.3}$$

Taking highest order terms of both sides and using (4.25), we find that

$$\boldsymbol{\beta}^*(\mathcal{S}^{k}) + \mathcal{I}^{k}|_z = \mathcal{T}^{k}. \tag{5.4}$$

We claim that, to prove freeness at order $k+1$, we only need to show that

$$\boldsymbol{\beta}^*(\mathcal{S}^{k+1}) + \mathcal{I}^{k+1}|_z = \mathcal{T}^{k+1}. \tag{5.5}$$

Indeed, if the latter equality holds, then any $P \in \mathcal{T}^{k+1}$ can be written as

$$P = \boldsymbol{\beta}^*(Q) + Y \qquad \text{for} \qquad Q \in \mathcal{S}^{k+1}, \qquad Y \in \mathcal{I}^{k+1}|_z. \tag{5.6}$$

But then there exists $L \in \mathcal{L}^{\leq k+1}|_z$ with highest order term $\mathbf{H}(L) = Y$, and so

$$L = Y + V \qquad \text{for some} \qquad V \in \mathcal{T}^{\leq k}.$$

Further,

$$\mathbf{p}^*(Q) = \boldsymbol{\beta}^*(Q) + U \qquad \text{for some} \qquad U \in \mathcal{T}^{\leq k}.$$

We use the induction hypothesis (5.3) to write

$$U + V = \mathbf{p}^*(W) + Z \qquad \text{for} \qquad W \in \mathcal{S}^{\leq k}, \qquad Z \in \mathcal{L}^{\leq k}|_z.$$

Thus, comparing with (5.6),

$$P = \mathbf{p}^*(Q) + L - (U+V) = \mathbf{p}^*(Q - W) + (L - Z) \in \mathbf{p}^*(\mathcal{S}^{\leq k+1}) + \mathcal{L}^{\leq k+1}|_z.$$

This completes the induction step.

To prove the claimed equality (5.5), first observe that (5.4) implies that any polynomial $P \in \mathcal{T}^k$ can be written in the form

$$P(t,T) = \boldsymbol{\beta}^*(Q(s,S)) + Y(t,T), \qquad \text{where} \qquad Q \in \mathcal{S}^k, \qquad Y \in \mathcal{I}^k|_z. \tag{5.7}$$

Since $k \geq 1$, we can (non-uniquely) write

$$Q(s,S) = \sum_{i=1}^{p} s_i \, Q_i(s,S) \qquad \text{for some} \qquad Q_i \in \mathcal{S}^{k-1},$$

and hence, substituting back into (5.7),

$$P(t,T) = \sum_{i=1}^{p} \beta_i(t) \, R_i(t,T) + Y(t,T), \quad \text{where} \quad R_i = \boldsymbol{\beta}^*(Q_i) \in \mathcal{T}^{k-1}, \quad Y \in \mathcal{I}^k|_z. \tag{5.8}$$

Every polynomial $\widehat{P} \in \mathcal{T}^{k+1}$ can be written as

$$\widehat{P}(t,T) = \sum_{j=1}^{p+q} t_j P_j(t,T), \qquad \text{for some} \qquad P_1, \ldots, P_{p+q} \in \mathcal{T}^k.$$

Applying (5.8) to each summand $P_j(t,T)$, and using the fact that $\mathcal{I}|_z$ is a submodule, we obtain

$$\widehat{P}(t,T) = \sum_{i=1}^{p} \beta_i(t) \, \widehat{R}_i(t,T) + \widehat{Y}(t,T), \qquad \text{where} \qquad \widehat{R}_i \in \mathcal{T}^k, \qquad \widehat{Y} \in \mathcal{I}^{k+1}|_z. \tag{5.9}$$

On the other hand, by (5.7), we can write each

$$\widehat{R}_i(t,T) = \boldsymbol{\beta}^*(\widehat{X}_i(s,S)) + \widehat{Z}_i(t,T), \qquad \text{where} \qquad \widehat{X}_i \in \mathcal{S}^k, \qquad \widehat{Z}_i \in \mathcal{I}^k|_z.$$

Then, substituting back into (5.9), we find

$$\widehat{P}(t,T) = \boldsymbol{\beta}^* \left(\sum_{i=1}^{p} s_i \, \widehat{X}_i(s,S) \right) + \left(\sum_{i=1}^{p} \beta_i(t) \, \widehat{Z}_i(t,T) + \widehat{Y}(t,T) \right) \in \boldsymbol{\beta}^*(\mathcal{S}^{k+1}) + \mathcal{I}^{k+1}|_z.$$

This validates (5.5), and hence justifies the induction step. *Q.E.D.*

Remark: We are able to prove a strengthened form of Theorem 5.4, that freeness of a pseudo-group action persists under prolongations, [**46**]. However, we will omit the proof as the result plays no role in the present paper.

Now comes the crucial fact: local freeness allows us to replace (4.30) by an equality, thereby bringing some remarkable algebraic structure into the picture.

Lemma 5.5. *Suppose $\mathcal{G}$ acts locally freely at $z^{(n)} \in \mathrm{J}^n(M,p)$. Then*

$$\mathcal{U}^k|_{z^{(k)}} = \mathcal{J}^k|_{z^{(k)}} \tag{5.10}$$

for all† $k > n$ *and all* $z^{(k)} \in \mathrm{J}^k(M,p)$ *with* $\widetilde{\pi}^k_n(z^{(k)}) = z^{(n)}$.

† The result does not hold when $k = n$; see Example 4.3.

Proof: Clearly, by an induction argument, it suffices to prove (5.10) when $k = n + 1$. In view of (4.30), we need to show that if $Q \in \mathcal{J}^{n+1}|_{z^{(n+1)}}$, then there exists $U \in \mathcal{S}^{\leq n}$ such that $Q + U \in \mathcal{Z}^{\leq n+1}|_{z^{(n+1)}}$, and hence $Q = \mathbf{H}(Q + U) \in \mathcal{U}^{n+1}|_{z^{(n+1)}}$. Let $P = \mathbf{p}^*(Q)$. In view of (4.25) and (4.29),

$$\mathbf{H}(P) = \mathbf{H}(\mathbf{p}^*(Q)) = \boldsymbol{\beta}^*(\mathbf{H}(Q)) = \boldsymbol{\beta}^*(Q) \in \mathcal{I}^{n+1}|_z,$$

and hence there exists $Y \in \mathcal{T}^{\leq n}$ such that $P + Y \in \mathcal{L}^{\leq n+1}|_z$. Now, by Lemma 5.3, local freeness implies that $Y = \mathbf{p}^*(U) + V$, where $U \in \mathcal{S}^{\leq n}$ and $V \in \mathcal{L}^{\leq n}|_z$. Thus,

$$\mathbf{p}^*(Q + U) = (P + Y) - V \in \mathcal{L}^{\leq n+1}|_z,$$

which, by (4.26), implies that $Q + U \in \mathcal{Z}^{\leq n+1}|_{z^{(n+1)}}$, as desired. *Q.E.D.*

Thus, while the prolonged symbol fiber $\mathcal{U}|_{z^{(\infty)}}$ is not, in general, a submodule, once the order is high enough — specifically strictly higher than the minimal order $n^\star$ of freeness — it does inherit a submodule structure. We will sometimes refer to $\mathcal{U}|_{z^{(\infty)}}$ as an *eventual submodule*, meaning that $\mathcal{U}^{\geq k}|_{z^{(k)}}$ is a submodule for k sufficiently large, in this case $k > n^\star$. We will subsequently exploit this "eventual" algebraic structure in our analysis of the algebra of differential invariants.

6. Algebraic Cross–Sections.

In general, the construction of a moving frame relies on the choice of a cross-section to the pseudo-group orbits, [**19**, **45**]. For our purposes, a *cross-section* is defined to be submanifold $K^n \subset \mathrm{J}^n(M, p)$ that satisfies the transversality condition

$$TK^n|_{z^{(n)}} \oplus \mathfrak{g}^{(n)}|_{z^{(n)}} = T\mathrm{J}^n(M, p)|_{z^{(n)}}, \qquad \text{for all} \qquad z^{(n)} \in K^n. \tag{6.1}$$

If the pseudo-group acts locally freely, each cross-section will define a locally equivariant moving frame, which is fully equivariant if the action is also free and the cross-section intersects each pseudo-group orbit in at most one point. Since we only deal with local equivariance here, we will ignore the latter, global constraint on the cross-section. Transversality is equivalent to the dual condition

$$(TK^n|_{z^{(n)}})^\perp \oplus \mathcal{Z}^{\leq n}|_{z^{(n)}} = \mathcal{S}^{\leq n}, \qquad \text{for all} \qquad z^{(n)} \in K^n, \tag{6.2}$$

meaning that the annihilator of the tangent space to the cross-section forms a complementary subspace to the prolonged pseudo-group annihilator (4.26).

In keeping with most applications†, we will only consider *coordinate cross-sections*, which are prescribed by setting an appropriate number of the jet coordinates $(x, u^{(n)})$ to assigned constant values. We will identify their differentials with monomials in $\mathcal{S}$, so that

$$dx^i \longleftrightarrow \tilde{s}_i, \qquad du^\alpha_J \longleftrightarrow s_J S^\alpha. \tag{6.3}$$

† The work of Mansfield, [**35**], is a notable exception. See also the cases of equi-affine, conformal and projective surfaces treated in [**27**, **43**]. The present methods can be straightforwardly adapted to more general cross-sections.

Thus, in view of (6.2), the coordinate cross-sections passing through a submanifold jet $z_0^{(n)} \in \mathrm{J}^n(M,p)$ are in one-to-one correspondence with monomial complements $\mathcal{K}^{\leq n}$ to the prolonged pseudo-group annihilator:

$$\mathcal{K}^{\leq n} \oplus \mathcal{Z}^{\leq n}|_{z_0^{(n)}} = \mathcal{S}^{\leq n}. \tag{6.4}$$

Indeed, if $\mathcal{K}^{\leq n}$ is spanned by the monomials $\tilde{s}_j, s_K S^\beta$, then the corresponding coordinate cross-section is

$$x^j = c^j, \qquad u_K^\beta = c_K^\beta, \qquad \text{for all} \qquad \tilde{s}_j,\ s_K S^\beta \in \mathcal{K}^{\leq n}, \tag{6.5}$$

where the constants $c^j = x_0^j,\ c_K^\beta = u_{K,0}^\beta$, are merely the values of the coordinate functions at the point $z_0^{(n)}$.

From here on, we fix a regular submanifold jet $z_0^{(\infty)} \in \mathrm{J}^\infty(M,p)$, and use $z_0^{(n)} = \widetilde{\pi}_n^\infty(z_0^{(\infty)})$ to denote its n^{th} order truncation. Theorem 5.4 guarantees that, for every $n > n^\star$, where $n^\star$ denotes the order of freeness of $\mathcal{G}$, the pseudo-group $\mathcal{G}$ acts locally freely at $z_0^{(n)} \in \mathrm{J}^n(M,p)$. Our intention is to construct a cross-section and associated moving frame in a suitable neighborhood of each $z_0^{(n)}$.

We will algorithmically construct a monomial complement $\mathcal{K} \subset \mathcal{S}$ to the prolonged annihilator $\mathcal{Z}|_{z_0^{(\infty)}}$ as follows. The first step is to fix a degree compatible term ordering† on the polynomial module $\widehat{\mathcal{S}}$, which we extend to $\mathcal{S}$ by making the extra monomials $\tilde{s}_i$ appear before all the others. Let $\mathcal{N}$ be the monomial subspace generated by the leading monomials of the polynomials in $\mathcal{Z}|_{z_0^{(\infty)}}$, or, equivalently, the prolonged symbol polynomials in $\mathcal{U}|_{z_0^{(\infty)}}$, cf. (4.28). Then $\mathcal{K}$ will be the *complementary monomial subspace* spanned by all monomials in $\mathcal{S}$ that are *not* in $\mathcal{N}$. To construct $\mathcal{K}$ by a finite algorithm, we use Lemma 5.5 to identify the higher order prolonged symbol spaces $\mathcal{U}^k|_{z_0^{(k)}}$ with the submodule $\mathcal{J}^k|_{z_0^{(k)}}$ for $k > n^\star$. Therefore, the leading monomials in the Gröbner basis for the submodule $\mathcal{J}|_{z_0^{(\infty)}}$ will completely prescribe the monomial subspaces $\mathcal{N}^k$ — and hence their complements $\mathcal{K}^k$ — for all $k > n^\star$. The lower order monomials in $\mathcal{N}^{\leq n^\star}$ are the leading monomials of a basis for the finite-dimensional space $\mathcal{Z}^{\leq n^\star}|_{z^{(n^\star)}}$ (or, equivalently, $\mathcal{U}^{\leq n^\star}|_{z^{(n^\star)}}$). Finally, $\mathcal{K}^{\leq n^\star}$ is the monomial complement to $\mathcal{N}^{\leq n^\star}$ in $\mathcal{S}^{\leq n^\star}$. If $\mathcal{G}$ does not act transitively on M, then $\mathcal{K}^{\leq n^\star}$ may include some of the order -1 monomials $\tilde{s}_j$ required to form a coordinate cross-section to the pseudo-group orbits on M.

For each $n^\star \leq n < \infty$, we let $K^n \subset \mathrm{J}^n(M,p)$ denote the coordinate cross-section passing through $z_0^{(n)}$ determined by the monomials in $\mathcal{K}^{\leq n}$, under the identification (6.3). Note that these cross-sections are compatible in the sense that $\widetilde{\pi}_n^k(K^k) = K^n$ for any $k \geq n \geq n^\star$. We will call the resulting direct limit, denoted by $K \subset \mathrm{J}^\infty(M,p)$, an *algebraic cross-section* to the prolonged pseudo-group orbits. In the next section, we will

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† In practical implementations of the algorithm, there are, presumably, advantages to choosing the term ordering on $\widehat{\mathcal{S}}$ to be appropriately "compatible" with the original term ordering on $\mathcal{T}$ under the linear pull-back map $\boldsymbol{\beta}^*$. However, this aspect remains to be fully explored.

use the algebraic cross-section K to construct an "algebraic moving frame" through the normalization process. We note that algebraic cross-sections satisfy the pseudo-group version of the minimal order cross-sections used in [**42**] to prove a corrected version of the theorem in [**19**] on generating differential invariants for finite-dimensional group actions.

Remark: It is worth mentioning that there are now two different identifications of one-forms on $\mathrm{J}^\infty(M,p)$ with polynomials in $\widehat{\mathcal{S}}$. The first, coming from the pairing (4.16), is used to construct $\mathcal{Z}$, $\mathcal{N}$, and hence $\mathcal{K}$. However, the resulting monomials in $\mathcal{K}$ are identified with a cross-section K via the more straightforward formulae (6.3). The choice of identification should be clear from the context.

Example 6.1. Consider the pseudo-group treated in Example 4.3. For the prolonged action on the surface jet bundles $\mathrm{J}^n(\mathbb{R}^3,2)$, the order of freeness is $n^\star = 2$. As noted above, the prolonged symbol submodule $\mathcal{J}$ is generated by $s_2^2 S$. Thus, for $k > n^\star = 2$, the leading monomial subspace $\mathcal{N}^k$ is spanned by $s_1^{k-i}\, s_2^i\, S$ for $i \geq 2$, and so its monomial complement $\mathcal{K}^k$ is spanned by $s_1^k S, s_1^{k-1} s_2\, S$. To complete the algebraic cross-section $\mathcal{K}$, since the low degree component of the annihilator is trivial, $\mathcal{Z}^{\leq 2} = \{0\}$, the low degree part of the complement, $\mathcal{K}^{\leq 2}$, is spanned by $\widetilde{s}_1, \widetilde{s}_2, S, s_1\, S, s_2\, S, s_1^2 S, s_1\, s_2\, S, s_2^2 S$. Choosing the regular jet $z_0^{(\infty)} \in \mathrm{J}^\infty(\mathbb{R}^3,2)$ with coordinates $x = y = u = u_x = u_y = u_{xx} = u_{xy} = 0$, $u_{yy} = 1$, $u_{x^k y^l} = 0$, $k + l \geq 3$, the monomial complement $\mathcal{K}$ corresponds to the coordinate cross-section

$$x = y = u = u_x = u_y = 0, \;\; u_{xx} = u_{xy} = 0, \;\; u_{yy} = 1, \;\; u_{x^k} = u_{x^{k-1}y} = 0, \;\; \text{for} \;\; k \geq 3, \tag{6.6}$$

that was used in [**45**; Example 11].

7. Moving Frames and Invariantization.

We next recall our definition, [**44, 45**], of a moving frame for the prolonged pseudo-group action on submanifolds. For each $n \geq 0$, let $\mathcal{D}^{(n)} = \mathcal{D}^{(n)}(M) \subset \mathrm{J}^n(M,M)$ denote the bundle or, more specifically, groupoid consisting of n^{th} order jets of local diffeomorphisms $\varphi\colon M \to M$, with *source map* $\boldsymbol{\sigma}^{(n)}\bigl(\mathrm{j}_n\varphi|_z\bigr) = z$ and *target map* $\boldsymbol{\tau}^{(n)}\bigl(\mathrm{j}_n\varphi|_z\bigr) = \varphi(z) = Z$. As above, let $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ denote the subbundle (sub-groupoid) consisting of all n-jets $\mathrm{j}_n\varphi$ of pseudo-group diffeomorphisms $\varphi \in \mathcal{G}$. Let $\mathcal{H}^{(n)} \subset \mathcal{E}^{(n)} \to \mathrm{J}^n(M,p)$ denote the bundles obtained by pulling back $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)} \to M$ via the projection $\widetilde{\pi}^n_0\colon \mathrm{J}^n(M,p) \to M$. Points $(z^{(n)}, g^{(n)}) \in \mathcal{E}^{(n)}$ consist of a submanifold jet $z^{(n)} \in \mathrm{J}^n(M,p)|_z$ along with a diffeomorphism jet $g^{(n)} \in \mathcal{D}^{(n)}|_z$ based at the same point $z = \widetilde{\pi}^n_0(z^{(n)}) = \boldsymbol{\sigma}^{(n)}(g^{(n)})$. The bundle $\mathcal{E}^{(n)}$ inherits a groupoid structure, and $\mathcal{H}^{(n)}$ is a sub-groupoid: The source map $\widetilde{\boldsymbol{\sigma}}^{(n)}\colon \mathcal{E}^{(n)} \to \mathrm{J}^n(M,p)$ is projection, $\widetilde{\boldsymbol{\sigma}}^{(n)}(z^{(n)}, g^{(n)}) = z^{(n)}$, while the target map $\widetilde{\boldsymbol{\tau}}^{(n)}\colon \mathcal{E}^{(n)} \to \mathrm{J}^n(M,p)$ can be identified with the prolonged action of diffeomorphisms on submanifold jets: $\widetilde{\boldsymbol{\tau}}^{(n)}(z^{(n)}, g^{(n)}) = g^{(n)} \cdot z^{(n)}$.

Definition 7.1. A *moving frame* $\rho^{(n)}$ of *order* n is a $\mathcal{G}^{(n)}$–equivariant local section of the bundle $\mathcal{H}^{(n)} \to \mathrm{J}^n(M,p)$, meaning that

$$\rho^{(n)}(g^{(n)} \cdot z^{(n)}) = \rho^{(n)}(z^{(n)}) \cdot (g^{(n)})^{-1}, \tag{7.1}$$

for all $g^{(n)} \in \mathcal{G}^{(n)}|_z$ with $z = \widetilde{\pi}^n_0(z^{(n)})$, such that both $z^{(n)}$ and $g^{(n)} \cdot z^{(n)}$ lie in the domain of definition of $\rho^{(n)}$.

Once we have fixed an algebraic cross-section, at each order n, the induced moving frame maps are constructed by the method of normalization of pseudo-group parameters, and hence are mutually compatible under the projections $\widetilde{\pi}^k_n\colon \mathrm{J}^k(M,p) \to \mathrm{J}^n(M,p)$, as described in detail in [**10**, **45**]. Briefly, let

$$\widehat{U}^\alpha_J = \widehat{F}^\alpha_J(x,u^{(k)},g^{(k)}), \qquad \alpha = 1,\ldots,q, \qquad 0 \le k = \#J \le n, \tag{7.2}$$

be the explicit formulas for the prolonged action of $\mathcal{G}^{(n)}$ on the fiber coordinates of $\mathrm{J}^n(M,p)$, where $g^{(k)}$ represent pseudo-group parameters of order $\le k$, that is, the k^{th} order truncation of the local fiber coordinates $g^{(\infty)}$ on the pseudo-group jet bundle $\mathcal{G}^{(\infty)}$. As in [**45**], we place hats over the transformed submanifold jet coordinates to avoid confusion with the diffeomorphism jet coordinates. The *normalization equations* corresponding to the coordinate cross-section (6.5) are

$$F^j(x,u,g^{(0)}) = c_j, \qquad \widehat{F}^\beta_K(x,u^{(k)},g^{(k)}) = c^\beta_K, \qquad \text{for all} \qquad \tilde{s}_j,\ s_K S^\beta \in \mathcal{K}^{\le n}. \tag{7.3}$$

Since, when $n \ge n^\star$, we are dealing with a bona fide cross-section, the Implicit Function Theorem guarantees that, near the identity jet, we can uniquely solve the normalization equations for the pseudo-group parameters

$$g^{(n)} = \widetilde{\rho}^{(n)}(x,u^{(n)}). \tag{7.4}$$

These formulas serve to prescribe the locally equivariant moving frame section

$$\rho^{(n)}(z^{(n)}) = (z^{(n)}, \widetilde{\rho}^{(n)}(z^{(n)})) \in \mathcal{H}^{(n)}.$$

Additional details and explicit examples can be found in [**10**, **45**].

Once constructed, the moving frame induces an invariantization process, mapping differential functions to differential invariants, differential forms to invariant differential forms, and so on. Invariantization is effected by replacing the pseudo-group parameters in a transformed object by their moving frame normalizations (7.4). Thus, the invariantization process

$$\iota\colon F(x,u^{(n)}) \longmapsto I(x,u^{(n)}) = F\big(\,\rho^{(n)}(x,u^{(n)}) \cdot (x,u^{(n)})\,\big) \tag{7.5}$$

maps the differential function F to the differential invariant $I = \iota(F)$. Geometrically, invariantization amounts to restricting the function to the cross-section, and then imposing invariance by requiring it to be constant along the pseudo-group orbits. As a result, ι defines an algebra morphism that projects the algebra of differential functions onto the algebra of differential invariants.

In particular,

$$\iota(x^i) = H^i, \qquad \iota(u^\alpha_J) = I^\alpha_J, \qquad i = 1,\ldots,n, \qquad \alpha = 1,\ldots,q, \qquad 0 \le \#J \le n, \tag{7.6}$$

will denote the *normalized differential invariants* obtained by invariantizing the submanifold jet coordinates on $\mathrm{J}^n(M,p)$. We use

$$\mathbf{I}^{(n)} = (H, I^{(n)}) = (\ \ldots\ H^i\ \ldots\ I^\alpha_J\ \ldots\ ) = \iota(x,u^{(n)}) = \iota(z^{(n)}) \tag{7.7}$$

to denote the complete collection of normalized differential invariants of order $\leq n$. Thus, invariantization of a differential function

$$F(z^{(n)}) = F(x, u^{(n)}) = F(\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ )$$

amounts to replacing each submanifold jet variable by its associated normalized differential invariant:

$$\iota(F) = F(\mathbf{I}^{(n)}) = F(H, I^{(n)}) = F(\ \ldots\ H^i\ \ldots\ I^\alpha_J\ \ldots\ ). \tag{7.8}$$

In particular, since differential invariants are not affected by the invariantization process, they have *precisely the same functional formula when written in terms of the normalized differential invariants*:

$$I(x, u^{(n)}) = \iota(I(x, u^{(n)})) = I(H, I^{(n)}) \qquad \text{whenever } I \text{ is a differential invariant.} \tag{7.9}$$

This trivial, but extremely useful fact, is known as the Replacement Theorem, [**19**].

Each normalized differential invariant is indexed by a monomial in $\mathcal{S}$, so H^i corresponds to $\tilde{s}_i$, while I^α_J corresponds to $s_J S^\alpha$. The complementary monomials indexing the cross-section coordinates (6.5) correspond to the constant *phantom differential invariants*, whose values equal the normalization constants:

$$H^j = c^j, \qquad I^\beta_K = c^\beta_K, \qquad \text{for all} \qquad \tilde{s}_j,\ s_K S^\beta \in \mathcal{K}. \tag{7.10}$$

The remaining monomials index the *basic differential invariants*

$$\mathbf{I}_{basic} = (\ \ldots\ H^i\ \ldots\ I^\alpha_J\ \ldots\ ), \qquad \text{for all} \qquad \tilde{s}_i,\ s_J S^\alpha \in \mathcal{N}. \tag{7.11}$$

Remark: If $\mathcal{G}$ acts transitively on $\mathrm{J}^k(M,p)$, then there are no non-constant differential invariants of order $\leq k$. In this case, every monomial in $\mathcal{S}$ of degree $\leq k$ belongs to $\mathcal{K}^{\leq k}$ and thus, because we are using an algebraic moving frame, corresponds to a phantom differential invariant.

Theorem 7.2. *Any differential invariant I defined near the cross-section can be locally uniquely written as a function $I = F(\mathbf{I}_{basic})$ of the functionally independent basic differential invariants* (7.11).

Definition 7.3. The *degree* of a basic differential invariant is defined to be $\deg I^\alpha_J = \#J$ for $s_J S^\alpha \in \mathcal{N}$, while $\deg H^i = 0$ for $\tilde{s}_i \in \mathcal{N}$. More generally, the *degree* of a differential invariant I is defined as

$$\deg I = \max\left(\{0\} \cup \left\{ \#J \ \middle|\ \frac{\partial I}{\partial I^\alpha_J} \not\equiv 0 \quad \text{for} \quad s_J S^\alpha \in \mathcal{N} \right\} \right). \tag{7.12}$$

Note that, for any $n \geq n^\star$, the basic differential invariants $\mathbf{I}^{(n)}_{basic}$ of degree $\leq n$ form a complete system of *functionally independent* differential invariants of order $\leq n$.

A differential invariant will be said to have *low* degree if $\deg I \leq n^\star$ and *high degree* if $\deg I \geq n^\star + 1$, where, as always, $n^\star$ denotes the order of freeness. In general, the order (meaning the highest order of jet coordinate u^α_J that it depends on) of a differential invariant equals its degree when it is of high degree; for the low degree invariants, the best that can be said is that their order is at most the order of freeness.

Lemma 7.4. *If I is a high degree differential invariant, then* $\deg I = \operatorname{ord} I > n^\star$. *On the other hand, if I is a low degree differential invariant, then all that can be said is that both* $\deg I$ *and* $\operatorname{ord} I \le n^\star$.

Proof: Whenever $n \ge n^\star$, the n^{th} order moving frame map (7.1) has order n. Thus, the order of any basic differential invariant I^α_J of degree $\#J = n$ is $\le \max\{n, n^\star\}$. Therefore, to establish the result we only need check that in the high degree case $\#J = n > n^\star$, we have $\operatorname{ord} I^\alpha_J = n$. But, when restricted to the cross-section, $I^\alpha_J \,|\, K^n = u^\alpha_J$ equals the corresponding non-constant jet coordinate, and hence must be of order exactly n. *Q.E.D.*

If $\mathcal{G}$ represents the action of a finite-dimensional Lie group, then, for n greater than the classical order of freeness, $\mathcal{N}^n = \widehat{\mathcal{S}}^{\,n}$ and $\mathcal{K}^n = \{0\}$, since we are working with a minimal order cross-section, and hence all group parameters have been normalized once we reach the order of freeness. In this case, the functionally independent differential invariants of sufficiently high order $n \gg 0$ are in one-to-one correspondence with *all* monomials in $\mathcal{S}^{\,n}$. On the other hand, for infinite-dimensional pseudo-groups, phantom differential invariants occur at all orders, but, assuming we choose an algebraic moving frame, those of order strictly greater than $n^\star$ are indexed by the leading monomials of polynomials in the prolonged symbol module $\mathcal{J}$.

To proceed further, let

$$\omega^i = \pi_H \big[\, \iota(dx^i) \,\big], \qquad i = 1, \ldots, p, \tag{7.13}$$

be the horizontal components of the invariantized one-forms dx^i; see [**10**, **45**] for details on the construction. Bear in mind that, since we have discarded their contact components, each ω^i is only invariant modulo contact forms when the group acts non-projectably. Thus, the collection (7.13) forms a *contact-invariant coframe*, cf. [**41**].

The *horizontal* (or *total*) *differential* of a differential function $F(x, u^{(n)})$ is given by

$$d_H\, F = \sum_{i=1}^{p} (\mathrm{D}_{x^i} F)\; dx^i = \sum_{i=1}^{p} (\mathcal{D}_i F)\, \omega^i, \tag{7.14}$$

where the second expression serves to define the invariant total differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$ that are dual to the contact-invariant coframe. In view of (7.13),

$$\mathcal{D}_i = \sum_{i=1}^{p} P^k_i \, \mathrm{D}_{x^k}, \qquad \text{where the coefficients satisfy} \qquad \operatorname{ord} P^k_i \le n^\star. \tag{7.15}$$

Consequently, for any differential function $F(x, u^{(n)})$,

$$\operatorname{ord} \mathcal{D}_i F \le \max\{\, n^\star, \operatorname{ord} F + 1\,\}. \tag{7.16}$$

As in [**45**], we will extend the invariantization process (7.5) to include the derivatives (jets) of vector field coefficients (3.2). Each vector field jet coordinate ζ^b_A serves to define a linear function on the space of vector fields $\mathcal{X}(M)$, and so could be regarded as a kind of covector or one-form. Its *horizontal invariantization* will be an invariant linear combination of the invariant horizontal one-forms (7.13), viz.

$$\gamma_A^b \equiv \pi_H \big[\, \iota(\zeta_A^b) \,\big] = \sum_{i=1}^{p} R_{A,i}^b \, \omega^i, \tag{7.17}$$

whose coefficients $R_{A,i}^b$ are certain differential invariants. In [**45**], these one-forms were identified as the horizontal components of the pull-backs, via the moving frame map, of the Maurer–Cartan forms for the pseudo-group, and are thus called the (horizontal) *invariantized Maurer–Cartan forms*. They are collectively denoted as

$$\gamma^{(n)} = (\ \ldots\ \gamma_A^b\ \ldots\ ) = \iota(\mathrm{j}_n \mathbf{v}). \tag{7.18}$$

In view of this identification, we will refer to the coefficients $R_{A,i}^b$ in (7.17) as the *Maurer–Cartan invariants*, [**25**, **27**, **42**, **43**]. Fortunately, we need not dwell on their underlying theoretical justification because, as we will establish in Lemma 8.3 below, *the explicit formulae* (7.17) *will be directly deduced from the recurrence formulae for the phantom differential invariants.*

Remark: As in (7.13), we will suppress the contact components of the invariantized Maurer–Cartan one-forms here. These do play a role in applications in the invariant calculus of variations, [**30**], but not in the structure of the differential invariant algebra, which is the present object of study.

More generally, let

$$L(x, u^{(n)}, \zeta^{(n)}) = \sum_{b=1}^{m} \sum_{\#A \leq n} h_b^A(x, u^{(n)})\, \zeta_A^b \tag{7.19}$$

be the local coordinate expression for a section of the pull-back of the bundle $(\mathrm{J}^n TM)^*$ to $\mathrm{J}^n(M,p)$ via the standard projection $\widetilde{\pi}_0^n \colon \mathrm{J}^n(M,p) \to M$. Its invariantization will be defined as the corresponding invariant linear combination

$$\iota\big(\, L(x, u^{(n)}, \zeta^{(n)})\,\big) = L(H, I^{(n)}, \gamma^{(n)}) = \sum_{b=1}^{m} \sum_{\#A \leq n} h_b^A(H, I^{(n)})\, \gamma_A^b \tag{7.20}$$

of invariantized Maurer–Cartan forms.

The Maurer–Cartan forms of a proper sub-pseudo-group $\mathcal{G} \subsetneq \mathcal{D}$ are not linearly independent. Remarkably, [**44**; Theorem 6.1], their dependencies are entirely prescribed by the pseudo-group's linear determining equations. These dependencies carry over to their invariantized counterparts (7.17). See [**45**] for a proof of this key result.

Theorem 7.5. *The invariantized Maurer–Cartan forms* (7.17) *are subject to the linear constraints*

$$L^{(n)}(H, I, \gamma^{(n)}) = L^{(n)}(\ \ldots\ H^i\ \ldots\ I^\alpha\ \ldots\ \gamma_A^b\ \ldots\ ) = 0, \tag{7.21}$$

obtained by invariantizing the linear determining equations (3.8) *for the pseudo-group.*

If $\mathcal{G}$ acts transitively on M, then the order zero differential invariants H^i, I^α are all constant, and so the Maurer–Cartan constraints (7.21) form a system of constant coefficient linear equations for the invariantized Maurer–Cartan forms. Intransitive actions are slightly more subtle, but still handled effectively by our approach.

8. Recurrence Formulae.

If I is any differential invariant, then so are its derivatives $\mathcal{D}_1 I, \ldots, \mathcal{D}_p I$, with respect to the invariant differential operators (7.15). In particular, the invariant derivatives of all the normalized differential invariants (7.6) are also differential invariants, and hence, by Theorem 7.2, can be locally re-expressed as functions of the normalized differential invariants. The resulting expressions

$$\mathcal{D}_i H^j = F^{\,j}_{\,i}(\;\ldots\; H^k \;\ldots\; I^\beta_K \;\ldots\;), \qquad\qquad \mathcal{D}_i I^\alpha_J = F^{\,\alpha}_{\,J,i}(\;\ldots\; H^k \;\ldots\; I^\beta_K \;\ldots\;), \qquad (8.1)$$

are known as *recurrence formulae*. The recurrence formulae are the master key that unlocks the structure of the differential invariant algebra $\mathcal{I}(\mathcal{G})$. Strikingly, they can be algorithmically determined using only linear algebra and differentiation, [**45**]. The only required ingredients are the choice of cross-section and the expressions for the infinitesimal determining equations for the pseudo-group. The construction does *not* require knowledge of the explicit formulas for either the pseudo-group transformations, or the moving frame, or even the differential invariants and invariant differential operators!

The recurrence formulae for the differentiated invariants are, in fact, particular consequences of a *universal recurrence formula* for the (horizontal) differential of any invariantized differential function.

Theorem 8.1. *If $F(x, u^{(n)})$ is any differential function, then*

$$d_H\, \iota(F) = \iota\big(\, d_H\, F + \mathbf{v}^{(\infty)}(F)\,\big), \qquad (8.2)$$

where $\mathbf{v}^{(\infty)}$ denotes the infinite prolongation (4.5) *of the vector field $\mathbf{v}$ in* (4.4).

Remark: The recurrence formula (8.2) also applies as stated when F represents a (horizontal) differential form; in this case the final term (8.5) represents its Lie derivative with respect to the prolonged vector field. Full details, including a proof of this key formula can be found in [**45**]. See also [**19**, **42**] for slightly different formulations for finite-dimensional group actions.

Let us interpret the three terms appearing in the recurrence formula (8.2). Using the final expression in (7.14), the left hand side is

$$d_H\, \iota(F) = \sum_{i=1}^{p} \mathcal{D}_i \big[\, \iota(F)\,\big]\, \omega^i. \qquad (8.3)$$

Similarly, using the middle expression in (7.14), the first term on the right hand side is

$$\iota(\, d_H\, F) = \sum_{i=1}^{p} \iota(\mathrm{D}_{x^i} F)\, \omega^i. \qquad (8.4)$$

The final term is obtained by invariantizing the action of the prolonged infinitesimal generator:

$$\mathbf{v}^{(n)}(F) = \sum_{i=1}^{p} \frac{\partial F}{\partial x^j}\, \xi^j + \sum_{\alpha=1}^{q} \sum_{\#J \leq n} \frac{\partial F}{\partial u^\alpha_J}\, \widehat{\varphi}^{\,\alpha}_J. \tag{8.5}$$

According to the prolongation formula (4.9), each prolonged vector field coefficient $\xi^j, \widehat{\varphi}^{\,\alpha}_J$ is a well-prescribed linear combination of the vector field jet coordinates ξ^i_A, φ^β_A. Thus, in accordance with (7.20), its invariantization is the corresponding invariant linear combination of the invariantized Maurer–Cartan forms (7.18). We denote these invariantized combinations by

$$\chi^j = \iota(\xi^j), \qquad \widehat{\psi}^{\,\alpha}_J = \iota(\widehat{\varphi}^{\,\alpha}_J) = \Phi^\alpha_J(I^{(n)}; \gamma^{(n)}), \tag{8.6}$$

where Φ^α_J is the coefficient function of the universal prolongation prescribed in (4.9). Observe that $\chi^j = \iota(\xi^j)$ and $\psi^\alpha = \widehat{\psi}^{\,\alpha} = \iota(\widehat{\varphi}^{\,\alpha})$ are order zero invariantized Maurer–Cartan forms, while, for $n = \#J \geq 1$, each $\widehat{\psi}^{\,\alpha}_J$ depends polynomially on the normalized differential invariants I^β_K, $1 \leq \#K \leq n$, and linearly on the basis invariantized Maurer–Cartan forms γ^b_A, $1 \leq \#A \leq n$.

In view of (4.10), the formulas (8.6) serve to define the *invariantized prolongation maps*

$$\widetilde{\mathbf{p}}^{(n)} \colon \gamma^{(n)} \longmapsto \psi^{(n)} = (\ \ldots\ \chi^j\ \ldots\ \widehat{\psi}^{\,\alpha}_J\ \ldots\ ) \qquad \text{for} \qquad \#J \leq n, \tag{8.7}$$

of orders $0 \leq n \leq \infty$, and, as above, we abbreviate $\widetilde{\mathbf{p}} = \widetilde{\mathbf{p}}^{(\infty)}$. Since the coefficients appearing in $\widetilde{\mathbf{p}}^{(n)}$ agree with those of the ordinary prolongation map $\mathbf{p}^{(n)}$ when restricted to the cross-section, local freeness, as per Definition 5.1, and (7.21), immediately imply the following.

Lemma 8.2. *For any $n \geq n^\star$, the order n invariantized prolongation map $\widetilde{\mathbf{p}}^{(n)}$ defines a monomorphism on the space of invariantized Maurer–Cartan forms:* $\ker \widetilde{\mathbf{p}}^{(n)} = \{\mathbf{0}\}$.

As a result, the final term in the universal recurrence formula (8.2) is

$$\iota\big(\,\mathbf{v}^{(n)}(F)\,\big) = \sum_{i=1}^{p} \iota\left(\frac{\partial F}{\partial x^j}\right)\chi^j \ + \sum_{\alpha=1}^{q} \sum_{\#J \leq n} \iota\left(\frac{\partial F}{\partial u^\alpha_J}\right)\widehat{\psi}^{\,\alpha}_J, \tag{8.8}$$

where each term is a linear combination of the invariantized Maurer–Cartan forms γ^b_A. Thus, after substituting (7.17), each of the one-forms (8.6) appearing in (8.8) is a linear combination of the invariantized horizontal one-forms (7.13):

$$\chi^j = \sum_{i=1}^{p} M^j_i\, \omega^i, \qquad \widehat{\psi}^{\,\alpha}_J = \sum_{i=1}^{p} M^\alpha_{J,i}\, \omega^i. \tag{8.9}$$

The coefficients $M^j_i, M^\alpha_{J,i}$ are differential invariants, and are certain invariant linear combinations of the as yet unknown Maurer–Cartan invariants $R^b_{A,i}$. Let us substitute the formulae (8.3, 4, 8, 9) into (8.2). Equating the resulting coefficients of the individual invariant horizontal forms ω^i produces the complete system of recurrence formulae

$$\mathcal{D}_i\, \iota(F) = \iota(\mathrm{D}_{x^i} F) + M_i[F], \tag{8.10}$$

in which the *correction term* is

$$M_i[F] = \sum_{i=1}^{p} \iota\left(\frac{\partial F}{\partial x^j}\right) M_i^j \; + \sum_{\alpha=1}^{q} \sum_{\#J \leq n} \iota\left(\frac{\partial F}{\partial u_J^\alpha}\right) M_{J,i}^\alpha. \tag{8.11}$$

In particular, taking F in (8.2) to be each of the jet coordinate functions results in the explicit recurrence formulae for the (horizontal) differentials of the normalized invariants:

$$\begin{aligned} d_H H^j &= \iota\left(dx^j + \xi^j\right) = \omega^j + \chi^j = \omega^j + \sum_{i=1}^{p} M_i^j\, \omega^i, \\ d_H I_J^\alpha &= \iota\left(du_J^\alpha + \widehat{\varphi}{}_J^\alpha\right) = \sum_{i=1}^{p} I_{J,i}^\alpha\, \omega^i + \widehat{\psi}{}_J^\alpha = \sum_{i=1}^{p} (I_{J,i}^\alpha + M_{J,i}^\alpha)\, \omega^i. \end{aligned} \tag{8.12}$$

The individual coefficients of the horizontal one-forms $\omega^1, \ldots, \omega^p$ in (8.12) implies that the system of *recurrence relations* (8.1) takes the form

$$\mathcal{D}_i H^j = \delta_i^j + M_i^j, \qquad\qquad \mathcal{D}_i I_J^\alpha = I_{J,i}^\alpha + M_{J,i}^\alpha, \tag{8.13}$$

where δ_i^j is the Kronecker delta, and the *correction terms* $M_i^j, M_{J,i}^\alpha$ are the as yet unknown coefficients in (8.9).

Taking into account our choice of algebraic cross-section, the recurrence formulae (8.12) (or, equivalently, (8.13)) naturally split into two genres. The phantom differential invariants (7.10) are, by definition, constant, and so have zero differential. These yield the *phantom recurrence formulae*

$$0 = \omega^j + \chi^j, \qquad 0 = \sum_{i=1}^{p} I_{K,i}^\beta\, \omega^i + \Phi_K^\beta(I^{(n)}; \gamma^{(n)}), \qquad \tilde{s}_j,\ s_K S^\beta \in \mathcal{K}. \tag{8.14}$$

Let us show that, as a direct consequence of local freeness of the prolonged pseudo-group action and the fact that we have a bona fide cross-section, the phantom recurrence formulae can be uniquely solved for the invariantized Maurer–Cartan forms, keeping in mind that the latter are subject to the invariantized determining equations (7.21).

Lemma 8.3. *If* $n \geq n^\star$, *the phantom recurrence formulae* (8.14) *of order* $\leq n$ *can be uniquely solved for the invariantized Maurer–Cartan forms* $\gamma^{(n)}$ *of order* $\leq n$. *Each such one-form is a linear combination of the invariantized horizontal one-forms,*

$$\gamma_A^b = \sum_{i=1}^{p} R_{A,i}^b\, \omega^i, \tag{8.15}$$

whose coefficients $R_{A,i}^b$, *the* Maurer–Cartan invariants, *are differential invariants of order* $\leq 1 + \max\{n^\star, \#A\}$.

Proof: Because the invariantized Maurer–Cartan forms (7.17) have been explicitly constructed using the methods in [**45**], the existence of a solution to the phantom recurrence equations (8.14) is not an issue. Thus, we need only establish uniqueness of the solution. But this is an immediate consequence of Lemma 8.2. The last statement in the theorem is a consequence of Lemma 7.4, which guarantees that, when $\# K \geq n^\star$, the order of any non-phantom differential invariants $I^\beta_{K,i}$ appearing in (8.14) equals $\# K + 1$. *Q.E.D.*

We substitute the resulting expressions (8.15) into the remaining recurrence formulae

$$d_H\, H^i = \omega^i + \chi^i, \qquad d_H\, I^\alpha_J = \sum_{i=1}^p I^\alpha_{J,i}\, \omega^i + \Phi^\alpha_J(I^{(n)}; \gamma^{(n)}), \qquad \tilde{s}_i,\ s_J S^\alpha \in \mathcal{N}, \qquad (8.16)$$

for the basic differential invariants. The individual coefficients of the ω^i will produce the recurrence formulae in the form (8.13) for the non-phantom differential invariants. Illustrative examples of this process can be found in [**10**, **45**, **54**].

It is worth pointing out that, since the prolonged vector field coefficients are polynomials in the jet coordinates u^α_J of order $\# J \geq 1$, their invariantizations are polynomial functions of the basic differential invariants I^α_J of degree $\# J \geq 1$. Since the correction terms are constructed by solving a linear system (8.14) for the invariantized Maurer–Cartan forms (8.15), the resulting Maurer–Cartan invariants $R^b_{A,i}$ are inevitably *rational functions* of these differential invariants. Thus, in almost all cases arising in applications, the resulting differential invariant algebra is endowed with an entirely rational algebraic recurrence structure.

Theorem 8.4. *If either $\mathcal{G}$ acts transitively on M, or its infinitesimal generators depend polynomially on the coordinates $z = (x, u) \in M$, then the recurrence formulae* (8.13) *depend rationally on the basic differential invariants.*

The hypotheses are not mutually exclusive. Thus, only when confronted with an intransitive pseudo-group that involves non-rational infinitesimal generator coefficients are we required to go beyond the rational algebraic category when analyzing the differential invariant algebra. And, even in this case, only the zero$^{\text{th}}$ order basic differential invariants will enter the recurrence formulae in a non-rational manner.

Because the basic differential invariants are functionally independent, their recurrence formula provide a complete set of identities that fix the structure of the differential invariant algebra $\mathcal{I}(\mathcal{G})$. Thus, to establish the Basis Theorems for the generating differential invariants and for the differential syzygies, we need only analyze the structure of these recurrence formulae.

However, at this stage a serious complication emerges: Because the invariantized Maurer–Cartan forms of order $n \geq n^\star$ are obtained by solving the linear system (8.14), their coefficients may depend on $(n+1)^{\text{st}}$ order differential invariants, and hence the correction term $M^\alpha_{J,i}$ in the resulting recurrence formula (8.13) for $\mathcal{D}_i I^\alpha_J$ may very well have the same order as the leading term $I^\alpha_{J,i}$. This possibility — which does not arise in the finite-dimensional Lie group situation, [**19**] — makes the determination of the differential algebraic structure of $\mathcal{I}(\mathcal{G})$ more subtle. Fortunately, this complication can be successfully circumvented by introducing an alternative collection of generating invariants that is better adapted to the underlying algebraic structure of the prolonged symbol module.

To proceed, we will need to invariantize the algebraic constructions developed in the first part of the paper. As in (7.8), the invariantization of any polynomial, map, etc., is found by replacing the submanifold jet coordinates by their normalized counterparts (7.6), and using (7.17) to invariantize vector field coefficients and their jets. As usual, the invariantized object coincides with its progenitor when restricted to the cross-section used to define the moving frame, and so enjoys the self-same algebraic properties.

Consider a parametrized polynomial

$$\eta(x,u;t,T) = \sum_{b=1}^{m} \sum_{\#A\le n} h_b^A(x,u)\, t_A T^b$$

that forms a section of the annihilator bundle $\mathcal{L}$. Its invariantization is the polynomial

$$\widetilde{\eta}(H,I;t,T) = \iota\bigl(\,\eta(x,u;t,T)\,\bigr) = \sum_{b=1}^{m} \sum_{\#A\le n} h_b^A(H,I)\, t_A T^b, \tag{8.17}$$

which is obtained by replacing the coordinates on M by their invariantizations $H^i = \iota(x^i)$, $I^\alpha = \iota(u^\alpha)$, which are constant if the pseudo-group acts transitively on M. The invariantized polynomial is a section of the *invariantized annihilator bundle* $\widetilde{\mathcal{L}}^{\le n}$, which can be identified as the pull-back of the restriction of $\mathcal{L}^{\le n}$ to the cross-section via the map $\widetilde{\boldsymbol{\tau}}^{(n)} \circ \rho^{(n)} \colon \mathrm{J}^n(M,p) \to K^n$. Just as the original annihilators characterize the linearized determining equations (3.8), these *invariantized annihilators* characterize the invariantized determining equations (7.21), which require

$$\langle\, \gamma^{(n)}\,; \widetilde{\eta}\,\rangle = 0 \qquad\qquad \text{for all sections } \ \widetilde{\eta} \ \text{ of } \ \widetilde{\mathcal{L}}^{\le n}.$$

In particular, if $\mathcal{G}$ acts transitively on M, then the generators (8.17) are constant coefficient polynomials, and so the fibers $\widetilde{\mathcal{L}}^{\le n}|_z \subset \mathcal{T}$ are all the same subspace, independent of the base point z. We further let $\widetilde{\mathcal{I}}^n = \iota(\mathcal{I}^n)$ denote the invariantized symbol submodule, which is also independent of the base point z when $\mathcal{G}$ acts transitively on M.

Similarly, we let

$$\widetilde{\sigma}(\mathbf{I}^{(k)}; s, S) = \sum_{\alpha=1}^{q} \sum_{\#J\le n} h_\alpha^J(\mathbf{I}^{(k)})\ s_J S^\alpha \in \widehat{\mathcal{S}}^{\,\le n} \tag{8.18}$$

be the invariantization of a symbol polynomial† (4.17), whose coefficients depend on the differential invariants of some (finite) order $\le k$. We associate to (8.18) the differential invariant

$$I_{\tilde\sigma} = \sum_{\alpha=1}^{q} \sum_{\#J\le n} h_\alpha^J(\mathbf{I}^{(k)})\, I_J^\alpha. \tag{8.19}$$

Note that, in view of Lemma 7.4,

$$\operatorname{ord} I_{\tilde\sigma} \le \max\{\, k, n^\star, \deg\widetilde{\sigma}\,\}.$$

† We could, of course, include terms involving the extra variables $\tilde{s}_j$ here, but, since these will not play a role in our subsequent development, it is simpler to omit them from the outset.

In particular, if both $k, n^\star < \deg \widetilde{\sigma}$, then $\operatorname{ord} I_{\tilde{\sigma}} = \deg \widetilde{\sigma}$ provided at least one coefficient $h_\alpha^J(\mathbf{I}^{(k)}) \not\equiv 0$ for some $s_J S^\alpha \in \mathcal{N}$ with $\# J = n = \deg \widetilde{\sigma}$. Thanks to the recurrence formulae (8.12),

$$\begin{aligned} d_H I_{\tilde{\sigma}} &= \sum_{\alpha=1}^{q} \sum_{\# J \geq 0} \left(h_\alpha^J \, d_H I_J^\alpha + I_J^\alpha \, d_H h_\alpha^J \right) \\ &= \sum_{\alpha=1}^{q} \sum_{\# J \geq 0} \left(\sum_{i=1}^{p} \left[h_\alpha^J I_{J,i}^\alpha + \mathcal{D}_i(h_\alpha^J) \, I_J^\alpha \right] \omega^i + h_\alpha^J \, \widehat{\psi}_J^\alpha \right) \\ &= \sum_{i=1}^{p} (I_{s_i \tilde{\sigma}} + I_{\mathcal{D}_i \tilde{\sigma}}) \, \omega^i + \langle \psi^{(\infty)} ; \widetilde{\sigma} \rangle, \end{aligned} \tag{8.20}$$

where the invariant differential operator $\mathcal{D}_i$ acts coefficient-wise on the parametrized polynomial (8.18):

$$\mathcal{D}_i \, \widetilde{\sigma}(\mathbf{I}^{(k+1)}; s, S) = \sum_{\alpha=1}^{q} \sum_{\# J \geq 0} \mathcal{D}_i \left[h_\alpha^J(\mathbf{I}^{(k)}) \right] s_J S^\alpha. \tag{8.21}$$

Next, let $\widetilde{\boldsymbol{\beta}} : \mathbb{R}^{2m} \to \mathbb{R}^m$ be the *invariantized symbol map*, which is obtained from $\boldsymbol{\beta}$ in (4.22) by invariantization:

$$\begin{aligned} s_i &= \widetilde{\beta}_i(t) = t_i + \sum_{\alpha=1}^{q} I_i^\alpha \, t_{p+\alpha}, & i &= 1, \ldots, p, \\ S^\alpha &= \widetilde{B}^\alpha(T) = T^{p+\alpha} - \sum_{i=1}^{p} I_i^\alpha \, T^i, & \alpha &= 1, \ldots, q, \end{aligned} \tag{8.22}$$

where $I_i^\alpha = \iota(u_i^\alpha)$ are the first order normalized differential invariants. In particular, if $\mathcal{G}$ acts transitively on $\mathrm{J}^1(M,p)$, then, by minimality of the algebraic moving frame, all the I_i^α are phantom differential invariants, and so in this case $\widetilde{\boldsymbol{\beta}}$ is a fixed linear map. Otherwise, when $\mathcal{G}$ acts intransitively on $\mathrm{J}^1(M,p)$, the map $\widetilde{\boldsymbol{\beta}}$ depends on the non-phantom differential invariants I_i^α, and so, by Lemma 7.4, on the submanifold jet coordinates of order at most $n^\star$ — the order of freeness of the pseudo-group. As with its progenitor $\boldsymbol{\beta}$, the map $\widetilde{\boldsymbol{\beta}}$ defines the "symbol" of the invariantized prolongation map (8.7).

Finally, we let $\widetilde{\mathcal{Z}} = (\widetilde{\mathbf{p}}^*)^{-1} \widetilde{\mathcal{L}}$, where $\widetilde{\mathbf{p}} = \widetilde{\mathbf{p}}^{(\infty)}$ is the invariantized prolongation map (8.7), denote the *invariantized prolonged annihilator subbundle*, so that, as in (4.27),

$$\langle \psi^{(\infty)} ; \widetilde{\tau} \rangle = 0 \qquad \text{if and only if} \qquad \widetilde{\tau} \text{ is a section of } \widetilde{\mathcal{Z}}. \tag{8.23}$$

We use the linear map (8.22) to define an invariantized version of the prolonged symbol submodule (4.29) at each point.

Definition 8.5. The *invariantized prolonged symbol submodule* is defined as

$$\widetilde{\mathcal{J}} = (\widetilde{\boldsymbol{\beta}}^*)^{-1}(\widetilde{\mathcal{I}}) = \left\{ \widetilde{\sigma}(s,S) \;\middle|\; \widetilde{\sigma}(\widetilde{\beta}(t), \widetilde{B}(T)) \in \widetilde{\mathcal{I}} \right\}. \tag{8.24}$$

Remark: Note that $\widetilde{\mathcal{I}}|_z$ depends on the degree 0 invariants (if any), while $\widetilde{\boldsymbol{\beta}}$ involves the degree 1 differential invariants. Thus, if $\mathcal{G}$ acts transitively on $\mathrm{J}^1(M,p)$, then the invariantized prolonged symbol submodule does not vary from point to point. In the intransitive case, we suppress the dependence of $\widetilde{\mathcal{J}}|_{z^{(\infty)}}$ and the polynomials therein on the non-constant differential invariants $\mathbf{I}^{(1)}(z^{(\infty)})$ of degree ≤ 1.

Since the invariantizations $\widetilde{\mathcal{Z}}, \widetilde{\mathcal{J}}$, etc. agree with their progenitors on the algebraic cross section K, by Lemma 5.5, the leading terms in any invariantized annihilating polynomial belong to the invariantized prolonged symbol module. In other words, we can decompose any non-zero $0 \neq \widetilde{\tau} \in \widetilde{\mathcal{Z}}^{\leq n}$ as

$$\widetilde{\tau}(s,S) = \widetilde{\sigma}(s,S) + \widetilde{\nu}(s,S) \in \widetilde{\mathcal{Z}}^{\leq n}, \qquad \text{where} \qquad \widetilde{\sigma} = \mathbf{H}(\widetilde{\tau}) \in \widetilde{\mathcal{J}}^{\,n}, \quad \widetilde{\nu} \in \mathcal{S}^{\,\leq n-1}. \tag{8.25}$$

Moreover, the differential invariant $I_{\tilde{\sigma}}$ associated with its symbol, cf. (8.19), is of high degree with

$$\operatorname{ord} I_{\tilde{\sigma}} = \deg I_{\tilde{\sigma}} = \deg \widetilde{\sigma} = n \geq n^\star + 1 \qquad \text{for} \qquad \widetilde{\sigma} \in \widetilde{\mathcal{J}}^{\,n}. \tag{8.26}$$

In view of (8.23), the recurrence formula (8.20) for such polynomials reduces to

$$d_H\, I_{\tilde{\sigma}} = \sum_{i=1}^{p} \left(I_{s_i \tilde{\sigma}} + I_{\mathcal{D}_i \tilde{\sigma}} \right) \omega^i - \langle\, \psi^{(n)}\,; \widetilde{\nu}\,\rangle. \tag{8.27}$$

In contrast to (8.13), the *correction term* in the algebraically adapted recurrence formula (8.27), namely $\sum I_{\mathcal{D}_i \tilde{\sigma}}\, \omega^i - \langle\, \psi^{(\infty)}\,; \widetilde{\nu}\,\rangle$, *is* of lower order than the *leading term* $\sum I_{s_i \tilde{\sigma}}\, \omega^i$, i.e., it depends on differential invariants of order $\leq n = \operatorname{ord} \widetilde{\sigma}$, whereas $\operatorname{ord} I_{s_i\, \tilde{\sigma}} = n+1$. Indeed, if $\mathcal{G}^{(1)}$ acts transitively on $\mathrm{J}^1(M,p)$, then $I_{\mathcal{D}_i \tilde{\sigma}} = 0$, while in the intransitive case $I_{\mathcal{D}_i \tilde{\sigma}}$ depends on the second order differentiated invariants $\mathcal{D}_i I^\alpha_j$, which, by (7.16), are differential functions of order $\leq \max\{2, n^\star+1\} \leq n$, since we are assuming that $n > n^\star \geq 1$. Furthermore, the final term $\langle\, \psi^{(\infty)}\,; \widetilde{\nu}\,\rangle$ is a linear combination of invariantized Maurer–Cartan forms of orders $\leq n-1$, which, according to Lemma 8.3, also depend on at most n^{th} order differential invariants provided $n > n^\star$. We conclude that, for such $\widetilde{\sigma}$, *the only terms on the right hand side of the recurrence formula* (8.27) *that can be of order* $n+1$ *are the leading coefficients,* $I_{s_i\, \tilde{\sigma}}$. Equating the coefficients of the forms ω^i in formula (8.27) leads to individual recurrence formulae

$$\mathcal{D}_i\, I_{\tilde{\sigma}} = I_{s_i\, \tilde{\sigma}} + M_{\tilde{\sigma},i} \equiv F_{\tilde{\sigma},i}(H, I^{(n+1)}), \tag{8.28}$$

in which, assuming (8.26), the *leading term* $I_{s_i\, \tilde{\sigma}}$ is a differential invariant of order $n+1$, while the *correction term* $M_{\tilde{\sigma},i}$ is of order $\leq n$. Iteration of the first order recurrence formulae (8.28) leads to the higher order recurrences

$$\mathcal{D}_J\, I_{\tilde{\sigma}} = I_{s_J \tilde{\sigma}} + M_{\tilde{\sigma},J} \equiv F_{\tilde{\sigma},J}(H, I^{(n+k)}), \tag{8.29}$$

whenever $J = (j_1, \ldots, j_k)$ is an *ordered* multi-index of order $k = \#J$,

$$\mathcal{D}_J = \mathcal{D}_{j_1} \mathcal{D}_{j_2}\ \cdots\ \mathcal{D}_{j_k}, \tag{8.30}$$

and, assuming $\deg \widetilde{\sigma} = n > n^\star$,

$$\operatorname{ord} M_{\tilde{\sigma},J} < n + \#J = \deg\bigl[\, s_J\, \widetilde{\sigma}(s,S)\,\bigr].$$

The invariant differential operators $\mathcal{D}_1,\ldots,\mathcal{D}_p$ do not necessarily commute, and so the order of the multi-index J in the recurrence formula (8.29) matters. In general, the invariant differential operators are subject to linear commutation relations of the form

$$\bigl[\,\mathcal{D}_i,\mathcal{D}_j\,\bigr] = \sum_{k=1}^{p} Y_{ij}^k\, \mathcal{D}_k, \qquad i,j = 1,\ldots,p, \tag{8.31}$$

where the coefficients $Y_{ij}^k = -\,Y_{ji}^k$ are certain differential invariants, called the *commutator invariants*. They are determined by the recurrence formulae for the invariant horizontal forms, cf. [**45**]; explicitly, according to (8.2) (as generalized to one-forms, and using the fact invariantization takes contact forms to contact forms, [**45**]),

$$\begin{aligned} d_H\, \omega^i &\equiv\; d\iota(dx^i) = \iota\bigl[\; d(dx^i) + \mathbf{v}(\,dx^i)\,\bigr] = \iota\,[\; d\xi^i\,] \\ &\equiv \iota \left[\, \sum_{j=1}^{p} D_j \xi^i\, dx^j \,\right] = \iota \left[\, \sum_{j=1}^{p} \left(\frac{\partial \xi^i}{\partial x^j} + \sum_{\alpha=1}^{q} \frac{\partial \xi^i}{\partial u^\alpha}\, u_j^\alpha \right) dx^j \,\right] \\ &\equiv \sum_{j=1}^{p} \left(\gamma_j^i + \sum_{\alpha=1}^{q} I_j^\alpha \gamma_\alpha^i \right) \wedge \omega^j, \end{aligned} \tag{8.32}$$

where $\equiv$ indicates equality modulo contact forms, and $\gamma_j^i, \gamma_\alpha^i$ denote the first order invariantized Maurer–Cartan forms (7.17) obtained by invariantizing the first order partial derivatives $\xi_j^i = \partial \xi^i/\partial x^j,\ \xi_\alpha^i = \partial\xi^i/\partial u^\alpha$, of the independent variable vector field coefficients. Replacing the invariantized Maurer–Cartan forms by their explicit formulas (8.15), as prescribed by our solution to the phantom recurrence relations (8.14), leads to the formulas

$$d_H\, \omega^k = -\sum_{i<j} Y_{ij}^k\, \omega^i \wedge \omega^j, \tag{8.33}$$

that serve to prescribe the commutator invariants in (8.31). As a consequence, each commutator invariant has order bounded by

$$\operatorname{ord} Y_{ij}^k \le n^\star + 1. \tag{8.34}$$

Moreover, Y_{ij}^k depends rationally on the basic differential invariants I_J^α of degree $\#J \ge 1$, and also rationally on H^i, I^α under the hypothesis of Theorem 8.4.

More generally, to each invariantized polynomial

$$\tilde{q}(s) = \iota[\,q(s)\,] = \sum_{\#J\,\le\, l} q_J(\mathbf{I}^{(j)}) s_J \in \mathbb{R}[s] \tag{8.35}$$

whose coefficients are differential invariants, we associate an invariant differential operator

$$\tilde{q}(\mathcal{D}) = \sum_{\#J\,\le\, l} q_J(\mathbf{I}^{(j)}) \mathcal{D}_J. \tag{8.36}$$

Since the multi-indices in (8.35) are unordered, the expression (8.36) is ambiguous. For specificity, we adopt the *normal ordering* convention that the sums range over non-decreasing multi-indices $1 \leq j_1 \leq j_2 \leq \cdots \leq j_k \leq p$, where $k = \#J$. Indeed, in view of (8.29), we can write

$$\tilde{q}(\mathcal{D})\, I_{\tilde{\sigma}} = I_{\tilde{q}\,\tilde{\sigma}} + M_{\tilde{\sigma},\tilde{q}}, \tag{8.37}$$

where $I_{\tilde{q}\,\tilde{\sigma}}$ is the differential invariant corresponding to the product polynomial $\tilde{q}(s)\,\widetilde{\sigma}(s,S)$. Moreover, provided $\deg \widetilde{\sigma} > n^\star$ and $k < \deg \tilde{q} + \deg \widetilde{\sigma}$, then

$$\operatorname{ord} M_{\tilde{\sigma},\tilde{q}} < \deg \tilde{q} + \deg \widetilde{\sigma} = \operatorname{ord} \tilde{q}(\mathcal{D})\, I_{\tilde{\sigma}} = \operatorname{ord} I_{\tilde{q}\,\tilde{\sigma}}, \qquad \widetilde{\sigma} \in \widetilde{\mathcal{J}}^{\,\geq n^\star+1} = \widetilde{\mathcal{J}} \,\cap\, \widehat{\mathcal{S}}^{\,\geq n^\star+1}. \tag{8.38}$$

We are now in a position to rigorously formulate a Constructive Basis Theorem for the differential invariant algebra of an eventually locally freely acting pseudo-group. The key is to use the differential invariants $I_{\tilde{\sigma}}$ corresponding to $\widetilde{\sigma} \in \widetilde{\mathcal{J}}$ instead of the less well-behaved high order basic differential invariants I^α_J.

Theorem 8.6. *Let $\mathcal{G}$ be a pseudo-group that acts locally freely on the submanifold jet bundle at order $n^\star$. Then the following differential invariants form a finite generating system for its differential invariant algebra $\mathcal{I}(\mathcal{G})$:*

(a) *$I_\nu = I_{\tilde{\sigma}_\nu}$ where $\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_l$, form a Gröbner basis for the high degree prolonged symbol module $\widetilde{\mathcal{J}}^{\,\geq n^\star+1}$ relative to our chosen term ordering, and,*

(b) *all basic differential invariants of low degree $n \leq n^\star$.*

Proof: Since we have included all the low degree differential invariants in our generating system, it suffices to show that every differential invariant $I_{\tilde{\sigma}}$ for $\widetilde{\sigma} \in \widetilde{\mathcal{J}}^{\,\geq n^\star+1}$ can be expressed in terms of the listed generators. Since we can write†

$$\widetilde{\sigma}(s,S) = \sum_{\nu=1}^{l} \tilde{q}_\nu(s)\, \widetilde{\sigma}_\nu(s,S)$$

as a linear combination of the Gröbner basis polynomials, the recurrence formula (8.37) implies that we can write the corresponding differential invariant

$$I_{\tilde{\sigma}} = \sum_{\nu=1}^{l} \left[\, \tilde{q}_\nu(\mathcal{D})\, I_\nu - M_{\tilde{\sigma},\tilde{q}_\nu} \,\right]$$

in terms of the differentiated Gröbner basis generators along with a correction term. Moreover, since $\deg I_\nu = \deg \widetilde{\sigma}_\nu > n^\star$, (8.38) implies that the correction terms is of lower order than $I_{\tilde{\sigma}}$. An evident induction on the degree of $\widetilde{\sigma}$ serves to establish the result. *Q.E.D.*

Remark: Typically, many of the listed generating differential invariants are redundant, as they can be written as combinations of invariant derivatives of other generating invariants. The explicit recurrence formulae for the differentiated invariants will allow one

† As per the remark following Definition 8.5, all the polynomials in this formula may depend on the basic invariants in $\mathbf{I}^{(1)}$, if any.

to systematically eliminate redundant differential invariants, and so produce a minimal generating system of differential invariants. However, establishing the minimality of a generating set remains a challenging problem.

Example 8.7. For the pseudo-group treated in Examples 4.3 and 6.1, the algebraic cross-section (6.6) leads to the normalization equations

$$\begin{gathered} X = 0, \qquad Y = 0, \qquad U = 0, \qquad \widehat{U}_Y = 0, \qquad \widehat{U}_X = 0, \qquad \widehat{U}_{XX} = 0, \\ \widehat{U}_{XY} = 0, \qquad \widehat{U}_{YY} = 1, \qquad \widehat{U}_{X^k} = 0, \qquad \widehat{U}_{X^{k-1}Y} = 0, \qquad k \geq 3. \end{gathered} \tag{8.39}$$

The explicit formulas for the prolonged pseudo-group action and for the resulting moving frame can be found in Example 11 of [**45**]. Recall that the order of freeness of this pseudo-group action is $n^\star = 2$. Since the high degree prolonged symbol submodule $\mathcal{J}^{\geq 3}$ is spanned by the monomials $s_1^i s_2^j S$ for $i+j \geq 3$, $j \geq 2$, the non-phantom differential invariants are obtained by invariantizing

$$I_{i,j} = \iota(u_{i,j}) = \iota\left(\frac{\partial^{i+j}u}{\partial x^i \partial y^j}\right), \qquad i+j \geq 3, \qquad j \geq 2. \tag{8.40}$$

In particular, by [**45**; eq. (3.24)],

$$I_{1,2} = \frac{u_{xyy} + u u_{yyy} + 2\,u_y u_{yy}}{u_{yy}^{3/2}}\,, \qquad I_{0,3} = \frac{u_{yyy}}{u_{yy}^{3/2}}\,. \tag{8.41}$$

The corresponding invariant total differential operators are, according to [**45**; eq. (3.26)],

$$\mathcal{D}_1 = \frac{1}{\sqrt{u_{yy}}}\,(\mathrm{D}_x + u\,\mathrm{D}_y), \qquad \mathcal{D}_2 = \frac{1}{\sqrt{u_{yy}}}\,\mathrm{D}_y. \tag{8.42}$$

Since we normalized both $I_{1,0} = \iota(u_x) = 0$, $I_{0,1} = \iota(u_y) = 0$, cf. (8.39), the invariantized linear maps (8.22) are trivial: $s_1 = \widetilde{\beta}^1(t) = t_1$, $s_2 = \widetilde{\beta}^2(t) = t_2$, $S = \widetilde{B}(T) = T$. Therefore, the Gröbner basis for $\widetilde{\mathcal{J}}^{\geq 3}$ consists of the monomials

$$\widetilde{\sigma}_1 = s_1 s_2^2\, S, \qquad \widetilde{\sigma}_2 = s_2^3\, S, \tag{8.43}$$

with corresponding differential invariants $I_1 = I_{1,2}$, $I_2 = I_{0,3}$ as in (8.41). Since there are no low order differential invariants, Theorem 8.6 immediately implies that I_1, I_2 generate the differential invariant algebra. This result can be confirmed by examination of the explicit recurrence relations, which can be found in [**45**; Example 32].

9. Syzygies.

In this final section, we resolve the classification problem for differential syzygies of the differential invariant algebra of an eventually freely acting pseudo-group. To this end, let

$$\mathbf{I}_{gen} = (\mathbf{I}_{low}; \mathbf{I}_{high}) = (\ \ldots\ H^j\ \ldots\ I_J^\alpha\ \ldots\ ;\ \ldots\ I_\nu\ \ldots\ ) \tag{9.1}$$

denote the generating differential invariants listed in Theorem 8.6, where $\mathbf{I}_{low}$ refers to all the low degree basic differential invariants, while $\mathbf{I}_{high}$ denotes the *high degree* generators, consisting of all the Gröbner basis invariants $I_\nu = I_{\tilde{\sigma}_\nu}$. We note that

$$\operatorname{ord} I \leq n^\star, \qquad I \in \mathbf{I}_{low}, \qquad\qquad \operatorname{ord} I = \deg I, \qquad I \in \mathbf{I}_{high}.$$

Let

$$\mathcal{D}^*\mathbf{I}_{gen} = (\mathcal{D}^*\mathbf{I}_{low}; \mathcal{D}^*\mathbf{I}_{high}) = (\ \ldots\ \mathcal{D}_K H^j\ \ldots\ \mathcal{D}_K I^\alpha_J\ \ldots\ ;\ \ldots\ \mathcal{D}_K I_\nu\ \ldots\ ) \tag{9.2}$$

denote all the invariantly differentiated generating invariants, where $K = (k_1, \ldots, k_l)$ ranges over all *ordered* multi-indices with $l = \#K \geq 0$.

To precisely define what is meant by a syzygy in the algebra of differential invariants, we introduce new variables

$$\begin{aligned} w &= (w_{low}; w_{high}) = (\ \ldots\ w^j\ \ldots\ w^\alpha_J\ \ldots\ ;\ \ldots\ w_\nu\ \ldots\ ), \\ w^* &= (w^*_{low}; w^*_{high}) = (\ \ldots\ w^j_L\ \ldots\ w^\alpha_{J,K}\ \ldots\ ;\ \ldots\ w_{\nu,N}\ \ldots\ ), \end{aligned} \tag{9.3}$$

representing, respectively, the generating invariants (9.1), and their invariant derivatives (9.2), so that L, K, N represent *ordered* multi-indices of order ≥ 0. Note that the variables in w also appear in w^*. We will refer to $w^j, w^j_L, w^\alpha_J, w^\alpha_{J,K}$, as *low degree* variables, and $w_\nu, w_{\nu,N}$, as *high degree* variables.

Definition 9.1. A *syzygy* among the generating differential invariants $\mathbf{I}_{gen}$ is represented by a nontrivial function

$$Z(w^*) = Z(\ \ldots\ w^j_L\ \ldots\ w^\alpha_{J,K}\ \ldots\ ;\ \ldots\ w_{\nu,N}\ \ldots\ ) \not\equiv 0, \tag{9.4}$$

depending on the formal variables (9.3), with the property that

$$Z(\mathcal{D}^*\mathbf{I}_{gen}) = Z(\ \ldots\ \mathcal{D}_L H^j\ \ldots\ \mathcal{D}_K I^\alpha_J\ \ldots\ ;\ \ldots\ \mathcal{D}_N I_\nu\ \ldots\ ) \equiv 0. \tag{9.5}$$

The *degree* of the syzygy (9.4) is the maximum of the degrees $\#L, \#J + \#K, \deg \widetilde{\sigma}_\nu + \#N$, of all the variables $w^j_L, w^\alpha_{J,K}, w_{\nu,N}$ that explicitly appear in it, that is, such that the partial derivative of Z with respect to the variable is not identically zero.

More explicitly, as a consequence of the recurrence formulae (8.28–29), we can write each differentiated generating invariant

$$\begin{aligned} \mathcal{D}_L H^j = F^j_L(\mathbf{I}_{basic}), \qquad \mathcal{D}_K I^\alpha_J &= F^\alpha_{J,K}(\mathbf{I}_{basic}), \qquad \ldots \\ \mathcal{D}_N I_\nu = F_{\nu,N}(\mathbf{I}_{basic}) &= I_{s_N \tilde{\sigma}_\nu} + M_{\nu,N}(\mathbf{I}_{basic}), \end{aligned} \tag{9.6}$$

locally uniquely as a function of (finitely many of) the basic differential invariants. Since the latter are functionally independent†, the syzygy (9.5) requires that

$$Z(\ \ldots\ F^j_L(\mathbf{I}_{basic})\ \ldots\ F^\alpha_{J,K}(\mathbf{I}_{basic})\ \ldots\ ;\ \ldots\ F_{\nu,N}(\mathbf{I}_{basic})\ \ldots\ ) \equiv 0.$$

† Thus, from an algebraic standpoint, one treats the basic differential invariants as independent algebraic variables. With this in mind, it is not necessary to introduce yet more symbols for them here.

If $Z(w^*)$ is any syzygy, so is the function obtained by formal invariant differentiation

$$Z_{,i}(w^*) = \sum_{j,L} \frac{\partial Z}{\partial w^j_L}\, w^j_{L,i} + \sum_{\alpha,J,K} \frac{\partial Z}{\partial w^\alpha_{J,K}}\, w^\alpha_{J,K,i} + \sum_{\nu,N} \frac{\partial Z}{\partial w_{\nu,N}}\, w_{\nu,N,i}. \tag{9.7}$$

Indeed, in view of (9.5),

$$Z_{,i}(\mathcal{D}^*\mathbf{I}_{gen}) = \mathcal{D}_i\big[\, Z(\mathcal{D}^*\mathbf{I}_{gen})\,\big] \equiv 0.$$

Higher order differentiated syzygies are denoted by

$$Z_{,I}(w^*) = Z_{,i_1,i_2,\ldots,i_j}(w^*) = (Z_{,i_2,\ldots,i_j})_{,i_1}(w^*) = \mathcal{D}_{i_1}\big(\, Z_{,i_2,\ldots,i_j}(w^*)\,\big) \tag{9.8}$$

where $I = (i_1, \ldots, i_j)$ is any ordered multi-index with $1 \le i_\kappa \le p$.

Syzygies can be grouped into two main classes. The first contains the *commutator syzygies*, reflecting the non-commutativity of the invariant differential operators. In general, if $K = (k_1, \ldots, k_l)$ is an *ordered multi-index*, then, as a consequence of (8.31),

$$\mathcal{D}_{\pi(K)} = \mathcal{D}_K + \sum_{\#J<\#K} Y^J_{\pi,K}\,\mathcal{D}_J, \tag{9.9}$$

for any permutation π of the entries of K. The right hand side of the commutator identity (9.9) is a linear combination of lower order invariant differential operators, whose coefficients $Y^J_{\pi,K}$ are combinations of invariant derivatives, of order $\le \#K-2$, of the commutator invariants Y^k_{ij}. For example,

$$\begin{aligned} \mathcal{D}_i\mathcal{D}_j\mathcal{D}_k &= \mathcal{D}_i\left(\mathcal{D}_k\mathcal{D}_j + \sum_{l=1}^{p} Y^l_{jk}\mathcal{D}_l\right) \\ &= \mathcal{D}_k\mathcal{D}_i\mathcal{D}_j + \sum_{l=1}^{p}\big[\, Y^l_{ik}\mathcal{D}_l\mathcal{D}_j + Y^l_{jk}\mathcal{D}_i\mathcal{D}_l + (\mathcal{D}_iY^l_{jk})\mathcal{D}_l\,\big]. \end{aligned} \tag{9.10}$$

The commutator formulae (9.9) produce an infinite number of *commutator syzygies*

$$\mathcal{D}_{\pi(K)}\, I = \mathcal{D}_K\, I + \sum_{\#J<\#K} Y^J_{\pi,K}\,\mathcal{D}_J\, I, \tag{9.11}$$

in which I is any one of our generating differential invariants, and J, K are assumed to be non-decreasing multi-indices, so that all invariant differential operators on the right hand side of the identity are in normal ordering. In view of (8.34), provided $\deg I > n^\star$, the degree of the summation terms on the right hand side of (9.11) is strictly less than

$$\deg \mathcal{D}_K\, I = \deg \mathcal{D}_{\pi(K)}\, I = \deg I + \#K.$$

In terms of our formal variables w^*, let

$$\begin{aligned} V^j_{K,\pi}(w^*) &= w^j_{\pi(K)} - w^j_K - W^j_{K,\pi}(w^*), \\ V^\alpha_{J,K,\pi}(w^*) &= w^\alpha_{J,\pi(K)} - w^\alpha_{J,K} - W^\alpha_{J,K,\pi}(w^*), \\ V_{\nu,K,\pi}(w^*) &= w_{\nu,\pi(K)} - w_{\nu,K} - W_{\nu,K,\pi}(w^*), \end{aligned} \tag{9.12}$$

represent the complete list of commutator syzygies (9.11) obtained by applying permutations of invariant differential operators to our generating differential invariants H^j, I_J^α, I_ν.

Warning: The higher order commutator syzygies *cannot* be generated by invariantly differentiating a finite number of low order ones, which is why they must be treated on a different footing. For example, the third order commutator syzygy

$$\mathcal{D}_i \mathcal{D}_j \mathcal{D}_k I = \mathcal{D}_j \mathcal{D}_i \mathcal{D}_k I + \sum_{l=1}^{p} Y_{ij}^l \mathcal{D}_l \mathcal{D}_k I$$

cannot be obtained by invariant differentiation of second order commutator syzygies. However, the commutator relations (9.9) *are* finitely generated as a two-sided ideal in the noncommutative algebra of invariant differential operators, and so in this extended sense, the commutator syzygies can be regarded as finitely generated by those in (8.31).

Definition 9.2. A collection $Z_1, \ldots, Z_k$ of syzygies is said to form a *generating system* if every syzygy can be written as a linear combination of them and finitely many of their derivatives, modulo the commutator syzygies (9.12):

$$\begin{aligned} Z(w^*) = \sum_{i,K} \; P_{K,i}(w^*)\, \mathcal{D}_K Z_i(w^*) + \sum_{\pi,K,j} \; Q_{K,\pi}^j(w^*)\, V_{K,\pi}^j(w^*) \\ + \sum_{J,\alpha,K,\pi} \; Q_{J,K,\pi}^\alpha(w^*)\, V_{J,K,\pi}^\alpha(w^*) + \sum_{\nu,K,\pi} \; Q_{\nu,K,\pi}(w^*)\, V_{\nu,K,\pi}(w^*). \end{aligned} \tag{9.13}$$

The second class, consisting of what we will call *essential syzygies*, is further subdivided into those of low and high degree. Let us write the recurrence formulae (8.1) for the first order derivatives of the low degree basic differential invariants $H^j, I_J^\alpha \in \mathbf{I}_{basic}$ as

$$\mathcal{D}_i H^j = F_{\;i}^{\,j}(\mathbf{I}_{basic}), \qquad \mathcal{D}_i I_J^\alpha = F_{\;J,i}^{\,\alpha}(\mathbf{I}_{basic}), \qquad \#J \leq n^\star. \tag{9.14}$$

Observe that, since we are only differentiating the invariant I_J^α once, the right hand side depends only on basic differential invariants of order $\leq n^\star + 1$. Those of order $\leq n^\star$ are just the low degree generating invariants, while those of order $n^\star + 1$ can be expressed in terms of the undifferentiated invariants I_ν corresponding to Gröbner basis polynomials $\widetilde{\sigma}_\nu \in \widetilde{\mathcal{J}}^{\,n^\star+1}$ and the low degree generating invariants. The corresponding *low degree syzygy generators* are

$$Z_i^j(w^*) = w_i^j - F_{\;i}^{\,j}(w), \qquad Z_{J,i}^\alpha(w^*) = w_{J,i}^\alpha - F_{\;J,i}^{\,\alpha}(w), \qquad \#J \leq n^\star. \tag{9.15}$$

Higher order invariant derivatives of low degree differential invariants are obtained by differentiating the first order syzygies (9.14):

$$\mathcal{D}_K \mathcal{D}_i I_J^\alpha = \mathcal{D}_K \big[\, F_{J,i}^\alpha(\mathbf{I}_{low}, \mathbf{I}_{high})\,\big]. \tag{9.16}$$

The right hand side can be written in terms of derivatives, of order $\leq \#K$, of the high and low degree invariants. The latter can, by an obvious induction, be expressed in terms of the low degree invariants, the high degree generating invariants, and the differentiated high degree generating invariants only, and so we can express any such derivative in the form

$$\mathcal{D}_K I_J^\alpha = \widehat{F}_{J,K}^\alpha(\mathbf{I}_{low}, \mathcal{D}^* \mathbf{I}_{high}), \qquad I_J^\alpha \in \mathbf{I}_{low}. \tag{9.17}$$

As a result, any syzygy $Z(w^*) = Z(w_{low}^*, w_{high}^*)$ can be replaced by a syzygy $\widetilde{Z}(w_{low}, w_{high}^*)$ that only involves the undifferentiated low degree generators.

The high degree syzygies are consequences of the algebraic syzygies among the high degree prolonged symbol polynomials. In terms of our Gröbner basis $\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_\ell$ for the high degree prolonged symbol module $\widetilde{\mathcal{J}}^{\,\geq n^\star+1}$, an *algebraic syzygy*, [**12**], is a non-zero ℓ-tuple of polynomials

$$0 \not\equiv \mathbf{q}(s) = (q_1(s), \ldots, q_\ell(s)) \in \mathbb{R}[s]^{\times \ell} \qquad \text{such that} \qquad \sum_{\nu=1}^{\ell} q_\nu(s)\, \widetilde{\sigma}_\nu(s,S) \equiv 0. \tag{9.18}$$

(In general, the Gröbner basis polynomials and hence their syzygies will depend on the basic invariants $\mathbf{I}^{(1)}$ of degree ≤ 1, if any. For clarity, we will suppress this dependency in our notation. As above, if $\mathcal{G}$ acts transitively on $\mathrm{J}^1(M,p)$, there are no such invariants to worry about.) In view of (8.37), each algebraic syzygy induces a corresponding differential syzygy among the Gröbner basis generating differential invariants $\mathbf{I} = (I_1, \ldots, I_\ell)$ of the form

$$\mathbf{q}(\mathcal{D}) \cdot \mathbf{I} = \sum_{\nu=1}^{\ell} q_\nu(\mathcal{D})\, I_\nu = W_{\mathbf{q}}(\mathcal{D}^* \mathbf{I}_{gen}), \tag{9.19}$$

where $\operatorname{ord} W_{\mathbf{q}} < \deg q_\nu + \operatorname{ord} I_\nu$, and we use our normal ordering convention (8.36) to specify the differential operators $q_\nu(\mathcal{D})$. Here $W_q(\mathcal{D}^* \mathbf{I}_{gen})$ denotes a differential invariant of degree strictly less than the terms on the left hand side, which can be explicitly determined by repeated application of the recurrence formulae.

Further, any non-trivial linear combination of algebraic syzygies,

$$0 \not\equiv \mathbf{q}(s) = \sum_{\kappa=1}^{k} r_\kappa(s)\, \mathbf{q}_\kappa(s), \qquad \text{where} \qquad \mathbf{r}(s) = (r_1(s), \ldots, r_k(s)) \not\equiv 0, \tag{9.20}$$

defines another algebraic syzygy. But the induced differential syzygy (9.19) can, modulo lower order terms, be obtained by invariantly differentiating the originating differential syzygies (9.19) with $\mathbf{q} = \mathbf{q}_\kappa$. Indeed, if $p(s) = r(s)\, q(s)$, and I is any generating differential invariant, then

$$p(\mathcal{D}) I = r(\mathcal{D})\, q(\mathcal{D}) I + K, \tag{9.21}$$

where K is a linear combination of commutator syzygies (9.11) resulting from the normal ordering of the product differential operator $r(\mathcal{D})\, q(\mathcal{D})$. Thus, the algebraic syzygy (9.20) produces a differential syzygy of the form

$$\sum_{\kappa=1}^{k} r_\kappa(\mathcal{D}) \big[\, \mathbf{q}_\kappa(\mathcal{D}) \cdot \mathbf{I} - W_{\mathbf{q}_\kappa}(\mathcal{D}^* \mathbf{I}_{gen}) \,\big] = W_{\mathbf{r},\mathbf{q}}(\mathcal{D}^* \mathbf{I}_{gen}),$$

in which

$$\operatorname{ord} W_{\mathbf{r},\mathbf{q}}(\mathcal{D}^*\mathbf{I}_{gen}) < \deg r_\kappa + \mathbf{q}_\kappa + \operatorname{ord}\mathbf{I}.$$

Hilbert's Syzygy Theorem, [**17**], states that there is a *finite* number of generating algebraic syzygies, $\mathbf{q}_1(s), \ldots, \mathbf{q}_k(s)$ such that any other algebraic syzygy can be written as a linear combination (9.20) of the generators for some $\mathbf{r}(s) = (r_1(s), \ldots, r_k(s)) \in \mathbb{R}[s]^{\times k}$. Moreover, the generating syzygies can be systematically constructed by Gröbner basis algorithms, [**15**]. The preceding argument shows that every high degree differential syzygy is, modulo the commutator syzygies and low degree syzygies, a differential consequence of the generating high degree differential syzygies. Thus, one can, by constructive algebra, find a finite system of generators for the differential syzygies among the high degree differential invariants. Our complete finite system of differential syzygies is then obtained by combining these with the low degree syzygies listed above.

Remark: The "higher syzygies" (syzygies of syzygies, etc.) appearing in Hilbert's Theorem on the resolution of ideals and modules, [**17**], will also impact the algebraic structure of the ring of invariant differential operators. However, we will not develop this line of investigation here.

We now explain why the two indicated classes of syzygies form a finite generating system, i.e., that all other syzygies are differential consequences thereof. Suppose $Z(w_{low}, w^*_{high})$ defines a syzygy

$$Z(\mathbf{I}_{low}; \mathcal{D}^*\mathbf{I}_{high}) \equiv 0, \tag{9.22}$$

which, without loss of generality, does not involve derivatives of the low degree generators. Our goal is to write the syzygy as a combination of derivatives of the listed generating differential syzygies and the commutator syzygies.

For this purpose, we invoke the following elementary lemma concerning linear algebraic syzygies. It is a particular case of Proposition 2.10 of [**40**].

Lemma 9.3. *Let $Z(x,y,z) \not\equiv 0$ be a function that depends smoothly on the variables $x = (x^1, \ldots, x^i)$, $y = (y^1, \ldots, y^j)$, $z = (z^1, \ldots, z^k)$. Suppose that, when subject to the constraints*

$$y = g(x), \qquad z = B(x)\,v + h(x), \tag{9.23}$$

where $v = (v^1, \ldots, v^l)$, and $B(x)$ is a given $k \times l$ matrix of constant rank for all x in some open domain, Z defines a syzygy, *in the sense that*

$$Z(x, g(x), B(x)\,v + h(x)) \equiv 0 \qquad \text{for all } x, v \text{ in some open domain.}$$

Then, locally,

$$Z(x,y,z) = \sum_{\nu=1}^{s} W_\nu(x,y,z)\, Z_\nu(x,y,z) + Z_0(x,y)$$

can be written as a combination of the generating syzygies

$$Z_\nu(x,y,z) = r_\nu(x)\,\big(\, z - h(x)\,\big),$$

in which $r_1(x), \ldots, r_s(x)$ form a basis for coker $B(x)$, *while $Z_0(x,y)$ defines a syzygy among the first two sets of variables only, meaning that $Z_0(x, g(x)) \equiv 0$.*

In our situation, suppose we have a syzygy Z of degree $n > n^\star$. We show that, modulo lower degree syzygies, it can be written in terms of the claimed generating syzygies. A straightforward induction will then complete the proof. To this end, we apply Lemma 9.3, under the following identifications of variables:

- z represents the highest degree variables $w_{\nu,N}$ appearing in Z, corresponding to the differentiated invariants $\mathcal{D}_N I_\nu$ of degree $n = \#N + \deg \widetilde{\sigma}_\nu$;
- y represents all the other variables $w_{\nu,M}$ consisting of lower order derivatives of the high degree differential invariants: $\mathcal{D}_M I_\nu$ with $\#M + \deg \widetilde{\sigma}_\nu < n$;
- v represents the variables w_L^α corresponding to the basic differential invariants I_L^α of degree $\#L = n$;
- x represents the variables w_J^α corresponding to the basic differential invariants I_J^α of all degrees (both low and high) $\#J < n$.
- The constraints (9.23) represent the recurrence formulae (9.6) for the differentiated invariants $\mathcal{D}_K I_\nu$; in particular, the second set refers to the formulae of highest degree $n = \#N + \deg \widetilde{\sigma}_\nu$.

To find the cokernel of the relevant matrix $B(w)$, let us write out the Gröbner basis polynomials explicitly:

$$\widetilde{\sigma}_\nu(s, S) = \sum_{\alpha=1}^{q} \sum_{\#J \leq n} C_{\alpha,\nu}^J \, s_J S^\alpha, \tag{9.24}$$

where the coefficients $C_{\alpha,\nu}^J$ may depend on the basic invariants $\mathbf{I}^{(1)}$ of degree ≤ 1 (if any). The associated differential invariant is, by (8.19),

$$I_\nu = I_{\widetilde{\sigma}_\nu} = \sum_{\alpha=1}^{q} \sum_{\#J \leq n} C_{\alpha,\nu}^J \, I_J^\alpha. \tag{9.25}$$

Thus, the corresponding syzygy (9.6) has the form

$$\mathcal{D}_N I_\nu = I_{s_N \widetilde{\sigma}_\nu} + M_{\nu,N}(\mathbf{I}_{basic}) = \sum_{\alpha=1}^{q} \sum_{\#J \leq n} C_{\alpha,\nu}^J \, I_{JN}^\alpha + M_{\nu,N}(\mathbf{I}_{basic}), \tag{9.26}$$

where the correction term $M_{\nu,N}(\mathbf{I}_{basic})$ is of lower order, and, from (9.24),

$$s_N \widetilde{\sigma}_\nu(s, S) = \sum_{\alpha=1}^{q} \sum_{\#J \leq n} C_{\alpha,\nu}^J \, s_{JN} S^\alpha. \tag{9.27}$$

Each row of the matrix B represents a top order syzygy (9.26), where $\deg \widetilde{\sigma}_\nu + \#N = n$. Its columns are indexed by the basic differential invariants I_K^α of degree (or order) $\#K = n$, and hence the coefficient $C_{a,\nu}^J$ appears in the column corresponding to I_{JN}^α; all other entries of this row are 0. Consequently, a vector $\mathbf{r} = (\ \ldots \ r_{N,\nu} \ \ldots \ )$, for $\#N = l$, belongs to the cokernel of B if and only if

$$\begin{aligned} 0 &= \sum_{\nu=1}^{\ell} \sum_{\#N=l} r_{N,\nu} \sum_{\alpha=1}^{q} \sum_{\#J\leq n} C^J_{\alpha,\nu}\, s_{JN} S^\alpha \\ &= \sum_{\nu=1}^{\ell} \sum_{\#N=l} r_{N,\nu} s_N \widetilde{\sigma}_\nu(s,S) = \sum_{\nu=1}^{\ell} q_\nu(s)\, \widetilde{\sigma}_\nu(s,S), \end{aligned}$$

where

$$q_\nu(s) = \sum_{\#N=l} r_{N,\nu} s_N.$$

Thus, the cokernel elements are in one-to-one correspondence with the algebraic syzygies among the Gröbner basis polynomials specified by $\mathbf{q}(s) = (q_1(s), \ldots, q_l(s))$, cf. (9.18). Since each algebraic syzygy corresponds to a combination of the essential differential syzygies and commutator syzygies, Lemma 9.3 implies that every syzygy of degree n can be written, modulo lower degree syzygies, in terms of the generating differential syzygies. An evident induction on degree will then establish a general Syzygy Theorem for the differential invariant algebra $\mathcal{I}(\mathcal{G})$ of an eventually locally freely acting analytic pseudo-group:

Theorem 9.4. *Let $\mathcal{G}$ be a Lie pseudo-group which acts locally freely on an open subset of the submanifold jet bundles $\mathrm{J}^n(M,p)$ for all $n \geq n^\star$. Then, every differential syzygy is a differential consequence of the syzygies among the differential invariants of order $\leq n^\star$, the finite generating system of algebraic syzygies among the Gröbner basis polynomials in $\widetilde{\mathcal{J}}^{\,\geq n^\star+1}$, and the commutator syzygies.*

One final observation: In all cases, the generating syzygies depend rationally on all variables with the possible exception of the undifferentiated differential invariants of degree 0. Thus, as in Theorem 8.4, if $\mathcal{G}$ acts transitively on M, or satisfies the hypothesis of that theorem, then all generating syzygies are rational functions of the variables w^*.

Remark: For finite-dimensional Lie group actions, another approach to the classification of syzygies in the differential invariant algebra appears in [**26**].

Example 9.5. For the pseudo-group in Examples 4.3, 6.1, and 8.7, the commutation relation for the invariant differential operators

$$[\,\mathcal{D}_1, \mathcal{D}_2\,] = I_2\, \mathcal{D}_1 - I_1\, \mathcal{D}_2 \tag{9.28}$$

can be deduced from the moving frame method, or simply by direct computation using the explicit formulas (8.42). There is a single generating syzygy among the Gröbner basis polynomials (8.43):

$$s_2\, \widetilde{\sigma}_1 - s_1\, \widetilde{\sigma}_2 = 0. \tag{9.29}$$

Since there are no low degree differential invariants, Theorem 9.4 implies that the syzygies among the differentiated invariants are all differential consequences of the commutation relation (9.28), as discussed above, along with the basic syzygy

$$\mathcal{D}_1 I_2 - \mathcal{D}_2 I_1 = 2. \tag{9.30}$$

The latter is a consequence of the recurrence formulae, [**45**; Example 32], or, simply, of a direct computation.

Lack of space prevents us from including any substantial new examples. Our previous work, [**10**, **45**], contains several worked examples, including that used by Kumpera, [**32**], to illustrate his approach (see also [**31**]), as well as the symmetry pseudo-group of the KP equation, which carries a Kac–Moody Lie algebra structure, [**13**]. Additional examples can be found in geometry, e.g., characteristic classes of foliations, cf. [**20**], conformal geometry and conformal field theory, [**16**, **18**], symplectic and Poisson geometry, [**36**, **40**], and the geometry of real hypersurfaces, [**11**], as well as symmetry groups of a wide variety of partial differential equations, [**28**, **40**], gauge theories, [**4**], fluid mechanics, [**2**, **7**], solitons and integrable systems, [**28**], image processing, [**53**, **57**], and geometric numerical integration, [**37**]. These will be the subject of future research.

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