On the Structure and Generators of Differential Invariant Algebras

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Abstract. The structure of algebras of differential invariants, particularly their generators, is investigated using the symbolic invariant calculus provided by the method of equivariant moving frames. We develop a computational algorithm that will, in many cases, determine whether a given set of differential invariants is generating. As an example, we establish a new result that the Gaussian curvature generates all the differential invariants for Euclidean surfaces in three-dimensional space.

1. Introduction.

The equivariant moving frame method, originally developed by Mark Fels and the author, [**1**, **17**] — see also Mansfield, [**10**] — provides a powerful algorithmic method for computing and studying differential invariants and, more generally, invariant differential forms, [**8**], of general Lie group actions. This paper focusses on the algebraic structures that are induced by the moving frame calculus, with particular attention paid to generators and relations. In the standard approach, one works in a differential geometric setting, and so the underlying category is smooth or analytic differential functions, classified up to functional independence. However, here we will take a more algebraic tack, and work in the category of polynomial functions, or, occasionally, rational functions. See also [**4**, **5**] for further development of the algebraic approach to moving frames.

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Remark: In this paper, the word "symbolic" is used in three different ways. The first is in the general computer algebra term "symbolic manipulation". Second is the "symbolic invariant calculus", a term inspired by [**10**], which is established by the method of moving frames, and effectively and completely determines the structure of the algebra of differential invariants and, more generally, invariant differential forms, purely symbolically, without any need for the explicit formulas for the moving frame, the differential invariants, the invariant differential forms, or the operators of invariant differentiation. Third is the "extended symbolic invariant calculus", which is an adaptation of the second usage, that is developed in Sections 6 and 7, and forms the basis of our computational algorithm.

The starting point is a smooth or analytic action of a real† r-dimensional Lie group G on a real m-dimensional manifold M. The action may be only local, and to avoid further complications with discrete symmetries, we will assume it to be connected, as in [**11**]. In the algebraic framework, we take M to be an open subset of $\mathbb{R}^m$, with fixed coordinates $z = (z^1, \ldots, z^m)$. We choose a basis for the infinitesimal generators

$$\mathbf{v}_\kappa = \sum_{i=1}^{m} \zeta_\kappa^i(z) \frac{\partial}{\partial z^i}, \qquad \kappa = 1, \ldots, r, \tag{1}$$

which are vector fields on M that span a Lie algebra isomorphic to the abstract Lie algebra $\mathfrak{g}$ of the Lie group G. For simplicity, we will assume that G acts *locally effectively on subsets*, [**13**], which is equivalent to requiring that its basis infinitesimal generators (1) be linearly independent vector fields when restricted to any open subset of M.

To ensure that the symbolic invariant calculus is fully algebraic, we will further assume that the group action is *infinitesimally algebraic*, meaning that either:

- G acts locally transitively on M, or, equivalently its infinitesimal generators $\mathbf{v}_1, \ldots, \mathbf{v}_r$ span the tangent space to M at all points; or,
- if intransitive, the coefficient functions $\zeta_\kappa^i(z)$ of the infinitesimal generators are polynomial functions of the coordinates on M.

In the latter case, we will also assume, in order to simplify the exposition, that G acts "locally transitively on the independent variables", in a sense defined at the beginning of Section 4. The preceding blanket assumptions hold in almost all examples of interest arising in applications.

Sections 3–5 review known facts and computational techniques from the method of moving frames. The new constructions and results appear in Sections 6–9, while Section 10 summarizes the resulting algorithm.

Remark: The methods to be presented can be extended to infinite-dimensional Lie pseudo-group actions. Although the constructions and underlying theory are significantly more complicated in the latter context, the resulting structure theory is of a very similar flavor; see [**19**, **20**, **21**] for details.

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† The constructions work in an identical fashion for complex Lie groups acting analytically on complex manifolds.

2. Multi-indices.

Let $p \geq 1$ be a fixed integer. A *p multi-index* is an ordered n-tuple $K = (k_1, \ldots, k_n)$ with $1 \leq k_\nu \leq p$, where $n = \# K$ is the *order* of K. We consider the empty 0-tuple $\mathrm{O} = ()$ to be the unique multi-index of order 0. Let $\mathbb{M}^{(n)}$ denote the set of all multi-indices of order $0 \leq k \leq n$. Note that $\mathbb{M}^{(n)}$ has cardinality $|\,\mathbb{M}^{(n)}\,| = 1 + p + \cdots + p^n = (p^{n+1} - 1)/(p-1)$. We further let $\mathbb{M} = \bigcup_{n \geq 0} \mathbb{M}^{(n)}$ denote the set of all p multi-indices.

A *symmetric p multi-index* J of order $n = \# J \geq 1$ is an unordered n-tuple $J = (j_1, \ldots, j_n)$ with $1 \leq j_\nu \leq p$, where we identify any two n-tuples that are obtained by permuting their indices. Thus any symmetric multi-index can be rearranged to be *non-decreasing*, meaning $j_i \leq j_{i+1}$ for $1 \leq i < \# J$. The empty order 0 multi-index O is considered to be symmetric. We let $\mathbb{S}^{(n)}$ denote the set of all symmetric multi-indices of order $0 \leq k \leq n$. Its cardinality is $|\,\mathbb{S}^{(n)}\,| = \binom{n+p}{p}$. Let $\mathbb{S} = \bigcup_{n \geq 0} \mathbb{S}^{(n)}$ denote the set of all symmetric p multi-indices.

3. The Jet Calculus.

Given the action of a Lie group on an m-dimensional manifold M, we are interested in the induced action on p-dimensional submanifolds $N \subset M$ for some fixed $1 \leq p < m$. We split the coordinates on $M \subset \mathbb{R}^m$ into independent and dependent variables

$$z = (x, u) = \{\, x^1, \ldots, x^p, u^1, \ldots, u^q \,\},$$

where $p + q = m$. We will restrict our attention to submanifolds that can be identified with graphs of smooth functions $u = f(x)$. For details, including extensions to general p-dimensional submanifolds, see [**11**].

The corresponding *jet space* of order $0 \leq n \leq \infty$, denoted by $\mathrm{J}^n = \mathrm{J}^n(M, p)$, is defined as the space of equivalence classes of p-dimensional submanifolds under the equivalence relation of n^{th} order contact. It has induced local coordinates

$$(x, u^{(n)}) = (\ldots x^i \ldots u^\alpha_J \ldots), \qquad i = 1, \ldots, p, \qquad \alpha = 1, \ldots, q, \qquad J \in \mathbb{S}^{(n)},$$

where we identify $u^\alpha_J = u^\alpha_{j_1 \ldots j_k}$, where $k = \# J$, with the partial derivative $\partial^k u^\alpha / \partial x^J$, so the equality of mixed partials is reflected in the fact that J is a symmetric multi-index. The dependent variables $u^\alpha = u^\alpha_{\mathrm{O}}$ are identified as those jet coordinates with empty multi-index $\mathrm{O} = ()$, so that $\mathrm{J}^0 \simeq M$. By a *differential function* (respectively, *differential polynomial*) we mean a smooth (respectively, polynomial) function $F(x, u^{(n)})$ of the jet coordinates.

In the jet space calculus, the *total derivative operators* $D_1, \ldots, D_p$ are derivations that act on differential functions (polynomials) by differentiating with respect to the independent variables $x^1, \ldots, x^p$, treating the jet variables u^α_J as functions thereof; they are thus characterized by their action on the individual jet coordinates:

$$D_i x^j = \delta^i_j, \quad D_i u^\alpha_J = u^\alpha_{J,i}, \quad i, j = 1, \ldots, p, \quad \alpha = 1, \ldots, q, \quad J \in \mathbb{S},$$

where δ^i_j is the Kronecker delta, and, given $J = (j_1, \ldots, j_k) \in \mathbb{S}$, we define the symmetric multi-index $(J, i) = (j_1, \ldots, j_k, i) \in \mathbb{S}$ of order $k+1$. Thus, we can write

$$D_i = \frac{\partial}{\partial x^i} \;+\; \sum_{\alpha=1}^{q} \sum_{J \in \mathbb{S}} u^\alpha_{J,i} \frac{\partial}{\partial u^\alpha_J}\,, \qquad\qquad i = 1, \ldots, p. \tag{2}$$

The total derivative operators mutually commute:

$$[\,D_i, D_j\,] = D_i\, D_j - D_j\, D_i = 0.$$

Higher order total derivatives are obtained by composition

$$D_J = D_{j_1} \;\cdots\; D_{j_k}, \qquad J = (j_1, \ldots, j_k) \in \mathbb{S}, \tag{3}$$

where commutativity is reflected in the fact that J is taken to be a symmetric multi-index. In particular, $D_{\mathrm{O}} = \mathbb{1}$ is the identity operator.

The induced action of the Lie group G on p-dimensional submanifolds induces an action on the jet spaces J^n, called the *prolonged action*. Its infinitesimal generators have the form

$$\mathbf{v}_\kappa = \sum_{i=1}^{p} \xi^i_\kappa(x,u)\,\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{J\in\mathbb{S}} \varphi^\alpha_{J,\kappa}(x,u^{(\#J)})\,\frac{\partial}{\partial u^\alpha_J}\,, \qquad \kappa = 1,\ldots,r, \tag{4}$$

where, by the well-known *prolongation formula*, [**11**],

$$\varphi^\alpha_{J,\kappa} = \mathbf{v}_\kappa(u^\alpha_J) = D_J \left(\varphi^\alpha_\kappa - \sum_{i=1}^{p} \xi^i_\kappa\, u^\alpha_i \right) + \sum_{i=1}^{p} \xi^i_\kappa u^\alpha_{J,i}. \tag{5}$$

Note: In view of the formula (2) for the total derivatives, the coefficients $\varphi^\alpha_{J,\kappa}$ depend polynomially on the jet coordinates u^β_K of orders $\#K \geq 1$. Hence, under our assumption that the action of G is infinitesimally algebraic, each prolonged infinitesimal generator (4) is a derivation of the space of differential polynomials.

A *differential invariant* is, by definition, an invariant differential function $I(x,u^{(n)})$. The infinitesimal invariance condition requires

$$\mathbf{v}_\kappa(I) = 0, \qquad \kappa = 1,\ldots,r,$$

which, by connectivity of the (prolonged) group action, is necessary and sufficient for invariance of the function I. One method for determining the invariants is to solve this system of homogeneous linear partial differential equations, [**11**]. However, the moving frame method is more direct and also has the advantage of being purely algebraic, and hence can be readily implemented in standard computer algebra systems.

4. Invariantization.

In addition to assuming that G acts infinitesimally algebraically on M, we will also, merely for the purpose of simplifying the notation and presentation, assume that it acts "locally transitively on the independent variables", meaning that the projected infinitesimal generators

$$\widehat{\mathbf{v}}_\kappa = \sum_{i=1}^{p} \xi^i_\kappa(x,u)\,\frac{\partial}{\partial x^i}\,, \qquad \kappa = 1,\ldots,r, \tag{6}$$

span a subspace of dimension p at each point $(x,u) \in M$. If G itself acts locally transitively on M, this condition is automatically satisfied.

By a general result, [**12**], local effectiveness implies that the prolonged group action is locally free[†] on a dense open subset of a jet space of sufficiently high order, say s. By a *local cross-section*, we mean a submanifold $\mathcal{K} \subset \mathrm{J}^s$ of complementary dimension that intersects the prolonged group orbits transversally in at most one point. Such a cross-section is defined by the equations

$$Z^\sigma(x, u^{(s)}) = c^\sigma, \qquad \sigma = 1, \ldots, r, \tag{7}$$

prescribed by r independent differential functions $Z^1, \ldots, Z^r$ of order $\leq s$ and r constants $c^1, \ldots, c^r \in \mathbb{R}$. To remain in the algebraic category, we assume that the Z^σ are polynomial functions of the jet coordinates. The simplest, and by far the most common, choice is when the Z^σ's are individual jet coordinates, in which case (7) is said to define a *coordinate cross-section*. Our blanket assumption that G acts locally transitively on the independent variables implies that we can, and will, always select the first p cross-section functions to be the independent variables: $Z^i = x^i$ for $i = 1, \ldots, p$. If G acts transitively on M, then we will select the next $q = m - p$ of them to be the dependent variables: $Z^{\alpha+p} = u^\alpha$ for $\alpha = 1, \ldots, q$. The construction of the moving frame map from the cross-section equations (7) follows as in [**1, 17**]; since we do not require these formulas in the symbolic calculus employed here, we will not dwell on the details.

Specification of the cross-section and consequent moving frame induces a process of *invariantization*, denoted by ι, that associates to each differential function F the unique differential invariant $I = \iota(F)$ that agrees with F on the cross-section. In particular, if I is a differential invariant, then $\iota(I) = I$. Thus, the invariantization process defines a projection from the algebra of differential functions to the algebra of differential invariants: $\iota(\iota(F)) = \iota(F)$. Moreover, it clearly respects all algebraic operations, and hence defines an algebra morphism. On the other hand, the resulting differential invariants are not necessarily polynomial in the jet coordinates, being prescribed by the moving frame solution to the polynomial cross-section equations, (7). If the group acts algebraically (which is not guaranteed by our assumptions on its infinitesimal generators), then the resulting differential invariants are algebraic functions of the jet coordinates, [**4, 5**]. See [**9**] for a (non-constructive) version based on rational differential invariants. In the symbolic moving frame calculus, the explicit formulas for the differential invariants are not required, although they can, at least modulo algebraic complications, be explicitly constructed through an application of the invariantization process.

In particular, the invariantization of each differential function used to define the cross-section (7) is the corresponding normalization constant:

$$\iota(Z^\sigma) = c^\sigma, \qquad \sigma = 1, \ldots, r. \tag{8}$$

These are commonly referred to as the *phantom differential invariants*. Thus, in view of our specified choice of cross-section as predicated on the assumption that the group acts locally transitively on the independent variables, all the independent variables invariantize to constants:

$$\iota(x^i) = c^i, \qquad i = 1, \ldots, p, \tag{9}$$

[†] A group action is *locally free* if the isotropy subgroup at each point is discrete.

being the first p of the phantom differential invariants (8). The *basic differential invariants* are obtained by invariantization of the remaining jet coordinates:

$$I^\alpha_J = \iota(u^\alpha_J), \qquad \alpha = 1,\ldots,q, \qquad J \in \mathbb{S}. \tag{10}$$

If G acts transitively, then, again by our assumption on the form of the cross-section, all the $I^\alpha = \iota(u^\alpha)$ are also constant phantom invariants. Since the invariantization process respects all algebraic operations, if

$$F(x, u^{(n)}) = F(\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ )$$

is any differential function, then

$$\iota(F) = F(\ \ldots\ \iota(x^i)\ \ldots\ \iota(u^\alpha_J)\ \ldots\ ) = F(\ \ldots\ c^i\ \ldots\ I^\alpha_J\ \ldots\ ). \tag{11}$$

In particular, if $J(x, u^{(n)})$ is any differential invariant, then

$$J(\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ ) = J(\ \ldots\ c^i\ \ldots\ I^\alpha_J\ \ldots\ ). \tag{12}$$

Equation (12) is known as the *Replacement Rule*, and allows one to immediately and uniquely "rewrite" any differential invariant in terms of the basic differential invariants (10), merely by replacing each jet coordinate by its corresponding basic differential invariant. Thus, the basic differential invariants form a complete system of differential invariants in the sense that any other differential invariant is a function thereof. Interestingly, even though the basic differential invariants need not be polynomial or even algebraic functions, every polynomial (algebraic) differential invariant can be written as a polynomial (algebraic) function thereof.

On the other hand, the basic differential invariants are not functionally independent, but are subject to the r polynomial equations provided by the invariantized cross-section relations (8):

$$\begin{aligned} &\iota(Z^i) = \iota(x^i) = c^i, && i = 1,\ldots,p, \\ &Z^\sigma(\ \ldots\ c^i\ \ldots\ I^\alpha_J\ \ldots\ ) = c^\sigma, && \sigma = p+1,\ldots,r, \end{aligned} \tag{13}$$

which form a complete system of functional (polynomial) relations. In particular, if we are using a coordinate cross-section, then the non-phantom basic differential invariants provide a complete system of functionally independent differential invariants, in the sense that any other differential invariant can be locally uniquely written as a function (not necessarily polynomial) thereof.

In the sequel, we let

$$\mathcal{I}^{(n)} = \iota(u^{(n)}) = \left\{\ I^\alpha_J = \iota(u^\alpha_J)\ \middle|\ \alpha = 1,\ldots,q, \quad J \in \mathbb{S}^{(n)}\ \right\} \tag{14}$$

denote the basic differential invariants obtained by invariantizing the dependent variable jet coordinates of order $\leq n$, including all such constant phantom invariants. Observe that, since the moving frame has order s, the order of each I^α_J is $\leq \max\{s, \#J\}$.

The *invariant differential operators* are obtained by invariantizing the total derivative operators (2):

$$\mathcal{D}_i = \iota(D_i), \qquad i = 1,\ldots,p. \tag{15}$$

As before, in the symbolic moving frame calculus, there is no need for their explicit formulas, although these can (modulo computational complications) be found through an explicit implementation of the invariantization process, [**1**]. Invariance means that if I is any differential invariant, so is $\mathcal{D}_i I$. The invariant differential operators produced by the moving frame construction do not, in general, commute; see equation (22) below for details. Higher order invariant differential operators are obtained by iteration:

$$\mathcal{D}_K = \mathcal{D}_{k_1} \mathcal{D}_{k_2} \;\cdots\; \mathcal{D}_{k_l}, \qquad\qquad K = (k_1, \ldots, k_l) \in \mathbb{M}, \tag{16}$$

where the non-commutativity of the $\mathcal{D}_i$'s is reflected in the fact that K is an ordered multi-index. As before, $\mathcal{D}_O = 1\!\!1$ is the identity map.

The *differential invariant algebra* will mean the algebra generated by the basic differential invariants, which could be polynomial, rational, or smooth functions thereof, depending on the context, along with the invariant differential operators. In the algorithm described below, we will restrict attention to the polynomial category.

The fundamental Lie–Tresse Theorem, [**1**, **9**, **12**, **21**], states that the differential invariant algebra is generated by a finite number of generating differential invariants through the operations of invariant differentiation.

Theorem 1. *Given a Lie group action on submanifolds of dimension p as above, there exist a finite number of* generating differential invariants *$I^1, \ldots, I^l$ such that every differential invariant can be locally expressed as a function of them and their invariant derivatives, namely $\mathcal{D}_K I^\sigma$ for $K \in \mathbb{M}$ and $\sigma = 1, \ldots, l$.*

The Lie–Tresse Theorem can be viewed, in a certain sense, as the analogue of the Hilbert Basis Theorem for differential invariant algebras. The moving frame recurrence formulas can be used to prove Theorem 1 constructively, in that they identify a set of generating differential invariants; see below. A significant problem, and the main focus of the latter part of this paper, is to find minimal generating sets of differential invariants since those identified via the moving frame calculus are typically far from minimal, and contain many redundancies. There is also an analogue of the Hilbert Syzygy Theorem for differential invariant algebras; see [**21**] for details.

5. The Recurrence Formulae.

Besides the systematic and algorithmic methods underlying its construction, the most important new contribution of the equivariant moving frame method, [**1**, **17**], is the general recurrence formula, which we now state for differential functions. See [**8**] for the extension to invariant differential forms.

While, as we noted above, the invariantization process respects all algebraic operations, it does not respect differentiation. The recurrence formula tells us how the operations of invariantization and differentiation are related.

Theorem 2. *Given $1 \leq i \leq p$, let $\mathcal{D}_i = \iota(D_i)$ be the invariant differential operator* (15) *produced by the moving frame invariantization process. Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be the prolonged infinitesimal generators* (4) *of the group action. Let F be a differential function* *and $\iota(F)$ its moving frame invariantization. Then*

$$\mathcal{D}_i\big[\,\iota(F)\,\big] = \iota\big[\,D_i(F)\,\big] + \sum_{\kappa=1}^{r} R_i^\kappa\,\iota\big[\,\mathbf{v}_\kappa(F)\,\big], \qquad i = 1,\ldots,p, \tag{17}$$

for certain differential invariants

$$\mathcal{R} = \{\ R_i^\kappa \ \mid\ \kappa = 1,\ldots,r, \quad i = 1,\ldots,p\ \}. \tag{18}$$

In particular, setting $F = u_J^\alpha$ in (17) leads to the recurrence formulae for the basic differential invariants:

$$\mathcal{D}_i I_J^\alpha = I_{J,i}^\alpha + \sum_{\kappa=1}^{r} R_i^\kappa\,\iota(\varphi_{J,\kappa}^\alpha), \tag{19}$$

where $\varphi_{J,\kappa}^\alpha$ are the prolonged infinitesimal generator coefficients (5).

The differential invariants R_i^κ are known as the *Maurer–Cartan invariants* since they appear as the coefficients of the pull-backs of the Maurer–Cartan forms on the Lie group G under the equivariant moving frame map, [**1**]. Fortunately, we do not need to know or understand this fact since the Maurer–Cartan invariants can be effectively computed by solving the *phantom recurrence formulae*. Namely, setting $F = Z^\sigma$ to be the cross-section differential functions in (17), and noting that $\iota(Z^\sigma) = c^\sigma$ is constant, we deduce

$$0 = \iota\big[\,D_i(Z^\sigma)\,\big] + \sum_{\kappa=1}^{r} R_i^\kappa\,\iota\big[\,\mathbf{v}_\kappa(Z^\sigma)\,\big], \qquad i = 1,\ldots,p. \tag{20}$$

For each fixed $i = 1,\ldots,p$, the corresponding phantom recurrence formulae (20) are a system of r linear algebraic equations for the r Maurer–Cartan invariants R_i^κ, $\kappa = 1,\ldots,r$. The condition that (7) define a valid cross-section implies that these p linear systems all have a unique solution. Thus, under our assumptions on the group action, the coefficients of the phantom recurrence formulae (20) are polynomial functions of the basic differential invariants, which implies that the Maurer–Cartan invariants $\mathcal{R}$ are *rational functions* of the basic differential invariants $\mathcal{I}^{(s)}$.

As noted above, the invariant differential operators produced by the moving frame construction do not, in general, commute. Their commutators can be written in the following form:

$$[\,\mathcal{D}_j, \mathcal{D}_k\,] = \mathcal{D}_j\,\mathcal{D}_k - \mathcal{D}_k\,\mathcal{D}_j = \sum_{i=1}^{p} Y_{jk}^i \mathcal{D}_i, \qquad j,k = 1,\ldots,p, \tag{21}$$

where the coefficients

$$Y_{jk}^i = -\,Y_{kj}^i = \sum_{\kappa=1}^{r} \big[\,R_k^\kappa\,\iota(D_j\xi_\kappa^i) - R_j^\kappa\,\iota(D_k\xi_\kappa^i)\,\big], \qquad i,j,k = 1,\ldots,p, \tag{22}$$

are certain differential invariants known as the *commutator invariants*. See [**1**, **8**] for details on the derivation of this formula.

6. The Symbolic Invariant Calculus.

The upshot of the preceding developments is that, remarkably, we *do not need to know* the actual formulas for the moving frame, nor the differential invariants, nor the invariant differential operators, in order to determine the structure of the resulting differential invariant algebra! In other words, we can work entirely *symbolically* when analyzing the differential invariant algebra, whose structure is entirely determined by the recurrence formulae (19, 20) and the commutator formulae (21, 22). Let us now formalize this procedure.

To this end, and under our blanket assumptions on the Lie group action and choice of moving frame cross-section, we introduce new "symbolic" variables

$$v = (\ \ldots\ v^\alpha_J\ \ldots\ ), \qquad \alpha = 1, \ldots, q, \qquad J \in \mathbb{S},$$

which will serve to represent the basic differential invariants: $v^\alpha_J \longleftrightarrow I^\alpha_J$. We will also set

$$v^{(n)} = (\ \ldots\ v^\alpha_J\ \ldots\ ), \qquad \alpha = 1, \ldots, q, \qquad J \in \mathbb{S}^{(n)},$$

for $0 \leq n \leq \infty$, so that $v = v^{(\infty)}$. Let us define the *symbolic invariantization process* $\widetilde{\iota}$, acting on differential functions $F(x, u^{(n)})$, by the following rule based on (11):

$$\widetilde{\iota}\left[\, F(x, u^{(n)})\,\right] = F(\ \ldots\ \widetilde{\iota}(x^i)\ \ldots\ \widetilde{\iota}(u^\alpha_J)\ \ldots\ ) = F(\ \ldots\ c^i\ \ldots\ v^\alpha_J\ \ldots\ ) = F(v). \tag{23}$$

As such the symbolic variables will be subject to the polynomial cross-section relations

$$Z^\sigma(v) = c^\sigma, \qquad \sigma = p+1, \ldots, r, \tag{24}$$

which are based on (7), keeping (9) in mind. The algebraic variety defined by the polynomial equations (24) will be called the *cross-section variety*. All symbolic calculations take place on this variety. As noted before, the simplest case is when we choose a coordinate cross-section, in which case the variables v^α_J that correspond to the jet coordinates u^α_J used to specify the cross-section are constant. Thus, in this case, the cross-section variety is simply an affine subspace.

As we saw above, the differential invariant algebra structure is completely encoded by the recurrence relations, specifically (19), which determine how the invariant differential operators act on the basic differential invariants. Rather than use the invariant differential operators directly, it will help to replace them by symbolic derivations. Namely, for $i = 1, \ldots, p$, let $\widetilde{\mathcal{D}}_i$ be the derivation defined by its action on the symbolic variables:

$$\widetilde{\mathcal{D}}_i\, v^\alpha_J = v^\alpha_{J,i} + \sum_{\kappa=1}^{r} \widetilde{R}^\kappa_i\, \widetilde{\iota}(\varphi^\alpha_{J,\kappa}), \tag{25}$$

where $\varphi^\alpha_{J,\kappa}$ are the prolonged infinitesimal generator coefficients (5), while $\widetilde{R}^\kappa_i = \widetilde{\iota}(R^\kappa_i)$ are the *symbolic Maurer–Cartan invariants*, which can be obtained by replacing the basic differential invariants in the formulae for the Maurer–Cartan invariants R^κ_i by their symbolic counterparts, $I^\alpha_J \longmapsto v^\alpha_J$, or, equivalently, by solving the linear system of equations

$$0 = \widetilde{\iota}\left[\, D_i Z^\sigma\,\right] + \sum_{\kappa=1}^{r} \widetilde{R}^\kappa_i\, \widetilde{\iota}\left[\, \mathbf{v}_\kappa(Z^\sigma)\,\right], \qquad \sigma = 1, \ldots, r, \qquad i = 1, \ldots, p, \tag{26}$$

associated with the (symbolic) phantom invariants, cf. (20). Since, under our assumptions on the group action, the coefficients of the linear system are polynomials in the symbolic variables v, the Maurer–Cartan invariants will be rational functions of v. As above, the calculations are performed on the cross-section variety (24).

As before, the symbolic invariant derivations so constructed will not, in general, commute. Their commutators follow from (21, 22):

$$[\,\widetilde{\mathcal{D}}_j, \widetilde{\mathcal{D}}_k\,] = \widetilde{\mathcal{D}}_j\,\widetilde{\mathcal{D}}_k - \widetilde{\mathcal{D}}_k\,\widetilde{\mathcal{D}}_j = \sum_{i=1}^{p} \widetilde{Y}^i_{jk}\widetilde{\mathcal{D}}_i, \tag{27}$$

where

$$\widetilde{Y}^i_{jk} = \widetilde{\iota}(Y^i_{jk}) = \sum_{\kappa=1}^{r} \left[\, \widetilde{R}^\kappa_k\,\widetilde{\iota}(D_j\xi^i_\kappa) - \widetilde{R}^\kappa_j\,\widetilde{\iota}(D_k\xi^i_\kappa)\,\right]. \tag{28}$$

are the *symbolic commutator invariants*. We recursively construct their higher order counterparts

$$\widetilde{\mathcal{D}}_K = \widetilde{\mathcal{D}}_{k_1}\ \cdots\ \widetilde{\mathcal{D}}_{k_l}, \qquad K \in \mathbb{M}^{(n)}, \qquad 0 \le l = \#K \le n, \tag{29}$$

keeping in mind that, owing to their non-commutativity, the multi-index K is *unordered*. (For completeness, $\widetilde{\mathcal{D}}_O = \mathbb{1}$ is the identity operator.) On the other hand, by invoking the commutator relations (27), one can adapt a Poincaré–Birkhoff–Witt type argument, [**7**], to restrict to only nondecreasing multi-indices, although this appears unnecessary, modulo possibly exploiting it in order to speed up the computational algorithm.

7. The Extended Symbolic Invariant Calculus.

The fact that the symbolic Maurer–Cartan invariants are, in general, rational functions of the symbolic variables v takes us outside our polynomial "comfort zone". Moreover, the algorithm to be developed below will ask that we not explicitly compute them via solving the phantom recurrence formulas (26) in advance. Instead, to maintain polynomiality, we will introduce a further set of symbolic variables w^κ_i to represent each Maurer–Cartan invariant R^κ_i, and rewrite (19) in the form

$$\widetilde{\mathcal{D}}_i\, v^\alpha_J = v^\alpha_{J,i} + \sum_{\kappa=1}^{r} w^\kappa_i\,\widetilde{\iota}(\varphi^\alpha_{J,\kappa}). \tag{30}$$

These new symbolic variables will be subject to the linear algebraic constraints

$$0 = C^\sigma_i(v,w) \equiv \widetilde{\iota}\left[\, D_i Z^\sigma\,\right] + \sum_{\kappa=1}^{r} w^\kappa_i\,\widetilde{\iota}\left[\,\mathbf{v}_\kappa(Z^\sigma)\,\right], \qquad \sigma = 1,\ldots,r, \qquad i = 1,\ldots,p, \tag{31}$$

corresponding to (26), whose coefficients depend polynomially on v. Solving this linear system will recover the symbolic Maurer–Cartan invariants $\widetilde{R}^\kappa_i$, as constructed in the preceding section, but here we will not do this, and instead work on the polynomial subvariety it defines.

We will also need to symbolically differentiate the variables representing the Maurer–Cartan invariants, and hence include further symbolic variables

$$w = (\ \ldots \ w^\kappa_{i;K} \ \ldots \ ), \qquad \kappa = 1, \ldots, r, \qquad i = 1, \ldots, p, \qquad K \in \mathbb{M}, \tag{32}$$

where K is an ordered multi-index owing to the non-commutativity of the symbolic invariant derivations. We also set

$$w^{(n)} = (\ \ldots \ w^\kappa_{i;K} \ \ldots \ ), \qquad \kappa = 1, \ldots, r, \qquad i = 1, \ldots, p, \qquad K \in \mathbb{M}^{(n)}, \tag{33}$$

for $0 \le n \le \infty$, so that, for instance, $w^{(0)} = (\ \ldots \ w^\kappa_i \ \ldots \ )$ represents the undifferentiated Maurer–Cartan invariants $\mathcal{R}$, while $w = w^{(\infty)}$.

We extend the symbolic invariant derivations (25) to the polynomial algebra generated by (v, w) by setting

$$\widetilde{\mathcal{D}}_j \, w^\kappa_{i;K} = w^\kappa_{i;j,K}. \tag{34}$$

Their commutators are as in (27) above, but now we express the symbolic commutator invariants in terms of the symbolic Maurer–Cartan variables:

$$\widetilde{Y}^i_{jk} = \widetilde{\iota}(Y^i_{jk}) = \sum_{\kappa=1}^r \left[\, w^\kappa_k \, \widetilde{\iota}(D_j \xi^i_\kappa) - w^\kappa_j \, \widetilde{\iota}(D_k \xi^i_\kappa) \, \right]. \tag{35}$$

The symbolic differentiated Maurer–Cartan invariants (34) are subject to a system of linear constraints, with polynomially v dependent coefficients, which are obtained by symbolically differentiating (31):

$$\begin{gathered} 0 = C^\sigma_{i;K}(v, w) \equiv \widetilde{\mathcal{D}}_K C^\sigma_i(v, w) = \widetilde{\mathcal{D}}_K \left(\widetilde{\iota}\left[\, D_i(Z^\sigma) \, \right] + \sum_{\kappa=1}^r \, w^\kappa_i \, \widetilde{\iota}\left[\, \mathbf{v}_\kappa(Z^\sigma) \, \right] \right), \\ \sigma = 1, \ldots, r, \quad i = 1, \ldots, p, \quad K \in \mathbb{M}. \end{gathered} \tag{36}$$

We will call the subvariety determined by $(23, 31, 36)$ the *extended cross-section variety*. As above, one can appeal to the commutation formulae (27) to restrict to non-decreasing multi-indices K, but we will not use this option in what follows.

8. Independence.

Let us review a basic result on functional dependence that will be used in the sequel. Given a smooth function $f \colon \mathbb{R}^m \to \mathbb{R}^k$ depending on $x = (x^1, \ldots, x^m) \in \mathbb{R}^m$, we denote its $k \times m$ *Jacobian matrix* by

$$\nabla f = \left(\frac{\partial f^i}{\partial x^j} \right). \tag{37}$$

Theorem 3. *The components of $f = (f^1(x), \ldots, f^k(x))$ are functionally independent if and only if their Jacobian matrix has* $\operatorname{rank} \nabla f = k$.

See [**11**; § 2.1] for details, including a precise definition of functional independence. For our purposes, the following corollary will be of crucial importance.

Proposition 4. *Let M be an m-dimensional manifold. Suppose that $f\colon M \to \mathbb{R}^k$ and $g\colon M \to \mathbb{R}^l$ are smooth functions. Assume that the rank of their Jacobian matrices ∇f and ∇g are constant. Then we can locally write $f = h \circ g$ where $h\colon \mathbb{R}^l \to \mathbb{R}^k$ is smooth if and only if*

$$\operatorname{rank}\begin{pmatrix} \nabla f \\ \nabla g \end{pmatrix} = \operatorname{rank} \nabla g. \tag{38}$$

More generally, suppose

$$M = \{\, x \in \mathbb{R}^n \mid c(x) = 0 \,\}$$

is a submanifold defined by the vanishing of a function $c\colon \mathbb{R}^n \to \mathbb{R}^j$. We assume that ∇c is also of constant rank in an open neighborhood of M. Suppose $f\colon \mathbb{R}^n \to \mathbb{R}^k$ and $g\colon \mathbb{R}^n \to \mathbb{R}^l$. Then Proposition 4 becomes the statement that, locally,

$$f \mid M = h \circ g \mid M \qquad \text{if and only if} \qquad \operatorname{rank}\begin{pmatrix} \nabla f \\ \nabla g \\ \nabla c \end{pmatrix} = \operatorname{rank}\begin{pmatrix} \nabla g \\ \nabla c \end{pmatrix} \quad \text{on } M. \tag{39}$$

In other words, given $y^i = f^i(x^1, \ldots, x^n)$ for $i = 1, \ldots, k$, and $z^j = g^j(x^1, \ldots, x^n)$ for $j = 1, \ldots, l$, and assuming the Jacobian matrices have constant rank, then, locally, we can write $y^i = h^i(z^1, \ldots, z^l)$ for $i = 1, \ldots, k$ on the submanifold M defined by $c(x) = 0$ if and only if condition (39) holds on M.

9. Generating Differential Invariants.

We now turn to the problem of finding generating sets of differential invariants, in accordance with the Lie–Tresse Theorem 1. There are two a priori known generating sets of differential invariants. First:

Theorem 5. *If the moving frame has order s, then $\mathcal{I}^{(s+1)}$ is a generating set.*

The proof relies on the structure of the basic recurrence formulae (19), the key observation being that if $k = \#J \geq s$, then the only term on the right hand side of order $k+1$ is the leading term $I^\alpha_{J,i}$ — all the summation terms, including the Maurer–Cartan invariants, are of order $\leq k$. See also [**14**] for further details. The next result is due to Hubert, [**3**], and is again based on an analysis of the recurrence relations.

Theorem 6. *The invariants $\mathcal{I}^{(0)} \cup \mathcal{R}$ form a generating set.*

In particular, if G acts transitively, then the invariants $\mathcal{I}^{(0)} = \iota(u)$ are all phantom and hence constant and therefore in this case the Maurer–Cartan invariants $\mathcal{R}$ form a generating set.

In both cases, the generating sets are, typically, far from minimal and there are many redundancies. Hence, the quest is to find minimal generating sets. Unfortunately, apart from the case of curves, where $p = 1$, there is as yet no general construction of minimal generating sets or computational test that will ensure whether or not a given generating set is minimal — except in the obvious situation where one can find a single generator. In low dimensional examples, e.g., surfaces in $\mathbb{R}^3$, this happens surprisingly often, cf. [**6**, **15**, **16**, **22**].

To this end, we will now describe an algorithm[†] for determining if a given set of differential invariants

$$J = (J^1, \ldots, J^l)$$

forms a generating set. We will work in the extended symbolic invariant calculus, as presented in Section 7. The proposed generating differential invariants are represented symbolically by functions

$$J(v) = \big(J^1(v), \ \ldots \ , J^l(v) \big) \tag{40}$$

depending on a finite number of the symbolic variables v^α_J. To remain in the polynomial category, we assume that these are polynomials. In most cases, they are, in fact, individual v^α_J's or perhaps simple combinations thereof. We could also allow them to depend on the symbolic Maurer–Cartan variables w; this will not change the ensuing argument. Let

$$J^\nu_K(v, w) = \widetilde{\mathcal{D}}_K J^\nu, \qquad \nu = 1, \ldots, l, \qquad K \in \mathbb{M}, \tag{41}$$

be the symbolic derivatives of the proposed generating invariants. We will call $\#K$ the *level* of the differentiated symbolic invariant (41).

Now suppose that

$$I(v, w) = \big(I^1(v, w), \ \ldots \ , I^k(v, w) \big) \tag{42}$$

is a known generating set, represented symbolically. A simple choice based on Theorem 5, and the one preferred here, is to set $I = v^{(s+1)}$ where s is the order of the moving frame. Alternatively, one could invoke Theorem 6 and take $I = w^{(0)}$ to be the (symbolic) Maurer–Cartan invariants. Typically, there are obvious redundancies among these generating invariants, including those prescribed by the extended cross-section variety (31, 36), and one can use these to reduce their initial number in order to streamline the ensuing computations. Clearly the J's are generating if we can write each I^σ as a function of the J^ν_K's, as always when restricted to the extended cross-section variety. If any I^σ already appears among the J^ν's, this requirement is automatic and so these can also be set aside when implementing the ensuing algorithm.

We now invoke Proposition 4, in the reformulation given at the very end of Section 8. The variables x represent the symbolic variables v, w. Of course, there are infinitely many of the latter; however, each function depends on only finitely many of them, and so, in any finite calculation, one can ignore all symbolic variables of a sufficiently higher order. The functions $y = f(x)$ will represent the generating invariants in (42), so $y = I(v, w)$, which can be reduced by discarding redundancies as discussed above, and we let $\widetilde{I}$ denote the remaining differential invariants. The functions $z = g(x)$ will represent the proposed

† See Section 10 for a summary of the algorithm to be developed here.

generating differential invariants (40) and their derivatives (41) up to a specified level $n \geq 0$, so

$$z = J^{(n)}(v,w) = (\ \ldots\ J^{\nu}_{K}(v,w)\ \ldots\ ), \qquad \nu = 1,\ldots,l, \qquad K \in \mathbb{M}^{(n)}. \tag{43}$$

The polynomial constraints $c(x) = 0$ represent the extended cross-section variety (36) up to level n, so

$$0 = C^{(n)}(v,w) = (\ \ldots\ C^{\sigma}_{i;K}(v,w)\ \ldots\ ), \quad \sigma = 1,\ldots,r, \quad i = 1,\ldots,p, \quad K \in \mathbb{M}^{(n)}. \tag{44}$$

Thus, according to (39), we need to compute the gradients (Jacobian matrices) of the right hand sides of (42, 43, 44) with respect to the v's and w's, whereby $\nabla = (\nabla_v, \nabla_w)$, and we set

$$\mathbb{J}^{(n)} = \begin{pmatrix} \nabla J^{(n)} \\ \nabla C^{(n)} \end{pmatrix}, \qquad \mathbb{I}^{(n)} = \begin{pmatrix} \nabla \widetilde{I} \\ \nabla J^{(n)} \\ \nabla C^{(n)} \end{pmatrix}. \tag{45}$$

As a direct corollary of (39), we have established our desired criterion.

Theorem 7. *The differential invariants $\{J^1,\ldots,J^l\}$ form a generating set if and only if*

$$\operatorname{rank} \mathbb{I}^{(n)} = \operatorname{rank} \mathbb{J}^{(n)} \tag{46}$$

for some level $n \geq 0$.

Indeed, if (46) holds, then Proposition 4 implies that, on the extended cross-section variety, we can express all the components of the known generating set $I(v,w)$ in terms of the differentiated invariants $J^{\nu}_{K}(v,w) = \widetilde{\mathcal{D}}_K J^{\nu}$, which implies that J is also a generating set of differential invariants.

Remark: Ideally, the rank criterion (46) should be checked symbolically. In practice, this is beyond the current capabilities of MATHEMATICA, and so instead it is checked by making several substitutions of random integers for the variables in the matrices. While not 100% foolproof, this method works well in all calculations performed to date.

Here is the one example that has been computed so far. Although not so complicated, it's starting to reach the limits of what MATHEMATICA is capable of — although a more clever programming scheme might push it a bit further. It would also be good to reprogram this in a more powerful computer algebra system.

Example 8. Consider the action of the Euclidean group $\mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$, consisting of all rigid motions, on surfaces $S \subset \mathbb{R}^3$. For simplicity, we assume the surface is given by the graph of a function $u = f(x,y)$. The corresponding local coordinates on the surface jet bundle are $x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots$, and, in general, $u_{jk} = D_x^j D_y^k u$. The total derivative operators are

$$\begin{aligned} D_x &= \partial_x + u_x \partial_u + u_{xx}\partial_{u_x} + u_{xy}\partial_{u_y} + u_{xxx}\partial_{u_{xx}} + u_{xxy}\partial_{u_{xy}} + u_{xyy}\partial_{u_{yy}} + \ \cdots\ , \\ D_y &= \partial_y + u_y \partial_u + u_{xy}\partial_{u_x} + u_{yy}\partial_{u_y} + u_{xxy}\partial_{u_{xx}} + u_{xyy}\partial_{u_{xy}} + u_{yyy}\partial_{u_{yy}} + \ \cdots\ . \end{aligned} \tag{47}$$

The classical moving frame construction, [**2**, **15**], relies on the cross-section

$$x = y = u = u_x = u_y = u_{xy} = 0, \tag{48}$$

of order $s = 2$, which is a valid cross-section provided $u_{xx} \neq u_{yy}$. The resulting fundamental differential invariants are denoted as $I_{jk} = \iota(u_{jk})$. In particular,

$$\kappa_1 = I_{20} = \iota(u_{xx}), \qquad \kappa_2 = I_{02} = \iota(u_{yy}),$$

are the *principal curvatures*; the moving frame is valid provided $\kappa_1 \neq \kappa_2$, meaning that we are at a non-umbilic point. The *mean* and *Gaussian curvature* invariants

$$H = \tfrac{1}{2}(\kappa_1 + \kappa_2), \qquad K = \kappa_1 \kappa_2,$$

are often used as convenient alternatives. Higher order differential invariants are obtained by invariant differentiation† using $\mathcal{D}_1 = \iota(D_x)$, $\mathcal{D}_2 = \iota(D_y)$. We caution the reader that the action of SE(3) is only locally free on the second order jet space, and this implies some residual discrete ambiguities remaining in the resulting normalized differential invariants; for example, rotating the surface 90° around its normal interchanges the principal curvatures, while rotating it 180° through its tangent plane changes their signs. This ambiguity, however, does not affect the ensuing calculations. Since we are working entirely symbolically, we do not require the explicit formulas for the moving frame, nor the principal curvature invariants, nor the invariant differential operators. A complete derivation of all the non-symbolic formulas for the equivariant moving frame, differential invariants, invariant differential operators, etc., can be found in [**18**].

A basis for the prolonged infinitesimal generators is provided by the following six vector fields‡:

$$\mathbf{v}_4 = \partial_x, \qquad \mathbf{v}_5 = \partial_y, \qquad \mathbf{v}_6 = \partial_u, \tag{49}$$

representing infinitesimal translations, and

$$\begin{aligned}
\mathbf{v}_1 &= -y\partial_x + x\partial_y - u_y\partial_{u_x} + u_x\partial_{u_y} - 2u_{xy}\partial_{u_{xx}} + (u_{xx} - u_{yy})\partial_{u_{xy}} + 2u_{xy}\partial_{u_{yy}} + \cdots, \\
\mathbf{v}_2 &= -u\partial_x + x\partial_u + (1 + u_x^2)\partial_{u_x} + u_x u_y \partial_{u_y} \\
&\qquad + 3u_x u_{xx}\partial_{u_{xx}} + (u_y u_{xx} + 2u_x u_{xy})\partial_{u_{xy}} + (2u_y u_{xy} + u_x u_{yy})\partial_{u_{yy}} + \cdots, \\
\mathbf{v}_3 &= -u\partial_y + y\partial_u + u_x u_y\partial_{u_x} + (1 + u_y^2)\partial_{u_y} \\
&\qquad + (u_y u_{xx} + 2u_x u_{xy})\partial_{u_{xx}} + (2u_y u_{xy} + u_x u_{yy})\partial_{u_{xy}} + 3u_y u_{yy}\partial_{u_{yy}} + \cdots,
\end{aligned} \tag{50}$$

representing infinitesimal rotations, where we just display the terms up to second order, although it is straightforward to prolong further, to any desired order, using (5).

† These are related to, but not the same as, the operators of covariant differentiation, since the latter do not take differential invariants to (scalar) differential invariants.

‡ The system for numbering the $\mathbf{v}_\kappa$ is for later convenience.

The phantom recurrence formulae[†] are

$$\begin{aligned} 0 &= \mathcal{D}_1 I_{10} = I_{20} + R_1^2, & 0 &= \mathcal{D}_2 I_{10} = R_2^2, \\ 0 &= \mathcal{D}_1 I_{01} = R_1^3, & 0 &= \mathcal{D}_2 I_{01} = I_{02} + R_2^3, \\ 0 &= \mathcal{D}_1 I_{11} = I_{21} + (I_{20} - I_{02}) R_1^1, & 0 &= \mathcal{D}_2 I_{11} = I_{12} + (I_{20} - I_{02}) R_2^1, \end{aligned} \tag{51}$$

and can easily be solved for the (rotational) Maurer–Cartan invariants R_i^κ. However, since we are working in the extended symbolic calculus, these are not needed here.

The generating differential invariants $\mathcal{I}^{(s+1)} = \mathcal{I}^{(3)}$ guaranteed by Theorem 5 are I_{20}, I_{02} and the 4 third order invariants $I_{30}, I_{21}, I_{12}, I_{03}$. However, the order two basic recurrence formulae have the very simple form

$$\mathcal{D}_1 I_{20} = I_{30}, \qquad \mathcal{D}_2 I_{20} = I_{21}, \qquad \mathcal{D}_1 I_{02} = I_{12}, \qquad \mathcal{D}_2 I_{02} = I_{03}, \tag{52}$$

because the third order coefficients of the prolonged infinitesimal generators $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ all vanish on the chosen cross-section. Thus it is obvious that we can generate all of the third order differential invariants from $\mathcal{I} = \{ I_{20}, I_{02} \}$, meaning that the principal curvatures (or, equivalently, the Gauss and mean curvature) form a generating set.

In [**15**], it was proved, by cleverly manipulating the higher order recurrence formulae and the commutator relations, that, in fact, a minimal generating set is provided by merely the mean curvature H alone. (We know that this is minimal because it consists of a single differential invariant.) Indeed, for suitably generic surfaces, there is a universal formula expressing the Gauss curvature as a rational function of H and its invariant derivatives.

Let us instead apply the computational algorithm based on Theorem 7. By this means, we not only reconfirm the preceding result that the mean curvature generates, but also prove that either principal curvature — κ_1 or κ_2 – is also a minimal generating set, as is the Gauss curvature K. The latter result comes as a surprise, since it implies that the mean curvature, which is an extrinsic invariant that depends upon the embedding of the surface in Euclidean space, can be expressed in terms of the Gauss curvature, which is an intrinsic invariant as a consequence of Gauss' Theorema Egregium, [**2**], and its invariant derivatives. Of course, the explanation is that the invariant differential operators do not preserve intrinsicness. Thus, it would be of interest to further develop a classification scheme for distinguishing intrinsic and extrinsic higher order differential invariants.

Note: Technically, we should work symbolically by replacing the I's by v's and the R's by w's. But, while this makes the symbolic algorithm easier to explain, in practice whether we call the symbolic variables v, w or I, R makes no difference.

In detail, using my MATHEMATICA code[‡] to compute the symbolic Jacobian matrices

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[†] For completeness, we should also include those of order 0, i.e. for $K_1 = \iota(x) = 0$, $K_2 = \iota(y) = 0$, $I_{00} = \iota(u) = 0$; however, these are only used to determine the translational Maurer–Cartan invariants, namely, R_i^κ for $\kappa = 4,5,6$ and $i = 1,2$, which do not appear anywhere else, and hence play no role in the ensuing calculations. This always happens when the transformation group includes translations.

[‡] The software packages and details of the computations are available on the author's website: `https://www-users.cse.umn.edu/~olver/omath.html`.

and then computing their ranks by substituting random integers (a few times just to make sure), we find the following.

For $\mathcal{J} = \{\, 2\,H = \kappa_1 + \kappa_2 = I_{20} + I_{02} \,\}$ and $\widetilde{\mathcal{I}} = \{\, \kappa_2 = I_{02} \,\}$:

level	size $\mathbb{J}^{(k)}$	rank $\mathbb{J}^{(k)}$	size $\mathbb{I}^{(k)}$	rank $\mathbb{I}^{(k)}$
0	13×18	13	14×18	14
1	39×47	39	40×47	40
2	91×101	91	92×101	92
3	195×204	195	196×204	195
4	403×404	394	404×404	394

Since the ranks are equal at level 3 (and so the level 4 computation is unnecessary, but was performed as a check on the algorithm), by Theorem 7, we can write κ_2 in terms of the third order invariant derivatives of H, which is thus generating, in accordance with the result found in [**15**]. Interestingly, the explicit formula that was found there by manipulation of the recurrence formula involves the fourth order derivatives of H, and hence there is an as yet unknown formula for K involving at most third order derivatives of H. (This is not a contradiction, owing to the many syzygies among the differentiated invariants.)

For $\mathcal{J} = \{\, \kappa_1 = I_{20} \,\}$ and $\widetilde{\mathcal{I}} = \{\, \kappa_2 = I_{02} \,\}$:

level	size $\mathbb{J}^{(k)}$	rank $\mathbb{J}^{(k)}$	size $\mathbb{I}^{(k)}$	rank $\mathbb{I}^{(k)}$
0	13×18	13	14×18	14
1	39×47	39	40×47	40
2	91×101	91	92×101	92
3	195×204	194	196×204	194
4	403×404	393	404×404	393

It is interesting that the level 3 and 4 rows have a (slightly) different rank than the previous case. As before, the ranks are equal at level 3, and thus, we can write κ_2 in terms of the third order derivatives of κ_1, which is thus generating. Switching the principal curvatures implies that κ_2 is also generating. This is a new result.

Finally, when $\mathcal{J} = \{\, K = \kappa_1\,\kappa_2 = I_{20}\,I_{02} \,\}$ and $\widetilde{\mathcal{I}} = \{\, \kappa_2 = I_{02} \,\}$, the table is the same as in the first case, which implies that we can write κ_2 in terms of the third order derivatives of the Gauss curvature K, which is thus generating, and hence there is a previously unknown formula for H in terms of derivatives of K, valid for suitably generic surfaces. As noted above, this is a surprising new result, and it would be instructive to construct the explicit formula, which has yet to be done.

Interesting results on generating differential invariants for surfaces in other three-dimensional Klein geometries can be found in [**6**, **15**, **16**, **22**].

10. The Algorithm.

We close by summarizing the above constructions in the form of an algorithm for determining whether a collection of differential invariants forms a generating set.

(1) Input the infinitesimal generators of the action of the Lie group. Their coefficients form the entries of the associated *Lie matrix*.

(2) Input the level n of the computation and the order k of the cross-section.

(3) Compute the prolonged infinitesimal generators up to order $n + k + 1$ using (5).

(4) Input the cross-section, as in (7). Ensure that this is a valid cross-section by checking that the Lie matrix has rank $r = \dim G$ when restricted to the cross-section. If not, terminate the calculation.

(5) Compute the recurrence formulas up to order $n + k + 1$ in the form (30), including the linear algebraic constraints (31) following from the cross-section specification.

(6) Compute the commutators in the symbolic form (35).

(7) Compute the higher order constraints (36) up to level n.

(8) Choose a known generating set of differential invariants represented symbolically as in (42). In the implementation used in the example, these are the ones given in Theorem 5, eliminating obvious redundancies to streamline the computation.

(9) Input the proposed generating differential invariants represented symbolically as in (40), and then compute their invariant derivatives (43) up to level n.

(10) Compute the Jacobian matrices (45). If the rank condition (46) is satisfied, then the chosen differential invariants form a generating set. If not, then either they are not generating, or one needs to choose a higher level n. In practice, since computing the ranks of the symbolic matrices (45) is too computationally intensive, one substitutes random integers for the variables they depend on, and compares the ranks of the corresponding integer matrices, repeating this computation several times to be sure. Of course, with poor choices of random integers, this final numerical step may be misleading, but in the implementation this is not observed, and the ranks are almost always independent of the random choice.

Thus, if successful, the algorithm will confirm that one has a generating set. If unsuccessful, one can try a higher level. Unfortunately, I do not know a bound on the level required to be sure whether or not the selected differential invariants are generating; establishing this is a significant and apparently difficult open problem.

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