Differential Invariants of Surfaces

Peter J. Olver†
School of Mathematics
University of Minnesota
Minneapolis, MN 55455
olver@math.umn.edu
http://www.math.umn.edu/∼olver

Abstract. The algebra of differential invariants of a suitably generic surface $S \subset \mathbb{R}^3$, under either the usual Euclidean or equi-affine group actions, is shown to be generated, through invariant differentiation, by a single differential invariant. For Euclidean surfaces, the generating invariant is the mean curvature, and, as a consequence, the Gauss curvature can be expressed as an explicit rational function of the invariant derivatives, with respect to the Frenet frame, of the mean curvature. For equi-affine surfaces, the generating invariant is the third order Pick invariant. The proofs are based on the new, equivariant approach to the method of moving frames.

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1. Introduction.

According to Cartan, the local geometry of submanifolds under transformation groups, including equivalence and symmetry properties, are entirely governed by their differential invariants. Familiar examples are curvature and torsion of a curve in three-dimensional Euclidean space, and the Gauss(ian) and mean curvatures of a surface, [**3**, **10**, **16**]. A less familiar but still well-studied example is the Pick invariant of a (non-singular) surface $S \subset \mathbb{R}^3$, which is the simplest differential invariant under the action of the equi-affine group consisting of all volume-preserving affine transformations, [**1**, **6**, **9**, **15**, **16**].

In general, given an r-dimensional Lie group G acting on an m-dimensional manifold M, we are interested in studying its induced action on submanifolds $S \subset M$ of a prescribed dimension, say $p < m$. To this end, we prolong the group action to the (extended) submanifold jet bundles $\mathrm{J}^n = \mathrm{J}^n(M,p)$ of order $n \geq 0$, [**10**]. A *differential invariant* is a (perhaps locally defined) real-valued function $I\colon \mathrm{J}^n \to \mathbb{R}$ that is invariant under the prolonged group action. Any finite-dimensional Lie group action admits an infinite number of functionally independent differential invariants of progressively higher and higher order. Moreover, there always exist $p = \dim S$ linearly independent invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$. The *Fundamental Basis Theorem*, first formulated by Lie, [**8**; p. 760], states that all the differential invariants can be generated from a finite number of low order invariants by repeated invariant differentiation. A modern statement and proof of Lie's Theorem can be found, for instance, in [**10**]. For curves, the invariant differentiation is with respect to the group-invariant arc length parameter; for Euclidean surfaces, they are with respect to the diagonalizing Frenet frame, [**3**, **7**].

A basic question, then, is to find a minimal set of generating differential invariants. For curves, where $p = 1$, the answer is well known, [**10**]. Under mild restrictions on the group action (specifically transitivity and no pseudo-stabilization under prolongation), there are exactly $m - 1$ generating differential invariants, and any other differential invariant is a function of the generating invariants and their successive derivatives with respect to arc length. Thus, for instance, the differential invariants of a space curve $C \subset \mathbb{R}^3$ under the action of the Euclidean group SE(3), are generated by $m - 1 = 2$ differential invariants, namely its curvature and torsion.

For higher dimensional submanifolds, the minimal number of generating differential invariants cannot be fixed a priori, but depends the particularities of the group action and, in fact, can be arbitrarily large, even for surfaces in three-dimensional space, [**13**]. In particular, it is well known that the Euclidean differential invariants of a surfaces $S \subset \mathbb{R}^3$ are all obtained by differentiating the Gauss and mean curvatures with respect to the Frenet frame, cf. [**3**, **10**]. But, surprisingly, these two curvature invariants do *not* form a minimal generating system! The goal of this paper is to prove that, for suitably generic surfaces in $\mathbb{R}^3$:

- The algebra of Euclidean differential invariants is generated by the mean curvature alone through invariant differentiation. In particular, the Gauss curvature can be expressed as an explicit rational function of invariant derivatives of the mean curvature.
- The algebra of equi-affine differential invariants is generated by the Pick invariant alone through invariant differentiation.

Thus, surprisingly, for both the Euclidean and equi-affine actions on $\mathbb{R}^3$, the local geometry, equivalence, and symmetry properties of generic surfaces are entirely prescribed by a single fundamental differential invariant. In the Euclidean case, the result simply follows from combining the commutator relation for the invariant differential operators with the Codazzi equation. In the equi-affine case, the proof is based on the equivariant approach to Cartan's method of moving frames that has been developed over the last decade by the author and various collaborators, [**2**, **12**, **13**]. (The Euclidean result can also be deduced from the equivariant moving frame method.)

One immediate advantage of the equivariant approach to moving frames is that it is no longer tied to classical geometrically-based actions, but can, in fact, be directly applied to *any* finite-dimensional Lie transformation group. Further, extensions to infinite-dimensional pseudo-groups have been developed in [**14**]. In geometrical contexts, the equivariant approach mimics the classical moving frame construction, [**3**, **6**], but goes significantly further, in that it supplies us with the complete and explicit structure of the underlying algebra of differential invariants through the so-called recurrence relations, [**2**, **13**]. Surprisingly, these fundamental relations can be determined using only the (prolonged) infinitesimal generators of the group action and the moving frame normalization equations. One does *not* need to know the explicit formulas for either the group action, or the moving frame, or even the differential invariants and invariant differential operators, in order to completely characterize the differential invariant algebra they generate!

In recent work with Hubert, [**5**], these techniques have been extended to prove that, under both the conformal or projective groups, the differential invariant algebras of surfaces $S \subset \mathbb{R}^3$ are generated by a single differential invariant. The development of general algorithms for pinpointing minimal systems of generating differential invariants is under active investigation.

2. Euclidean Surfaces.

We begin with the standard action of the special Euclidean group $\mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$, consisting of all rigid, orientation-preserving motions, on surfaces $S \subset \mathbb{R}^3$. The classical moving frame construction, [**3**; Chapter 10], or its equivariant reformulation, [**7**; Example 9.9], produces the well known principal curvatures κ^1, κ^2, whose symmetric combinations

$$H = \tfrac{1}{2}\,(\kappa^1 + \kappa^2), \qquad\qquad K = \kappa^1\,\kappa^2, \tag{1}$$

are, respectively the *mean curvature* and *Gauss curvature* differential invariants. (Technically, since H can change its sign under a 180° rotation that preserves the tangent plane, only H^2 is a true invariant. However, we can, in accordance with standard practice, safely ignore this minor technicality in our development.)

Let $\mathcal{D}_1, \mathcal{D}_2$ denote the dual invariant differential operators, which are prescribed by differentiation with respect to the Frenet frame that diagonalizes the first fundamental form of the surface. It is well known, [**3**, **7**], that the algebra of differential invariants of a Euclidean surface is generated by its mean and Gauss curvatures, in the sense that any other differential invariant I can be expressed as a function of them and finitely many of their iterated invariant derivatives:

$$I = \Phi(\ \ldots\ \mathcal{D}_J H\ \ \ldots\ \mathcal{D}_J K\ \ \ldots\ ). \tag{2}$$

Here, we employ the notation

$$\mathcal{D}_J = \mathcal{D}_{j_1}\mathcal{D}_{j_2} \cdots \mathcal{D}_{j_k} \quad \text{for} \quad J = (j_1, \ldots, j_k) \quad \text{with each} \quad j_\nu = 1 \text{ or } 2,$$

for invariant differential operators of order $0 \le k = \#J$. Keep in mind that the invariant differential operators do not commute (see below for details), and so J is an *ordered* multi-index.

The differentiated invariants are not functionally independent, and there is a single fundamental differential relation or "syzygy", namely the *Codazzi equation*, which can be expressed in terms of the principal curvatures, [**7**]:

$$\kappa^1_{,22} - \kappa^2_{,11} + \frac{\kappa^1_{,1}\kappa^2_{,1} + \kappa^1_{,2}\kappa^2_{,2} - 2(\kappa^2_{,1})^2 - 2(\kappa^1_{,2})^2}{\kappa^1 - \kappa^2} - \kappa^1\kappa^2(\kappa^1 - \kappa^2) = 0. \tag{3}$$

All other syzygies follow from the Codazzi syzygy through invariant differentiation, [**7**]. The Codazzi equation can, in fact, be straightforwardly deduced from the infinitesimal moving frame analysis, [**7**], by comparing the recurrence formulae for the differentiated invariants $\kappa^1_{,22}$ and $\kappa^2_{,11}$. Note that the denominator in (3) vanishes at umbilic points on the surface, where the principal curvatures coincide $\kappa^1 = \kappa^2$, and the classical moving frame is not valid. We avoid such singular points in our subsequent computations.

The main result of this section is as follows:

Theorem 1. *The algebra of Euclidean differential invariants for a non-degenerate surface is generated by its mean curvature through invariant differentiation.*

Proof: The term "non-degenerate" will be explained during the course of the proof.

To establish the result, it suffices to write the Gauss curvature K in terms of H and its invariant derivatives. Now, according to the moving frame computations in [**3**; pp. 233–4] or [**7**; Example 9.9], the invariant differential operators satisfy the commutation relation

$$\left[\, \mathcal{D}_1, \mathcal{D}_2 \,\right] = \mathcal{D}_1\,\mathcal{D}_2 - \mathcal{D}_2\,\mathcal{D}_1 = Z_2\,\mathcal{D}_1 - Z_1\,\mathcal{D}_2, \tag{4}$$

where

$$Z_1 = \frac{\kappa^2_{,1}}{\kappa^1 - \kappa^2}\,, \qquad Z_2 = \frac{\kappa^1_{,2}}{\kappa^2 - \kappa^1}\,, \tag{5}$$

will be called the *commutator invariants*. An easy computation shows that the Codazzi syzygy (3) can be written compactly as

$$K = \kappa^1\kappa^2 = -\,(\mathcal{D}_1 + Z_1)Z_1 - (\mathcal{D}_2 + Z_2)Z_2. \tag{6}$$

We note that the latter identity immediately establishes Gauss' Theorema Egregium, [**3**, **16**]. Indeed, the invariant differentiations, and hence the commutator invariants, depend only on the first fundamental form, and so are intrinsic to the surface.

As a consequence, in order to express the Gauss curvature K in terms of invariant derivatives of the mean curvature H, it suffices to write commutator invariants Z_1, Z_2 in this manner. To this end, we note that the commutator identity (4) can be applied to any differential invariant. In particular,

$$\mathcal{D}_1\mathcal{D}_2 H - \mathcal{D}_2\mathcal{D}_1 H = Z_2\,\mathcal{D}_1 H - Z_1\,\mathcal{D}_2 H, \tag{7}$$

and, furthermore,

$$\mathcal{D}_1\mathcal{D}_2\mathcal{D}_J H - \mathcal{D}_2\mathcal{D}_1\mathcal{D}_J H = Z_2\,\mathcal{D}_1\mathcal{D}_J H - Z_1\,\mathcal{D}_2\mathcal{D}_J H \tag{8}$$

for any invariant derivative of the mean curvature. As long as at least one of the following 2×2 determinants is nonzero:

$$\det\begin{pmatrix} \mathcal{D}_1 H & \mathcal{D}_2 H \\ \mathcal{D}_1\mathcal{D}_J H & \mathcal{D}_2\mathcal{D}_J H \end{pmatrix} \neq 0, \tag{9}$$

we can solve (7–8) for the commutator invariants Z_1, Z_2 as rational functions of the invariant derivatives of H. In particular, if (9) holds at the minimal order, i.e., $\#J = 1$, we can write

$$Z_k = \frac{(\mathcal{D}_k\mathcal{D}_j H)(\mathcal{D}_1\,\mathcal{D}_2 H - \mathcal{D}_2\mathcal{D}_1 H) - (\mathcal{D}_k H)(\mathcal{D}_1\mathcal{D}_2\mathcal{D}_j H - \mathcal{D}_2\mathcal{D}_1\mathcal{D}_j H)}{(\mathcal{D}_1 H)(\mathcal{D}_2\mathcal{D}_j H) - (\mathcal{D}_2 H)(\mathcal{D}_1\mathcal{D}_j H)}, \quad k = 1, 2, \tag{10}$$

where the index j is allowed to be either 1 or 2. Plugging these expressions into the right hand side of the Codazzi identity (6) produces an explicit formula for the Gauss curvature as a rational function of the invariant derivatives, of order ≤ 4, of the mean curvature, valid for all surfaces satisfying the nondegeneracy condition

$$(\mathcal{D}_1 H)(\mathcal{D}_2\mathcal{D}_j H) \neq (\mathcal{D}_2 H)(\mathcal{D}_1\mathcal{D}_j H). \tag{11}$$

Moreover, by inspecting its dependence on the highest order derivatives of the surface parametrization, the nondegeneracy conditions (11) or (9) are easily seen to hold for suitably generic surfaces. This completes the demonstration of Theorem 1, where "nondegenerate" means that the surface is not umbilic, and satisfies (9) for at least one J. *Q.E.D.*

Remark: If H is constant, the determinants (9) are all 0 and so the preceding argument breaks down. In our terminology, constant mean curvature surfaces are degenerate. Indeed, such a surface need not have constant Gauss curvature, and so are not covered by the theorem. An interesting challenge is to classify all degenerate surfaces, which are characterized by the vanishing of certain fairly complicated nonlinear partial differential equations. It is possible that, among the non-umbilic surfaces, only those with constant mean curvature satisfy all of the the degeneracy conditions.

Finally, we remark that we *cannot* generate all the differential invariants by invariant differentiation of the Gauss curvature. Indeed, as noted above, all such differential invariants are intrinsic, depending only on the induced surface metric, whereas the mean curvature is an extrinsic surface invariant, that depends on its embedding into $\mathbb{R}^3$ which precludes its expression in terms of derivatives of the Gauss curvature. (It is instructive to try to mimic the preceding construction starting with K instead of H to see where the argument breaks down.)

3. Equi–Affine Surfaces.

Let us now turn to the geometry of surfaces $S \subset \mathbb{R}^3$ under the standard action of the *equi-affine group* $\mathrm{SA}(3) = \mathrm{SL}(3) \ltimes \mathbb{R}^3$ consisting of all (oriented) volume-preserving affine transformations:

$$g \cdot z = A z + b, \qquad \text{where} \qquad \begin{array}{c} g = (A, b) \in \mathrm{SA}(3), \qquad \det A = 1, \\ z = (x, y, u)^T \in \mathbb{R}^3. \end{array} \tag{12}$$

Theorem 2. *The algebra of differential invariants for nondegenerate surfaces under the action of the equi-affine group is generated by a single third order differential invariant, known as the Pick invariant, through invariant differentiation.*

As before, the term "nondegenerate" will be explained during the course of the proof. In particular, surfaces with constant Pick invariant are degenerate, and hence not covered by the result. If *all* its equi-affine differential invariants are constant, then Cartan's Theorem, [**2**], implies that the surface must be the orbit of a suitable two-parameter subgroup of SA(3), e.g., an ellipsoid or hyperboloid. However, because of the degeneracy, it is possible for a surface to have constant Pick invariant and yet not all of its higher order differential invariants be constant. See [**1**, **6**, **9**, **15**] for details on the classification of surfaces with constant Pick invariant.

We will be working under the assumption that the surface S is locally given by the graph of a function $u = f(x, y)$. But this is purely for computational convenience: All calculations and results are readily be extended to general parametrized surfaces, modulo the action of the infinite-dimensional reparametrization pseudo-group, cf. [**2**]. (The equi-affine action on surfaces with a fixed parametrization leads to a different system of differential invariants. The latter can also be straightforwardly handled by the equivariant moving frame methodology, but will not concern us here.)

Let $\mathrm{J}^n = \mathrm{J}^n(\mathbb{R}^3, 2)$ denote the n^{th} order surface jet bundle, with the usual induced coordinates $z^{(n)} = (x, y, u, u_x, u_y, u_{xx}, \ \ldots\ , u_{jk}, \ \ldots\ )$ for $j + k \leq n$, whose fiber coordinates u_{jk} represent the partial derivatives $\partial^{j+k} u / \partial x^j \partial y^k$. The induced action of SA(3) on J^n is obtained by the standard prolongation process, [**10**], (or, more prosaically, by implicit differentiation). The explicit formulas are easily established but, for the present purposes, not required.

According to [**2**], an n^{th} order *right moving frame* is a (locally defined) equivariant map $\rho\colon \mathrm{J}^n \to \mathrm{SA}(3)$, whence $\rho(g^{(n)} \cdot z^{(n)}) = \rho(z^{(n)}) \cdot g^{-1}$ for all $g \in \mathrm{SA}(3)$ and all jets $z^{(n)} \in \mathrm{J}^n$ in the domain of ρ. Classical moving frames, as in [**3**, **6**], can all be interpreted as left equivariant maps to the group, and so can be obtained by composing the right-equivariant version with the group inversion map $g \mapsto g^{-1}$.

The existence of a moving frame requires that the prolonged group action be free and regular, [**2**]. Since

$$\dim \mathrm{SA}(3) = 11, \qquad \text{while} \qquad \dim \mathrm{J}^n = 2 + \binom{n+2}{2} = \tfrac{1}{2} n^2 + \tfrac{3}{2} n + 3,$$

a necessary condition for the existence of an equi-affine moving frame is that the jet order $n \geq 3$. Indeed, the prolonged action of SA(3) is locally free on the dense open subset

$$V^3 = \{\, u_{xx} u_{yy} - u_{xy}^2 \neq 0, \ P \neq 0 \,\} \subset \mathrm{J}^3 \tag{13}$$

of jets of *non-singular*† *surfaces*. Here P refers to the third order Pick invariant, to be defined in (21) below.

A moving frame is uniquely prescribed by the choice of a cross-section to the group orbits through Cartan's normalization procedure, [**2**]. Since the n–jet of a function can be identified with its n^{th} order Taylor polynomial, the choice of cross-section normalization is equivalent to specification of a *normal form* for the leading terms in the Taylor expansion of the functional equation $u = f(x, y)$ defining the surface. In the non-singular regime, there are two standard nondegenerate normal forms:

Hyperbolic case: Assuming $u_{xx}u_{yy} - u_{xy}^2 < 0$, we define the cross-section $K \subset V^3$ by the equations

$$\begin{aligned} x = y = u = u_x = u_y = u_{xy} = 0, \qquad u_{xx} = 1, \qquad u_{yy} = -1, \\ u_{xyy} = u_{xxx}, \qquad\qquad u_{xxy} = u_{yyy} = 0. \end{aligned} \tag{14}$$

This corresponds to the power series normal form

$$u(x, y) = \tfrac{1}{2}(x^2 - y^2) + \tfrac{1}{6}c(x^3 + 3xy^2) + \cdots \tag{15}$$

for the surface at the distinguished point $\mathbf{0} = (0, 0, 0)$. A hyperbolic surface is *nonsingular* if and only if $c \neq 0$.

Elliptic case: Assuming $u_{xx}u_{yy} - u_{xy}^2 > 0$, we use

$$\begin{aligned} x = y = u = u_x = u_y = u_{xy} = 0, \qquad u_{xx} = 1, \qquad u_{yy} = 1, \\ u_{xyy} = -u_{xxx}, \qquad\qquad u_{xxy} = u_{yyy} = 0, \end{aligned} \tag{16}$$

to define the cross-section, corresponding to the power series normal form

$$u(x, y) = \tfrac{1}{2}(x^2 + y^2) + \tfrac{1}{6}c(x^3 - 3xy^2) + \cdots . \tag{17}$$

Non-singularity of the elliptic surface again requires $c \neq 0$.

In both cases, the coefficient c can be identified with the (square root of the) Pick invariant.

Remark: The *parabolic case*, where $u_{xx}u_{yy} - u_{xy}^2 \equiv 0$, requires a higher order moving frame, and the geometric and differential invariant theoretic structure is quite different; for instance, there is no direct analog of the Pick invariant. A detailed analysis and classification of parabolic surfaces can be found in Jensen, [**6**; chapter VI].

Given a cross-section $K \subset \mathrm{J}^n$, the induced right moving frame $\rho\colon \mathrm{J}^n \to \mathrm{SA}(3)$, defined on a suitable open subset $V \subset \mathrm{J}^n$ containing K, is given by $\rho(z^{(n)}) = g$, where $g \in \mathrm{SA}(3)$ is the‡ group element that maps the jet $z^{(n)} \in V$ to the cross-section: $g^{(n)} \cdot z^{(n)} \in K$.

† The non-degenerate surfaces alluded to above are necessarily non-singular, but require an additional genericity constraint; see equation (41) below.

‡ Uniqueness requires that G act freely. For a locally free action, there remain discrete ambiguities that are dealt with by further prolongation. See [**11**] for some simple examples.

The moving frame induces an *invariantization process*, denoted by ι, that maps differential functions to differential invariants, differential forms to invariant differential forms, differential operators to invariant differential operators, and so on. Specifically, the invariantization of any differential function $F\colon \mathrm{J}^n \to \mathbb{R}$ is the unique differential invariant $I = \iota(F)$ that agrees with F when restricted to the cross-section: $I|K = F|K$. In particular, $\iota(I) = I$ if I is any differential invariant. Thus, invariantization prescribes a morphism that projects the algebra[§] of differential functions to the algebra of differential invariants.

In particular, invariantization of the basic jet coordinates results in the *normalized differential invariants*

$$H_1 = \iota(x) = 0, \qquad H_2 = \iota(y) = 0, \qquad I_{jk} = \iota(u_{jk}), \qquad j,k \geq 0. \tag{18}$$

The invariantizations of the combinations of variables appearing in the cross-section equations (14) or (16) will be constant, and are known as *phantom differential invariants*, while the remaining non-constant *basic differential invariants* form a complete system of functionally independent invariants for the prolonged group action. We use

$$I^{(n)} = (I_{00}, I_{10}, I_{01}, I_{20}, I_{11}, \ \ldots \ , I_{0n}) = \iota(u^{(n)}) \tag{19}$$

to denote all the normalized differential invariants, both phantom and basic, of order $\leq n$ obtained by invariantizing the dependent variable u and its derivatives.

To be specific, let us concentrate on the hyperbolic regime from now on, leaving the elliptic modifications until the end of the paper. For the hyperbolic cross-section (14), the phantom differential invariants are

$$\begin{gathered} H_1 = H_2 = I_{00} = I_{10} = I_{01} = I_{11} = I_{21} = I_{03} = 0, \\ I_{20} = 1, \qquad I_{02} = -1, \qquad I_{30} - I_{12} = 0. \end{gathered} \tag{20}$$

There is one nontrivial independent differential invariant of order 3:

$$P = I_{30} = \iota(u_{xxx}) = I_{12} = \iota(u_{xyy}). \tag{21}$$

which corresponds to the coefficient c in the normalized Taylor expansion (15). To avoid an ambiguous sign, resulting from the fact that the action of SA(3) on J^3 is only locally free, its square, P^2, is traditionally known as the *Pick invariant*, [**16**], although for brevity, we will often refer to P itself as the Pick invariant.

There are 5 functionally independent basic differential invariants of order 4, which we denote by

$$\begin{gathered} Q_0 = I_{40} = \iota(u_{xxxx}), \qquad Q_1 = I_{31} = \iota(u_{xxxy}), \qquad Q_2 = I_{22} = \iota(u_{xxyy}), \\ Q_3 = I_{13} = \iota(u_{xyyy}), \qquad Q_4 = I_{04} = \iota(u_{yyyy}), \end{gathered} \tag{22}$$

followed by 6 basic differential invariants of order 5, and, in general, $n+1$ independent differential invariants I_{jk} of order $n = j + k$. These can all be identified with the Taylor coefficients in the normalized series expansion (15).

[§] More rigorously, since such functions may be only locally defined, one should employ the language of sheaves, [**17**], rather than algebras. But this extra technicality can be avoided in concrete examples.

In addition, the two basic invariant differential operators are obtained by invariantizing the total derivatives $\mathcal{D}_1 = \iota(D_x)$, $\mathcal{D}_2 = \iota(D_y)$, or, equivalently, are given as the dual differentiations with respect to the contact-invariant coframe

$$\omega_1 = \iota(dx), \qquad \omega_2 = \iota(dy), \tag{23}$$

fixed by the moving frame. If F is any differential function, then its (horizontal†) differential

$$dF = (D_x F)\,dx + (D_y F)\,dy = (\mathcal{D}_1 F)\,\omega_1 + (\mathcal{D}_2 F)\,\omega_2. \tag{24}$$

In particular, the invariant differential operators map any non-phantom differential invariant I to a pair of independent higher order differential invariants $\mathcal{D}_1 I, \mathcal{D}_2 I$.

Since the prolonged equi-affine action is locally free almost everywhere on J^3, a general result in [**2**] implies that all the higher differential invariants can be generated by invariant differentiation of the 6 differential invariants $P, Q_0, \ldots, Q_4$ of order ≤ 4. This fact can also be deduced from the recurrence formulae presented below. Thus, to establish our claimed Theorem 2, we need only show that all the fourth order invariants Q_j can, in fact, be written as functions of the invariant derivatives of the third order Pick invariant P.

In general, the entire structure of the algebra of differential invariants follows from the general *recurrence formulae*, first established in [**2**], that relate the normalized and differentiated invariants. These formulae are explicitly constructed from the prolonged infinitesimal generators of the group action. In our case, the Lie algebra $\mathfrak{sa}(3)$ of infinitesimal generators of the equi-affine group is spanned by the following 11 vector fields:

$$\begin{gathered} \mathbf{v}_1 = x\,\partial_x - u\,\partial_u, \qquad \mathbf{v}_2 = y\,\partial_y - u\,\partial_u, \\ \mathbf{v}_3 = y\,\partial_x, \qquad \mathbf{v}_4 = u\,\partial_x, \qquad \mathbf{v}_5 = x\,\partial_y, \qquad \mathbf{v}_6 = u\,\partial_y, \qquad \mathbf{v}_7 = x\,\partial_u, \qquad \mathbf{v}_8 = y\,\partial_u, \\ \mathbf{w}_1 = \partial_x, \qquad \mathbf{w}_2 = \partial_y, \qquad \mathbf{w}_3 = \partial_u. \end{gathered} \tag{25}$$

We prolong each of these to the submanifold jet spaces J^n using the standard prolongation formula, [**10**]. The n^{th} prolongation of a vector field

$$\mathbf{v} = \xi(x,y,u)\,\frac{\partial}{\partial x} + \eta(x,y,u)\,\frac{\partial}{\partial y} + \varphi(x,y,u)\,\frac{\partial}{\partial u} \tag{26}$$

on $\mathbb{R}^3$ is the vector field

$$\mathbf{v}^{(n)} = \mathbf{v} + \sum_{1 \leq j+k \leq n} \varphi^{jk}(x,y,u^{(j+k)})\,\frac{\partial}{\partial u_{jk}} \tag{27}$$

on $\mathrm{J}^n = \mathrm{J}^n(\mathbb{R}^3, 2)$, whose coefficients are given by

$$\varphi^{jk} = D_x^j D_y^k \left(\varphi - \xi\,u_x - \eta\,u_y \right) + \xi\,u_{j+1,k} + \eta\,u_{k,j+1}. \tag{28}$$

† The term "horizontal" refers to the fact that we are ignoring any contact forms that appear in the invariantized one-forms, because they do not play a role in the present analysis. The contact components are, however, of importance when studying equi-affine invariant variational problems. See [**7**] for a complete development.

For conciseness, we do not write out the explicit formulas for the prolonged equi-affine infinitesimal generators (25) here, although they are easily calculated using (28).

Specializing the general moving frame recurrence formulae found in [**2**, **13**] to the present context, we have the following key result:

Theorem 3. *The* recurrence formulae *for the differentiated invariants are*

$$\begin{aligned} \mathcal{D}_1 I_{jk} &= I_{j+1,k} + \sum_{\kappa=1}^{8} \varphi_\kappa^{jk}(0,0,I^{(j+k)}) R_1^\kappa, \\ \mathcal{D}_2 I_{jk} &= I_{j,k+1} + \sum_{\kappa=1}^{8} \varphi_\kappa^{jk}(0,0,I^{(j+k)}) R_2^\kappa, \end{aligned} \qquad j+k \geq 1, \tag{29}$$

where R_i^κ are certain differential invariants.

In (29), $\varphi_\kappa^{jk}(0,0,I^{(j+k)}) = \iota\big(\varphi_\kappa^{jk}(x,y,u^{(j+k)})\big)$ indicates the invariantization of the prolonged vector field coefficient, obtained by replacing each jet coordinate $x, y, u, \ldots, u_{il}, \ldots$ by the corresponding differential invariant $H_1 = 0, H_2 = 0, I_{00} = 0, \ldots, , I_{il}, \ldots$, as in (18).

The differential invariant R_i^κ appearing in (29) arises as the coefficient of the invariant one-form ω_i, cf. (23), in the invariantized Maurer–Cartan form $\gamma^\kappa = \iota(\mu^\kappa)$ dual to the infinitesimal generator $\mathbf{v}_\kappa$. For this reason, $R_i = (R_i^1, \ \ldots \ , R_i^8)$, $i = 1, 2$, will be collectively known as the *Maurer–Cartan invariants*. A full explanation of this identification would require several paragraphs. Moreover, it is not needed when performing the actual computations. Indeed, the explicit formulas for the Maurer–Cartan invariants can be found directly from the recurrence formulas for the phantom differential invariants, irrespective of how they arise from the underlying theory. And so, in the interests of brevity, we refer the reader to [**2**, **13**] for the complete story.

Remark: In (29), we have omitted the recurrence formulas for the trivial order zero differential invariants $H_1 = H_2 = I_{00} = 0$, since they only affect the additional Maurer–Cartan invariants associated to the translational generators $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$. Since these infinitesimal generators have trivial prolongation, their Maurer–Cartan invariants do not appear in any of the higher order recurrence formulas (29).

In the hyperbolic regime, using the explicit formulas for the coefficients of the prolonged infinitesimal generators of SA(3), the resulting phantom recurrence formulae are

$$\begin{aligned} &0 = \mathcal{D}_1 I_{10} = 1 + R_1^7, && 0 = \mathcal{D}_2 I_{10} = R_2^7, \\ &0 = \mathcal{D}_1 I_{01} = R_1^8, && 0 = \mathcal{D}_2 I_{01} = -1 + R_2^8, \\ &0 = \mathcal{D}_1 I_{20} = I_{30} - 3R_1^1 - R_1^2, && 0 = \mathcal{D}_2 I_{20} = -3R_2^1 - R_2^2, \\ &0 = \mathcal{D}_1 I_{11} = -R_1^3 + R_1^5, && 0 = \mathcal{D}_2 I_{11} = I_{30} - R_2^3 + R_2^5, \\ &0 = \mathcal{D}_1 I_{02} = I_{12} + R_1^1 + 3R_1^2, && 0 = \mathcal{D}_2 I_{02} = R_2^1 + 3R_2^2, \\ &0 = \mathcal{D}_1 I_{21} = I_{31} - I_{30}R_1^3 - 2I_{30}R_1^5 + R_1^6, && 0 = \mathcal{D}_2 I_{21} = I_{22} - I_{30}R_2^3 - 2I_{30}R_2^5 + R_2^6, \\ &0 = \mathcal{D}_1 I_{03} = I_{13} - 3I_{30}R_2^3 - 3R_2^6, && 0 = \mathcal{D}_2 I_{03} = I_{04} - 3I_{30}R_2^3 - 3R_2^6. \end{aligned} \tag{30}$$

In addtion, we have the following recurrence formulae for the non-constant third order invariants

$$\begin{aligned} \mathcal{D}_1 I_{30} &= I_{40} - 4I_{30}R_1^1 - I_{30}R_1^2 - 3R_1^4, & \mathcal{D}_2 I_{30} &= I_{31} - 4I_{30}R_2^1 - I_{30}R_2^2 - 3R_2^4, \\ \mathcal{D}_1 I_{12} &= I_{22} - 2I_{30}R_1^1 - 3I_{30}R_1^2 + R_1^4, & \mathcal{D}_2 I_{12} &= I_{13} - 2I_{30}R_2^1 - 3I_{30}R_2^2 + R_2^4. \end{aligned} \tag{31}$$

Owing to our normalization condition (21),

$$\mathcal{D}_1 I_{30} = -\mathcal{D}_1 I_{12}, \qquad \mathcal{D}_2 I_{30} = -\mathcal{D}_2 I_{12}. \tag{32}$$

Solving the combined linear system (30–32) produces the explicit forms of the Maurer–Cartan invariants:

$$\begin{aligned} R_1 &= \left(\tfrac{1}{2} I_{30}, -\tfrac{1}{2} I_{30}, \frac{3I_{31} + I_{13}}{12 I_{30}}, \tfrac{1}{4} I_{40} - \tfrac{1}{4} I_{22} - \tfrac{1}{2} I_{30}^2, \frac{3I_{31} + I_{13}}{12 I_{30}}, -\tfrac{1}{4} I_{31} + \tfrac{1}{4} I_{13}, -1, 0 \right) \\ &= \left(\tfrac{1}{2} P, -\tfrac{1}{2} P, \frac{3Q_1 + Q_3}{12 P}, \tfrac{1}{4} Q_0 - \tfrac{1}{4} Q_2 - \tfrac{1}{2} P^2, \frac{3Q_1 + Q_3}{12 P}, -\tfrac{1}{4} Q_1 + \tfrac{1}{4} Q_3, -1, 0 \right), \\ R_2 &= \left(0, 0, \frac{3I_{22} + I_{04}}{12 I_{30}} + \tfrac{1}{2} I_{30}, \tfrac{1}{4} I_{31} - \tfrac{1}{4} I_{13}, \frac{3I_{22} + I_{04}}{12 I_{30}} - \tfrac{1}{2} I_{30}, 0, -\tfrac{1}{4} I_{22} + \tfrac{1}{4} I_{04} - \tfrac{1}{2} I_{30}^2, 0, 1 \right) \\ &= \left(0, 0, \frac{3Q_2 + Q_4}{12 P} + \tfrac{1}{2} P, \tfrac{1}{4} Q_1 - \tfrac{1}{4} Q_3, \frac{3Q_2 + Q_4}{12 P} - \tfrac{1}{2} P, -\tfrac{1}{4} Q_2 + \tfrac{1}{4} Q_4 - \tfrac{1}{2} P^2, 0, 1 \right). \end{aligned} \tag{33}$$

These expressions are then substituted back into the remaining recurrence formulae for the higher order differential invariants, thereby producing the complete system of recurrence relations among the normalized and differentiated invariants.

Our proof of Theorem 2 relies on a detailed analysis of these equi-affine recurrence relations. In particular, the recurrence formulae for the third and fourth order differential invariants are

$$\begin{aligned} \mathcal{D}_1 I_{30} &= I_{40} - 4I_{30}R_1^1 - I_{30}R_1^2 - 3R_1^4, \\ \mathcal{D}_2 I_{30} &= I_{31} - 4I_{30}R_2^1 - I_{30}R_2^2 - 3R_2^4, \\ \mathcal{D}_1 I_{40} &= I_{50} - 5I_{40}R_1^1 - I_{40}R_1^2 - 10I_{30}R_1^4 - 4I_{31}R_1^5, \\ \mathcal{D}_2 I_{40} &= I_{41} - 5I_{40}R_2^1 - I_{40}R_2^2 - 10I_{30}R_2^4 - 4I_{31}R_2^5, \\ \mathcal{D}_1 I_{31} &= I_{41} - 4I_{31}R_1^1 - 2I_{31}R_1^2 - I_{40}R_1^3 - 3I_{22}R_1^5 - 2I_{30}R_1^6, \\ \mathcal{D}_2 I_{31} &= I_{32} - 4I_{31}R_2^1 - 2I_{31}R_2^2 - I_{40}R_2^3 - 3I_{22}R_2^5 - 2I_{30}R_2^6, \\ \mathcal{D}_1 I_{22} &= I_{32} - 3I_{22}R_1^1 - 3I_{22}R_1^2 - 2I_{31}R_1^3 - 2I_{30}R_1^4 - 2I_{13}R_1^5, \\ \mathcal{D}_2 I_{22} &= I_{23} - 3I_{22}R_2^1 - 3I_{22}R_2^2 - 2I_{31}R_2^3 - 2I_{30}R_2^4 - 2I_{13}R_2^5, \\ \mathcal{D}_1 I_{13} &= I_{23} - 2I_{13}R_1^1 - 4I_{13}R_1^2 - 3I_{22}R_1^3 - I_{04}R_1^5 + 6I_{30}R_1^6, \\ \mathcal{D}_2 I_{13} &= I_{14} - 2I_{13}R_2^1 - 4I_{13}R_2^2 - 3I_{22}R_2^3 - I_{04}R_2^5 + 6I_{30}R_2^6, \\ \mathcal{D}_1 I_{04} &= I_{14} - I_{04}R_1^1 - 5I_{04}R_1^2 - 4I_{13}R_1^3 + 6I_{30}R_1^4, \\ \mathcal{D}_2 I_{04} &= I_{05} - I_{04}R_2^1 - 5I_{04}R_2^2 - 4I_{13}R_2^3 + 6I_{30}R_2^4, \end{aligned} \tag{34}$$

where we now replace the Maurer–Cartan invariants by their explicit formulas (33).

The Maurer–Cartan invariants (33) are all of order ≤ 4. Thus, whenever $n = j+k \geq 4$, the only differential invariant of order $n+1$ appearing on the right hand side of the recurrence formula (29) is the leading term — namely, $I_{j+1,k}$ or $I_{j,k+1}$. This immediately establishes, by a simple induction argument, our earlier claim that all of the differential invariants of order ≥ 5 can be written in terms of (iterated) invariant derivatives of the differential invariants of order 3 and 4, namely P and $Q_0, \ldots, Q_4$.

To find formulas for the fourth order invariants Q_i in terms of derivatives of the Pick invariant P, we proceed as follows. In view of (21, 22) and (33), the first two recurrence formulae (34) are

$$P_1 \equiv \mathcal{D}_1 P = \tfrac{1}{4} Q_0 + \tfrac{3}{4} Q_2, \qquad P_2 \equiv \mathcal{D}_2 P = \tfrac{1}{4} Q_1 + \tfrac{3}{4} Q_3. \tag{35}$$

Thus, we are already able to generate 2 linear combinations of the fourth order invariants.

Secondly, the invariant differential operators do not commute, but rather satisfy

$$\mathcal{D}_3 = \left[\, \mathcal{D}_1, \mathcal{D}_2 \,\right] = \mathcal{D}_1 \mathcal{D}_2 - \mathcal{D}_2 \mathcal{D}_1 = Y_1 \mathcal{D}_1 + Y_2 \mathcal{D}_2, \tag{36}$$

for certain differential invariants Y_1, Y_2. Specializing the general commutator formulas established in [**2**, **7**], the commutator invariants are given by†

$$\begin{aligned} Y_1 &= \sum_{\kappa=1}^{8} \left(\frac{\partial \xi_\kappa}{\partial x}(0,0,0)\, R_2^\kappa - \frac{\partial \xi_\kappa}{\partial y}(0,0,0)\, R_1^\kappa \right) = R_2^1 - R_1^3, \\ Y_2 &= \sum_{\kappa=1}^{8} \left(\frac{\partial \eta_\kappa}{\partial x}(0,0,0)\, R_2^\kappa - \frac{\partial \eta_\kappa}{\partial y}(0,0,0)\, R_1^\kappa \right) = R_2^5 - R_1^2. \end{aligned} \tag{37}$$

Substituting our formulas (33) for the Maurer–Cartan invariants, we deduce that the commutator invariants are

$$Y_1 = -\,\frac{3\,Q_1 + Q_3}{12\,P}\,, \qquad Y_2 = \frac{3\,Q_2 + Q_4}{12\,P}\,. \tag{38}$$

We now set

$$P_3 = \mathcal{D}_3 P = \mathcal{D}_1 \mathcal{D}_2 P - \mathcal{D}_2 \mathcal{D}_1 P = \mathcal{D}_1 P_2 - \mathcal{D}_2 P_1 = Y_1 P_1 + Y_2 P_2. \tag{39}$$

At this point we have constructed 3 independent fourth order differential invariants — namely P_1, P_2 and P_3 — by differentiation of the Pick invariant.

To obtain another fourth order invariant, we appeal to the same commutator trick used in the Euclidean case, cf. equations (7–10). We differentiate any of the three preceding fourth order invariants:

$$\mathcal{D}_3 P_j = Y_1 \mathcal{D}_1 P_j + Y_2 \mathcal{D}_2 P_j, \qquad j = 1, 2, 3. \tag{40}$$

† In more general contexts, the partial derivatives should be replaced by total derivatives with respect to x, y. Here, since we normalized both $I_{10} = \iota(u_x) = 0$ and $I_{01} = \iota(u_y) = 0$, the additional u derivative terms do not affrect the final formula.

As long as at least one of the following 2×2 determinants is nonzero:

$$\det \begin{pmatrix} P_1 & P_2 \\ \mathcal{D}_1 P_j & \mathcal{D}_2 P_j \end{pmatrix} \neq 0 \qquad \text{for} \qquad j = 1, 2, \text{ or } 3, \tag{41}$$

we can solve (39–40) for the two fourth order differential invariants Y_1, Y_2. An explicit computation based on the recurrence relations (34) confirms that none of these determinants is identically zero, and so, for generic non-singular surfaces, we can produce the invariants Y_1, Y_2 as certain rational combinations of the invariant derivatives of P up to order 3. The explicit formulas are similar to those in (10).

Note that if the Pick invariant is constant, the determinants (41) are all 0 and so the preceding argument breaks down. Indeed, it is possible that a surface with constant Pick invariant admit a non-constant fourth order differential invariant, [**6**]. An interesting challenge is to classify the degenerate equi-affine surfaces, for which all such determinants (41) are zero and so are characterized by the vanishing of certain fairly complicated polynomial combinations of the differential invariants. It is possible that, among the non-singular surfaces, only those with constant Pick invariant satisfy the degeneracy conditions, but so far I lack any supporting evidence.

Summarizing and slightly simplifying, we have succeeded in expressing the following fourth order differential invariants

$$S_1 = Q_0 + 3\,Q_2, \qquad S_2 = Q_1 + 3\,Q_3, \qquad S_3 = 3\,Q_1 + Q_3, \qquad S_4 = 3\,Q_2 + Q_4, \tag{42}$$

as certain rational combinations of the invariant derivatives of the Pick invariant of order ≤ 3. The first two are multiples of P_1, P_2, whereas the latter two are simply related to Y_1, Y_2. Observe that we can express Q_1 and Q_3 in terms of S_2 and S_3.

To construct the final fourth order invariant, we return to the recurrence formulas (34) for the Q_j's. A direct computation using (33) shows that

$$\begin{aligned} 12\,P\big(\,\mathcal{D}_1 S_4 - \mathcal{D}_2 S_3\,\big) &= 18\,P^2 (Q_0 - 2\,Q_2 + Q_4) - (18\,Q_1^2 + 36\,Q_1 Q_3 + 10\,Q_3^2) + \\ &\qquad + (9\,Q_0 Q_2 + 3\,Q_0 Q_4 + 36\,Q_2^2 + 15\,Q_2 Q_4 + Q_4^2) \\ &= 48\,P^2 Q_0 - 30\,P^2 S_1 + 18\,P^2 S_4 - 3\,S_2 S_3 - S_3^2 + 3\,S_1 S_4 + S_4^2. \end{aligned} \tag{43}$$

Since all terms except the first depend on previously computed fourth order differential invariants, we are able to write the invariant Q_0 as an explicit (complicated) rational combination of the invariant derivatives, of orders ≤ 4, of the Pick invariant. Combining this with our previously constructed fourth order invariants, (42), we have indeed produced 5 functionally independent fourth order differential invariants by successively differentiating the Pick invariant. This completes the proof of Theorem 2 in the hyperbolic regime.

The Elliptic Case: The calculations are very similar, and only requires changing some of the signs. The Maurer–Cartan invariants are

$$R_1 = \left(\tfrac{1}{2}\,P, -\tfrac{1}{2}\,P, \frac{3\,Q_1 - Q_3}{12\,P}\,, \tfrac{1}{4}\,Q_0 + \tfrac{1}{4}\,Q_2 - \tfrac{1}{2}\,P^2, \frac{-3\,Q_1 + Q_3}{12\,P}\,, -\tfrac{1}{4}\,Q_1 + \tfrac{1}{4}\,Q_3, -1, 0 \right),$$

$$R_2 = \left(0, 0, \frac{3\,Q_2 - Q_4}{12\,P} + \tfrac{1}{2}\,P, \tfrac{1}{4}\,Q_1 + \tfrac{1}{4}\,Q_3, \frac{3\,Q_2 - Q_4}{12\,P} - \tfrac{1}{2}\,P, \tfrac{1}{4}\,Q_2 + \tfrac{1}{4}\,Q_4 - \tfrac{1}{2}\,P^2, 0, 1 \right). \tag{44}$$

The first order derivatives of the Pick invariant $P = I_{30} = \iota(u_{xxx})$ are

$$P_1 = \mathcal{D}_1 P = \tfrac{1}{4}\,Q_0 - \tfrac{3}{4}\,Q_2, \qquad\qquad P_2 = \mathcal{D}_2 P = \tfrac{1}{4}\,Q_1 - \tfrac{3}{4}\,Q_3. \tag{45}$$

The commutation relation is

$$\mathcal{D}_3 = \big[\, \mathcal{D}_1, \mathcal{D}_2 \,\big] = Y_1 \mathcal{D}_1 + Y_2 \mathcal{D}_2, \tag{46}$$

where the commutator invariants are

$$Y_1 = -\,\frac{3\,Q_1 - Q_3}{12\,P}\,, \qquad\qquad Y_2 = -\,\frac{3\,Q_2 - Q_4}{12\,P}\,. \tag{47}$$

As before, we set $P_3 = \mathcal{D}_3 P = Y_1 P_1 + Y_2 P_2$, and can solve for Y_1, Y_2 provided one of the determinantal conditions (40) holds. At this stage we have produced the fourth order invariants

$$S_1 = Q_0 - 3\,Q_2, \qquad S_2 = Q_1 - 3\,Q_3, \qquad S_3 = 3\,Q_1 - Q_3, \qquad S_4 = 3\,Q_2 - Q_4. \tag{48}$$

Finally, the relation

$$\begin{aligned} 12\,P\big(\, \mathcal{D}_1 S_4 - \mathcal{D}_2 S_3 \,\big) &= -\,18\,P^2 (Q_0 + 2\,Q_2 + Q_4) - (18\,Q_1^2 - 36\,Q_1 Q_3 + 10\,Q_3^2) + \\ &\qquad + (9\,Q_0 Q_2 - 3\,Q_0 Q_4 - 36\,Q_2^2 + 15\,Q_2 Q_4 - Q_4^2) \\ &= -\,48\,P^2 Q_0 + 30\,P^2 S_1 + 18\,P^2 S_4 - 3\,S_2 S_3 - S_3^2 + 3\,S_1 S_4 - S_4^2 \end{aligned} \tag{49}$$

allows us to construct Q_0, and hence all of the fourth (and all higher) order differential invariants as rational invariant differential functions of the Pick invariant.

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