Invariant Numerical Approximations to Differential Invariant Signatures

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Abstract. In this paper we present a new approach to the approximation of differential invariants recently introduced by the authors. The fundamental idea is to introduce a completely invariant finite difference approximation to a high order differential invariant by a suitable combination of joint invariants of the underlying group action. Applications to curvature and arc length invariants associated with the Euclidean and affine groups, which play a fundamental role in computer vision are provided. Finally we discuss how, in consort with a general theorem of É. Cartan, the recognition of objects from their differential invariant signatures can be practically implemented in a fully group-invariant fashion.

[†] *Supported in Part by NSF Grant DMS 92–03398.*

[‡] *Supported in Part by NSF Grants DMS 92–04192 and 95–00931.*

[§] *Supported in Part by NSF Grant ECS–9122106, by the Air Force Office of Scientific Research F49620–94–1–00S8DEF, by the Army Research Office DAAL03-91-G-0019, DAAH04–93–G–0332, DAAH04–94–G–0054.*

December 5, 1995

1. Introduction.

Lie group methods have proven to play a vital role in modern research in computer vision. Indeed, certain visually-based symmetry groups and their associated differential invariants have, in recent years, assumed great significance in practical image processing and object recognition. Recent surveys can be found in the papers appearing in earlier volumes [**20**], [**23**], and, in particular, the review papers [**22**], [**28**]. For reasons of space, we shall not attempt to describe the applications of differential invariants to object recognition and geometric diffusion-based multi-scale smoothing, referring the interested reader to the preceding references and the other papers in this volume.

In order to construct a numerical approximation to the differential invariant I, we use a finite difference approach and introduce a mesh or discrete sequence of points $P_i \in \mathcal{C}$, $i = 0, 1, 2, \ldots$, to approximate the curve. The approximation scheme will be computed using appropriate combinations of the coordinates of the mesh points. The approximation will be invariant under the underlying group G, and hence its numerical values will not depend on the group transformations, provided it depends on the joint invariants of the mesh points. Again, the simplest example is provided by the Euclidean distance $\mathbf{d}(P, Q)$ between points in the plane, which depends on two points. Thus, *any G-invariant numerical approximation to a differential invariant must be governed by a function of the joint invariants of G.* For instance, any Euclidean invariant approximation to the curvature of a plane curve must be based on the distances between the mesh points.

Our approach to differential invariants in computer vision is governed by the following philosophy. We begin with a transformation group G acting on a space E, representing the image space, whose subsets are the objects of interest. In visual applications, the group G is typically either the Euclidean, affine, similarity, or projective group. We are particularly interested in how the geometry, in the sense of Klein, induced by the transformation group G applies to (smooth) submanifolds contained in the space E. A differential invariant I of G is a real-valued function, depending on the submanifold and its derivatives at a point, which is unaffected by the action of G. In general, a transformation group admits a finite number of fundamental differential invariants, $I_1, \ldots, I_N$, and a system of invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_n$, equal in number to the dimension of the submanifold, and such that every other differential invariant is a function of the fundamental differential invariants and their successive derivatives with respect to the invariant differential operators. This result dates back to the original work of S. Lie, [**17**]; see [**21**] for further historical remarks and a modern exposition. For example, in the case of Euclidean curves in the plane, the group action is provided by the Euclidean group consisting of translations and rotations, and every differential invariant is a function of the Euclidean curvature and its derivatives with respect to Euclidean arc length. Similarly, for affine planar geometry, the underlying group is the equi-affine group of area-preserving affine transformations, and every differential invariant of a curve is a function of the affine curvature and its various derivatives with respect to affine arc length.

The fact that, for transitive group actions, an object can be fully reconstructed, modulo group transformations, from a suitable collection of differential invariants, was proved in a general result of É. Cartan's. Thus, for example, a curve in the Euclidean plane is uniquely determined, modulo translation and rotation, from its curvature invariant κ and

its first derivative with respect to arc length κ_s. Thus, the curve is uniquely prescribed by its euclidean signature curve, parametrized by the two functions (κ, κ_s). Similarly, a curve in the affine plane is uniquely determined, modulo an affine transformation, by its *affine signature curve* which is the planar curve parametrized by its affine curvature and its derivative with respect to affine arc length. This definition of signature offers significant advantages over the traditional approach, cf. BKLP, which plot curvature as a function of arc length, in that *a*) it does not include the ambiguity in the choice of initial point on the curve, and *b*) it readily extends, via Cartan's general theorem, to surfaces and higher-dimensional submanifolds. In this paper, we present some preliminary pictures of Euclidean signature curves, leaving their general analysis for a later work.

In practical applications of invariant theory to computer vision, one is forced to practically compute a differential invariant, such as the curvature of a curve, by a discrete numerical approximation. A robust and efficient numerical implementation is crucial, but is a nontrivial problem in that the more important differential invariants depend on high order derivatives and are thus particularly sensitive to noise and round-off error. Although the differential invariants reflect the invariance of the image under a transformation group, most standard numerical approximation schemes fail to incorporate this symmetry. Consequently, two objects which are equivalent under a group transformation, while having the same differential invariants, may have unequal numerical versions, thereby complicating the implementation of their invariant characterization by signatures. In our approach, the problem of invariance of the numerical approximation is solved through the use of an explicitly group-invariant numerical scheme, based on suitable combinations of joint invariants based on the mesh points used to approximate the object in question. Thus, our schemes are *automatically* invariant under the prescribed transformation group.

Motivations for this approach come from a variety of sources. In modern numerical analysis, the introduction of numerical schemes that incorporate additional structure enjoyed by the problem being approximated have become quite popular in recent years. The first instances of such schemes are the symplectic integrators arising in Hamiltonian mechanics, and the closely allied energy conserving methods; see [**9**], [**18**], [**29**]. Closer in spirit are the invariant numerical schemes for solving partial differential equations studied by Shokin, [**25**], and Dorodnitsyn, [**10**].

More specifically, any discrete approximation scheme ultimately relies on introducing a mesh, or discrete number of points, in the submanifold. The approximation scheme will then rely on certain appropriate combinations of the coordinates of the mesh points. The approximation will be invariant under the underlying transformation group G, and hence its numerical values will not be affected by the group transformations, provided it depends on the various "joint invariants" of the mesh points. In general, if G is any group acting on a space E, then a joint invariant is a function $J(x^{(1)}, \dots, x^{(k)})$ depending on several points $x^{(i)} \in E$ having the property that its value is unchanged under *simultaneous* action of the group elements $g \in G$ on the point configuration, so that $J(g \cdot x^{(1)}, \dots, g \cdot x^{(k)}) = J(x^{(1)}, \dots, x^{(k)})$. For example, in the case of the Euclidean group, every joint invariant is given as a function of the Euclidean distances $\mathbf{d}(P, Q)$ between pairs of points P, Q, which are the fundamental joint invariants in this case. Similarly, in the case of the equi-affine group, the simplest joint invariant is the area $\mathbf{A}(P, Q, R)$ of the triangle whose vertices

are the three points P, Q, R, and, again, every other joint invariant is a function of these triangular area invariants. Results of M. Green [**12**], generalized in [**21**], relate the number of differential invariants of curves to the number of joint invariants of the group action, serving as an additional motivation for establishing a more practical connection between the two quantities — a bridge between the discrete and continuous invariant theory.

The recent preprint of Bruckstein, Rivlin, and Weiss, [**6**], is very close in spirit to ours; the crucial (and, in our opinion, tremendously important) distinction is that our approach allows arbitrary discretizations of the curve, whereas in their approach, the distances (or other joint invariant measures) between mesh points is required to be the same, making the actual discretization quite tricky to implement. By allowing arbitrary discretizations, we achieve a far broader range of discrete invariant signatures. Related ideas appear in the work of Cooper *et. al.*, [**15**], on a more algebraic use of joint invariants in computer vision; a direct comparison of our approach with theirs would be of great interest.

An intermediary role is played by the "semi-differential invariants" (or, as we would prefer they be known, "joint differential invariants") introduced into vision by Van Gool *et. al.*, [**19**]. Related work appears in the theory of "noise resistant" differential invariants developed by Weiss, [**30**], as well as the local invariant signatures of Bruckstein *et. al.*, [**3**], [**4**], [**5**]. In such an approach, one approximates a higher order differential invariants by a joint differential invariant depending on lower order derivatives evaluated at several points on the curve. In our view, this is only a partial resolution of the difficulty, since to compute any such semi-differential invariant, one must still evaluate each derivative that appears in it by a discrete approximation, and hence the original high order differential invariant is itself approximated in the end by a fully discrete finite difference version. In particular, to maintain invariance of the approximation, one must use a finite difference approximation to the semi-differential invariant by joint invariants, and so one always ends up analyzing the approximation of differential invariants by joint invariants anyway. (On the other hand, one can certainly motivate the construction of useful joint invariant approximations via semi-differential invariants.)

In this survey, we discuss the cases of planar curves under the Euclidean and affine groups in some detail, and conclude with an outline of the general theory. A more detailed version of these results, including a new approach to the affine geometry of convex curves, appears in the authors' recent paper [**7**].

2. Euclidean Curves in the Plane.

As our first example, we describe the geometry of curves in the Euclidean plane $E \simeq \mathbb{R}^2$. The underlying group is the Euclidean group $\mathrm{E}(2) = \mathrm{O}(2) \ltimes \mathbb{R}^2$ consisting of rotations, reflections, and translations. According to Weyl, [**32**], every joint invariant of the Euclidean group is a function of the Euclidean distances $\mathbf{d}(P, Q) = |P - Q|$ between points.

Consider a regular, smooth plane curve $\mathcal{C} \subset E$ of class C^2. The simplest differential invariant of the Euclidean group is the Euclidean curvature of $\mathcal{C}$, whose value at a point $P \in \mathcal{C}$ is defined as the reciprocal of the radius of the osculating circle to $\mathcal{C}$ at P. In terms of a coordinate system such that the part of $\mathcal{C}$ near P is represented by the graph of a

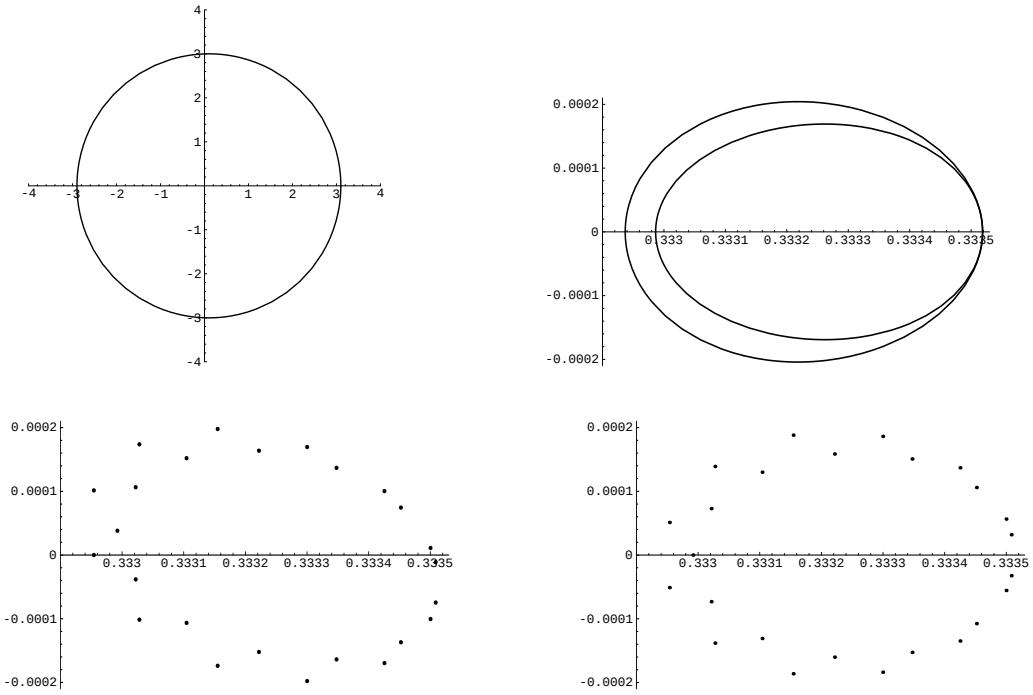

Figure 1. Euclidean Signature Curve for $r = 3 + \frac{1}{10}\cos\theta$.

function $y = u(x)$, then

$$\kappa = \frac{u_{xx}}{(1 + u_x^2)^{3/2}}. \tag{1}$$

The Euclidean arc length parameter is defined as $ds = \sqrt{1 + u_x^2}\, dx$, the right hand side representing the simplest invariant one-form for the Euclidean group.

Theorem 1. *Every differential invariant of curves in the Euclidean plane is a function of the Euclidean curvature and its derivatives with respect to Euclidean arc length:* $I = F(\kappa, \kappa_s, \kappa_{ss}, \ldots)$.

Although the successive derivatives of curvature with respect to arc length lead to an infinite hierarchy of higher and higher order differential invariants, as far as the characterization of the curve goes, one only needs to consider the first two: κ and κ_s. This fact motivates the following definition of the signature curve in the Euclidean case, and is formalized in the subsequent theorem, the proof of which follows from the more general results discussed below — see Theorem 16.

Definition 2. The *Euclidean signature curve* associated with a parametrized plane curve $\mathcal{C} = \{(x(t), y(t))\} \subset E$ is the curve $\mathcal{S} \subset Z \simeq \mathbb{R}^2$ parametrized by the curvature and its first derivative with respect to arc length: $\mathcal{S} = \{(\kappa(t), \kappa_s(t))\} \subset Z$.

Theorem 3. *Two smooth* (C^3) *curves* $\mathcal{C}$ *and* $\overline{\mathcal{C}}$ *can be mapped to each other by a Euclidean transformation,* $\overline{\mathcal{C}} = g \cdot \mathcal{C}$, $g \in \mathrm{E}(2)$, *if and only if their signature curves are identical:* $\overline{\mathcal{S}} = \mathcal{S}$.

In Figure 1, the top two pictures show a roughly circular curve on the left, and its Euclidean signature curve, parametrized by (κ, κ_s), on the right. Note particularly the

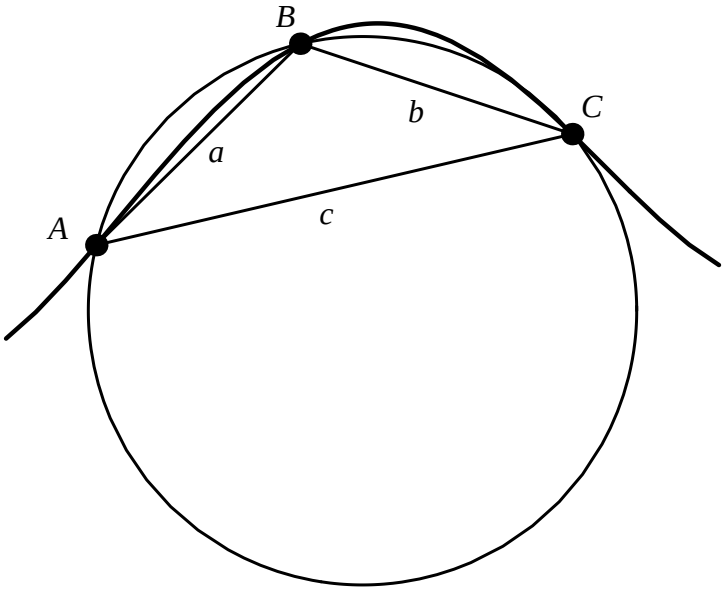

Figure 2. Euclidean Curvature Approximation.

scales on the signature plot. Indeed, the original curve is described, in polar coordinates, by $r = 3 + \frac{1}{10}\cos\theta$. Its near circularity is reflected by the fact that its signature curve is very small, concentrated near the point $\left(0, \frac{1}{3}\right)$, which is the signature "curve" for a circle of radius 3. The more a curve deviates from circularity, the less concentrated its signature curve becomes — see Figures 3 and 4 below.

As a first illustration of our general philosophy of approximating differential invariants by joint invariants, we describe how to use standard geometrical constructions to obtain a numerical approximation to the Euclidean curvature that is unaffected by rigid motions, so that any translated or rotated version of the curve will provide precisely the same numerical approximation for its curvature. We first approximate the parametrized curve by a sequence of mesh points $P_i \in \mathcal{C}$, not necessarily equally spaced. Our goal is to approximate the Euclidean curvature of $\mathcal{C}$ in a Euclidean invariant manner, and, in view of the characterization of Euclidean joint invariants, this requires the approximation to depend only on the distances $\mathbf{d}(P_i, P_j)$ between mesh points. Because the curvature is a second order differential function, the simplest approximation will require three mesh points. With this in mind, we now derive the basic approximation formula for the Euclidean curvature.

Let A, B, C be three successive points on the curve $\mathcal{C}$ such that the Euclidean distances are $a = \mathbf{d}(A, B)$, $b = \mathbf{d}(B, C)$, $c = \mathbf{d}(A, C)$, which are assumed to be small; see Figure 2. The key idea is to use the circle passing through the points A, B, C as our approximation to the osculating circle to the curve at B. Therefore, the reciprocal of its radius $r = r(A, B, C)$ will serve as an approximation to the curvature of the curve at B. Let Δ denote the area of the triangle whose vertices are A, B, C, and let $s = \frac{1}{2}(a + b + c)$ denote its semi-perimeter, so that $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$. We apply Heron's formula to compute the radius of

the circle passing through the points A, B, C, leading to the exact formula

$$\widetilde{\kappa}(A,B,C) = 4\,\frac{\Delta}{abc} = 4\,\frac{\sqrt{s(s-a)(s-b)(s-c)}}{abc} \tag{2}$$

for its curvature. Since formula (2) only depends on the Euclidean distances between the three points, it provides us with a completely Euclidean invariant numerical approximation to the curvature of $\mathcal{C}$ at the middle point B. In other words, the approximation for two curves related by a Euclidean motion will be *identical*.

We now need to analyze how closely the numerical approximation $\widetilde{\kappa}(A,B,C)$ is to the true curvature $\kappa(B)$ at the point B. Our analysis is based on a Taylor series expansion — see [**7**] for a detailed argument.

Theorem 4. *Let A, B, C be three successive points on the curve $\mathcal{C}$, and let a, b, c be their Euclidean distances. Let $\kappa = \kappa(B)$ denote the Euclidean curvature at B. Let $\widetilde{\kappa} = \widetilde{\kappa}(A,B,C)$ denote the curvature of the circle passing through the three points. Then the following expansion is valid:*

$$\begin{aligned}\widetilde{\kappa} = \kappa &+ \frac{1}{3}(b-a)\frac{d\kappa}{ds} + \frac{1}{12}(b^2 - ab + a^2)\frac{d^2\kappa}{ds^2} + \\ &+ \frac{1}{60}(b^3 - ab^2 + a^2b - a^3)\frac{d^2\kappa}{ds^2} + \frac{1}{120}(b-a)(3b^2 + 5ab + 3a^2)\kappa^2\frac{d\kappa}{ds} + \cdots.\end{aligned} \tag{3}$$

In particular, if we choose the points to be equal distance apart, meaning that $a = b$ then the first error term in the approximation (3) is of second order.

Remark: Since a, b, and c are Euclidean invariants, *every* coefficient of the powers $a^m b^n$ in the series expansion (3) must be a Euclidean differential invariant, and hence a function of κ and its arc length derivatives.

The same general method can also be used to find Euclidean-invariant numerical approximations for computing the higher order differential invariants $\kappa_s = d\kappa/ds$, etc. For example, to determine a fully Euclidean invariant finite difference approximation to κ_s, we approximate the Euclidean distance along the curve by the Euclidean distance between the individual mesh points. Thus, to approximate $\kappa_s(P_i)$ we use the finite difference quotient

$$\widetilde{\kappa}_s(P_{i-2}, P_{i-1}, P_i, P_{i+1}) = \frac{\widetilde{\kappa}(P_{i-1}, P_i, P_{i+1}) - \widetilde{\kappa}(P_{i-2}, P_{i-1}, P_i)}{\mathbf{d}(P_i, P_{i-1})}, \tag{4}$$

to approximate $\kappa_s(P_i)$. However, equation (4) suffers from a numerical bias owing to the asymmetry of points chosen to represent the curve near P_i. In the lower two pictures in Figure 1, the original curve has been discretized by choosing 25 points (equally spaced in the angular variable, but not equal Euclidean distance apart). The bottom left figure gives the discrete Euclidean signature curve based on (3) and (4). Note the bias in the vertical direction of the signature points, as opposed to the exact signature curve, which is symmetric about the κ axis.

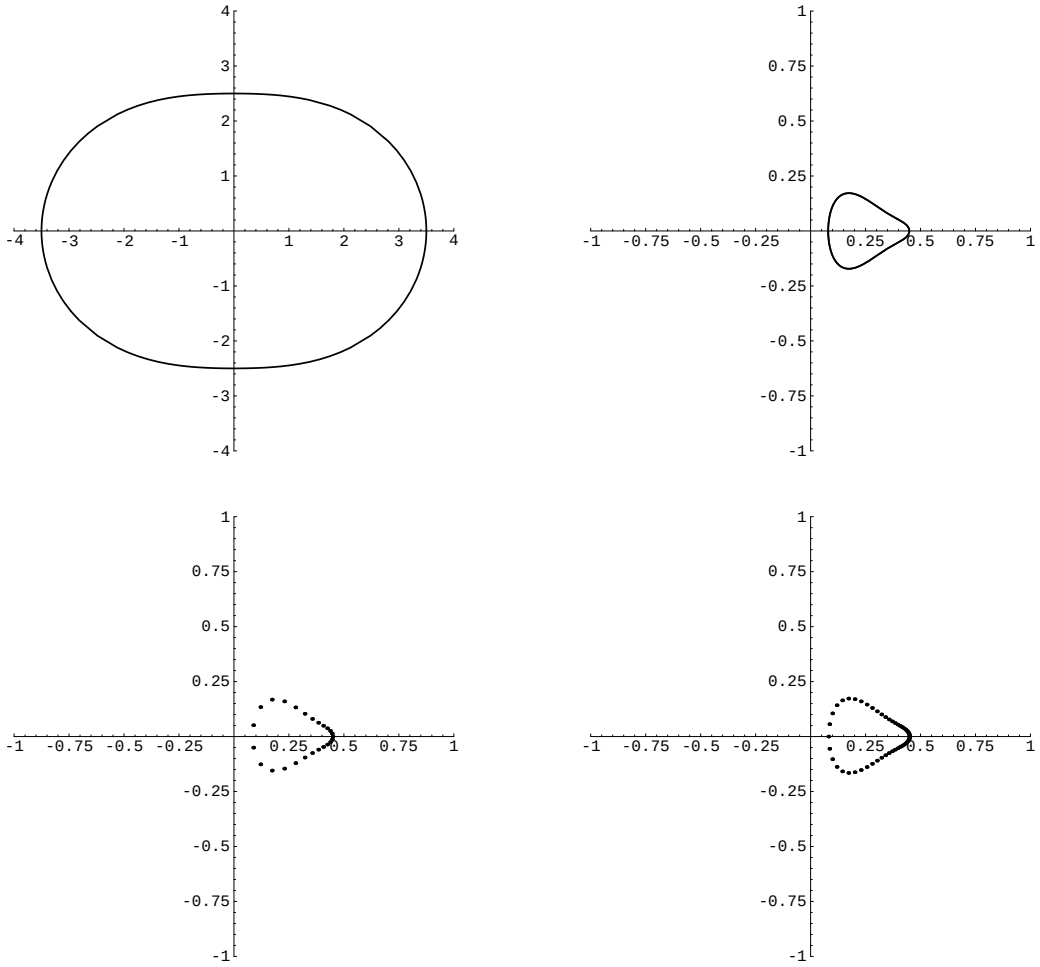

Figure 3. Euclidean Signature Curve for $r = 3 + \frac{1}{2}\cos 2\theta$.

To counteract such biases, we propose to use the centered difference formula

$$\widetilde{\kappa}_s(P_{i-2}, P_{i-1}, P_i, P_{i+1}, P_{i+2}) = \frac{\widetilde{\kappa}(P_i, P_{i+1}, P_{i+2}) - \widetilde{\kappa}(P_{i-2}, P_{i-1}, P_i)}{\mathbf{d}(P_{i+1}, P_{i-1})}. \tag{5}$$

as the Euclidean invariant approximation to the derivative κ_s. Thus, we obtain a fully Euclidean invariant discrete approximation to the Euclidean signature curve by using

$$\big(\widetilde{\kappa}(P_{i-1}, P_i, P_{i+1}), \widetilde{\kappa}_s(P_{i-2}, P_{i-1}, P_i, P_{i+1}, P_{i+2})\big) \tag{6}$$

as our approximating points. The bottom right picture in Figure 1 gives the corresponding unbiased discrete Euclidean signature curve based on the Euclidean invariant approximation (6). Similar preliminary numerical experiments indicate a very good agreement between the signature curve and its discrete counterpart. Examples of signature curves and their discrete approximations are displayed in Figure 3 and Figure 4. Note particularly the increase in complexity and size of the signature curve with the deviation of the original

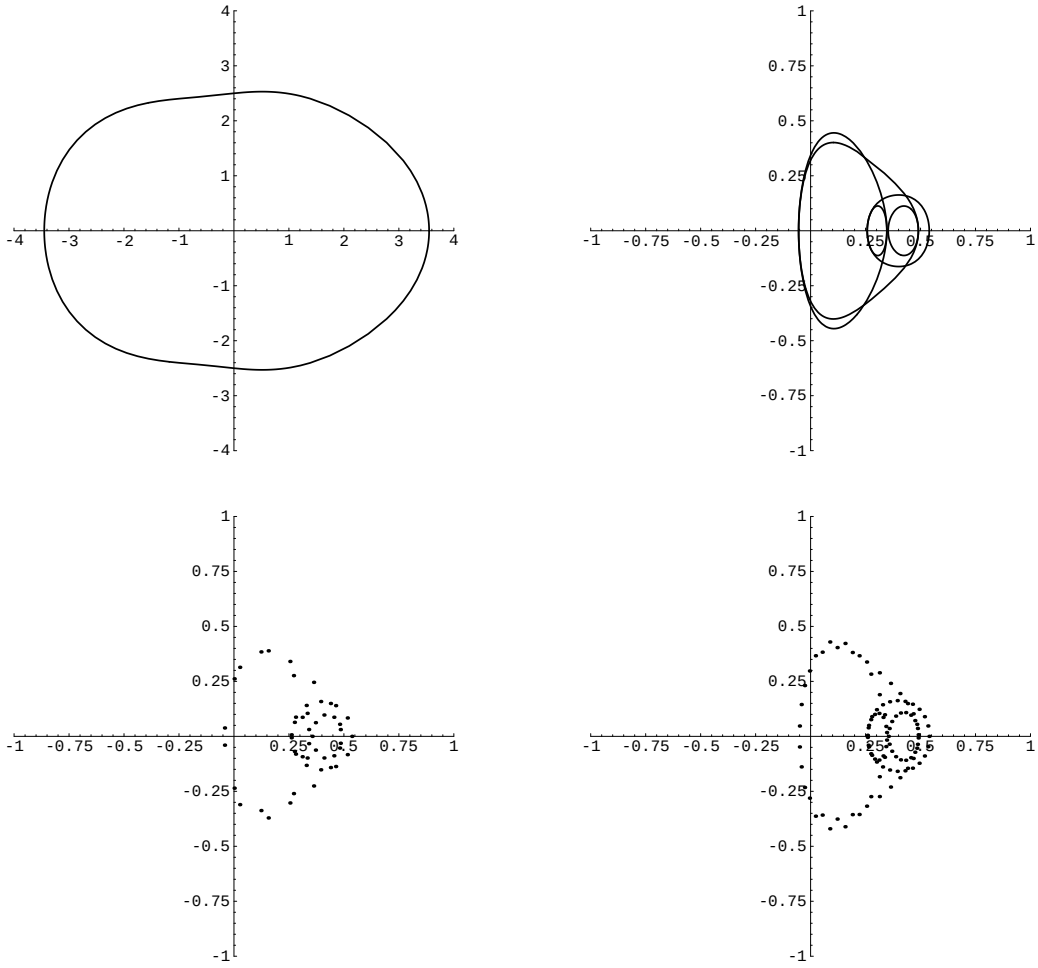

Figure 4. Euclidean Signature Curve for $r = 3 + \frac{1}{2}\cos 2\theta + \frac{1}{20}\cos 5\theta$.

curve from circularity. Angular Fourier modes are characterized by the signature curve's winding around the circular point $\left(0, \frac{1}{3}\right)$. The discrete approximations, based on 50 and 100 points in the original curve, are in excellent agreement with the true signature curve, represented in the top left picture.

3. Affine Curves in the Plane.

In our second example, we discuss a fully affine-invariant finite difference approximation to the affine curvature and arc length of a convex curve in the plane $E \simeq \mathbb{R}^2$. The underlying transformation group is the special affine (or equi-affine) group $\mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$ consisting of all area-preserving affine transformations: $\mathbf{x} \to A\mathbf{x} + b$, $\det A = 1$. Thus, the simplest joint affine invariant is the area of a triangle whose vertices are three given points. Given a configuration of points $P_i = (x_i, y_i) \in E$, we define

$$[ijk] = [P_i, P_j, P_k] = (P_i - P_j) \wedge (P_i - P_k) = \det \begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{vmatrix}, \tag{7}$$

so that $[ijk]$ equals the signed area of the parallelogram whose sides are $P_i - P_j$ and $P_i - P_k$, which is twice the signed area of the triangle whose vertices are P_i, P_j, P_k. (The area is positive if the triangle is traversed in a clockwise direction.) According to Weyl, [**32**], every joint affine invariant $I(P_1, \dots, P_n)$ depending on the points P_i is a function of these triangular areas $[ijk]$. Unlike Euclidean distances, the joint affine area invariants are not functionally independent, but are subject to certain relations or "syzygies", which are all consequences of the following:

$$\begin{aligned} [ijl] + [jkl] &= [ijk] + [ikl], \\ [ijk][ilm] - [ijl][ikm] + [ijm][ikl] &= 0. \end{aligned} \tag{8}$$

For example, in a configuration of five points $P_0, \dots, P_5$, there are 10 possible triangular areas, but only five independent ones.

Consider a regular, smooth convex plane curve $\mathcal{C} \subset E$ of class C^4. Affine geometry requires the (unfortunate) restriction to convex curves, although many applications to computer vision can dispense with this restriction. For example, the affine-invariant curvature flow can be extended to arbitrary curves by omitting the tangential component, cf. [**24**], [**22**]. The simplest differential invariant of the equi-affine group is its affine curvature. If we represent the curve as a graph, $y = u(x)$ then the affine curvature is the fourth order differential invariant

$$\kappa = \frac{3u_{xx}u_{xxxx} - 5u_{xxx}^2}{9(u_{xx})^{8/3}}. \tag{9}$$

Note that κ is undefined at inflection points, where $u_{xx} = 0$, corroborating our restriction to convex curves. Two smooth, convex curves passing through a common point P have the same equi-affine curvature at P if and only if they have fourth order contact at P. In particular, the curvature to a curve $\mathcal{C}$ at P equals the (constant) curvature of its osculating conic at P, which is defined as unique conic passing through P having fourth order contact with $\mathcal{C}$ at P. The affine arc length element

$$ds = \sqrt[3]{u_{xx}}\, dx \tag{10}$$

is the simplest invariant one-form.

Theorem 5. *Every equi-affine differential invariant for a curve in the plane is a function of the successive derivatives of affine curvature with respect to affine arc length:* $I = F(\kappa, \kappa_s, \kappa_{ss}, \dots)$.

Definition 6. The *affine signature curve* associated with a parametrized plane curve $\mathcal{C} = \{(x(t), y(t))\} \subset E$ is the curve $\mathcal{S} \subset Z \simeq \mathbb{R}^2$ parametrized by the affine curvature and its first derivative with respect to affine arc length: $\mathcal{S} = \{(\kappa(t), \kappa_s(t))\} \subset Z$.

Theorem 7. *Two smooth* (C^5) *curves* $\mathcal{C}$ *and* $\overline{\mathcal{C}}$ *can be mapped to each other by an affine transformation,* $\overline{\mathcal{C}} = g \cdot \mathcal{C}$, $g \in \mathrm{SA}(2)$, *if and only if their signature curves are identical:* $\overline{\mathcal{S}} = \mathcal{S}$.

As with Theorem 3, this result is a special case of Theorem 16 below.

Turning to the numerical approximations to affine differential invariants, our starting point is the result that one can approximate the (positive) affine curvature at a point of a plane curve by the affine curvature of the conic section passing through five nearby points. We will explicitly show how this may be used to produce an affine-invariant finite difference approximation to the affine curvature. We thus need to determine the formula for the affine curvature of such a conic.

Theorem 8. *The affine curvature of a nondegenerate conic $\mathcal{C}$ defined by the quadratic equation*

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0. \tag{11}$$

is given by

$$\kappa = \frac{S}{T^{2/3}}, \tag{12}$$

where

$$S = AC - B^2 = \det \begin{vmatrix} A & B \\ B & C \end{vmatrix}, \qquad T = \det \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix}. \tag{13}$$

Remark: Both S and T are equi-affine invariants of the conic. The invariant S vanishes if and only if the five points lie on a parabola. The invariant T vanishes if and only if the conic degenerates to a pair of lines, and hence fails our convexity hypothesis.

In particular, the equi-affine curvature of an ellipse in the plane is given by $\kappa = (\pi/\mathbf{A})^{2/3}$, where

$$\mathbf{A} = \pi \frac{T}{S^{3/2}} \tag{14}$$

is the area of the ellipse.

Five points in general position in the plane determine a unique conic section that passes through them. The explicit formula is not difficult to establish; see [**26**] for a proof of the following classical result.

Theorem 9. *Let $P_0, \ldots, P_4$ be five points in general position in the plane. There is then a unique conic section $\mathcal{C}$ passing through them, whose quadratic equation has the affine-invariant form*

$$[013][024][\mathbf{x}12][\mathbf{x}34] = [012][034][\mathbf{x}13][\mathbf{x}24], \tag{15}$$

where $\mathbf{x} = (x, y)$ is an arbitrary point on $\mathcal{C}$.

Combining Theorems 8 and 9, we deduce an explicit formula for the affine curvature of the conic passing through five given points. According to the general result about joint affine invariants, the resulting formula can be written in terms of the 10 triangular areas

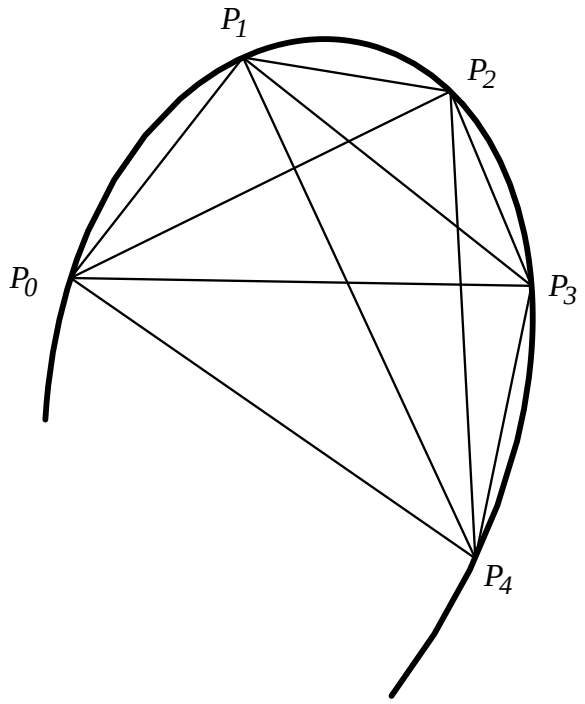

Figure 5. The Affine Pentagram.

determined by the points taken three at a time; see Figure 5. Substituting the formulas for the coefficients, we find a particularly nice affine-invariant expression for our first affine invariant

$$4T = \prod_{0 \le i < j < k \le 4} [ijk]; \tag{16}$$

in other words, to compute T, multiply together all 10 triangular areas in the pentagram described by the 5 points. The fact that T has such a form is not so surprising, since T vanishes if and only if the conic degenerates to a pair of lines, which requires that three of the five points lie on a line, meaning that $[ijk] = 0$ for some $i < j < k$. The simplest affine-invariant formula for S that we know is

$$\begin{aligned} 4S = {} & [013]^2[024]^2\big([124] - [123]\big)^2 + [012]^2[034]^2\big([134] + [123]\big)^2 - \\ & - 2[012][034][013][024]\big([123][234] + [124][134]\big). \end{aligned} \tag{17}$$

Formula (17) is not nearly as pleasant as (16), particularly because the right hand side appears to be asymmetrical with respect to permutations of the five points. However, S must clearly be symmetrical with respect to these permutations. Of course, the explanation lies in the syzygies (8) among the triangular areas, a judicious application of which suffices to demonstrate that (17) is symmetrical under permutation. A completely symmetrical formula for S can, of course, be obtained by symmetrizing (17), i.e., summing over all possible permutations of the set $\{0, 1, 2, 3, 4\}$ and dividing by $5! = 120$, although the result is much more complicated than (17). We have been unable to find a simple yet symmetrical version of the formula for S.

As in the Euclidean case, we are interested in finite difference numerical approximations to the affine curvature of a strongly convex plane curve $\mathcal{C}$ which are invariant under the special affine group. We thus approximate the parametrized curve $\mathbf{x}(t)$ by a sequence of mesh points $P_i = \mathbf{x}(t_i)$. Any affine-invariant numerical approximation to the affine curvature κ (as well as any other affine differential invariant $d^n\kappa/ds^n$) must be a function of the joint affine invariants of the mesh points, which means that it must be a function of the areas $[ijk]$ of the parallelograms (or triangles) described by the mesh points. Because the affine curvature is a fourth order differential function, the simplest approximation will require five mesh points, so that the approximation will depend on the ten triangular areas (or, more basically, the five independent areas) in the pentagram whose vertices are the five mesh points; see Figure 5.

With this in mind, let us number the five successive mesh points as P_0, P_1, P_2, P_3, P_4. (This is just for simplicity of exposition; of course, in general, one should replace the indices $0, \dots, 4$ by $i, i+1, i+2, i+3, i+4$.) Since we are assuming that $\mathcal{C}$ is convex, the mesh points are in general position. Let $\mathcal{C} = \mathcal{C}(P_0, P_1, P_2, P_3, P_4)$ be the unique conic passing through the mesh points. Let $\widetilde{\kappa} = \widetilde{\kappa}(P_0, P_1, P_2, P_3, P_4)$ denote the affine curvature of the conic $\mathcal{C}$, which we evaluate via the basic formula (12), where the invariants S, T are computed in terms of the triangular areas according to (17), (16). We regard $\widetilde{\kappa}$ as a numerical approximation to the affine curvature $\kappa = \kappa(P_2)$ of $\mathcal{C}$ at the middle point P_2. We now need to analyze how closely the numerical approximation $\widetilde{\kappa}$ is to the true curvature κ at the point P_2. Assuming the points are close together, we need to compute a Taylor series expansion of the distance $\widetilde{\kappa}$. An extensive MATHEMATICA computation produces the desired result; see [**7**] for details.

Theorem 10. *Let P_0, P_1, P_2, P_3, P_4 be five successive points on the convex curve $\mathcal{C}$. Let κ be the affine curvature of $\mathcal{C}$ at P_2, and let $\widetilde{\kappa} = \widetilde{\kappa}(P_0, P_1, P_2, P_3, P_4)$ denote the affine curvature of the conic section passing through the five points. Let*

$$L_i = \int_{P_2}^{P_i} ds, \qquad i = 0, \dots, 4, \tag{18}$$

denote the signed affine arc length of the conic from P_2 to P_i; in particular $L_2 = 0$. We assume that each L_i is small. Then the following expansion is valid:

$$\widetilde{\kappa} = \kappa + \frac{1}{5}\left(\sum_{i=0}^{4} L_i\right)\frac{d\kappa}{ds} + \frac{1}{30}\left(\sum_{0 \le i \le j \le 4} L_i L_j\right)\frac{d^2\kappa}{ds^2} + \cdots. \tag{19}$$

The higher order terms are cubic in the distances L_i.

Remark: The property of "being close" is therefore expressed in affine-invariant form as the statement that all the arc lengths $L_0, \dots, L_4$ are small. In this way, we are able to introduce a fully affine-invariant notion of "distance", albeit one that requires knowledge of five, rather than two, points.

As in the Euclidean case, we can similarly determine affine-invariant numerical approximations for computing the higher order differential invariants given by the successive derivatives of affine curvature with respect to affine arc length. The starting point is an affine-invariant approximation to the affine arc length measured along the given curve $\mathcal{C}$ to replace the Euclidean distance between the two mesh points as an approximation to the Euclidean arc length of the curve between the two mesh points. The obvious candidate for this is to measure distance along the interpolating conic section passing through five successive mesh points. Thus we need an explicit formula for the arc length of a conic section.

Theorem 11. *Let $\mathcal{C}$ be an ellipse defined by the quadratic equation* (11), *and let $S > 0$ and T be the two affine invariants defined by* (13). *Note that $CT < 0$, since otherwise* (11) *has no real solutions. Let $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$ be two points on $\mathcal{C}$. Then the affine arc length, measured along the conic, from P_0 to P_1 is given by*

$$\int_{P_0}^{P_1} ds = \frac{T^{1/3}}{S^{1/2}} \arcsin \sqrt{\frac{-CT}{S^2}} \left(x + \frac{CD - BE}{S} \right) \Bigg|_{x=x_0}^{x_1}. \tag{20}$$

A similar formula can be found in the hyperbolic case. The fact that the right hand side of (20) is affine-invariant is not so obvious from the explicit formula. In the elliptical case, its invariance follows from the following geometrical interpretation.

Corollary 12. *If $\mathcal{C}$ is an ellipse, then the affine arc length from P_0 to P_1 is equal to $2ST^{-2/3}\mathbf{A}(P_0, P_1)$, where $\mathbf{A}(P_0, P_1)$ denotes the area of the elliptical sector obtained by connecting P_0 and P_1 to the center of the ellipse by straight line segments.*

In particular, using equations (12), (14), we discover that the total affine arc length of an ellipse equals twice the cube root of its area:

$$\mathbf{L} = 2\sqrt[3]{\mathbf{A}} = -2\pi \frac{T^{1/3}}{S^{1/2}}. \tag{21}$$

It is a curious fact that, in equi-affine geometry, the arc length of an ellipse is given by elementary functions, while in Euclidean geometry, it must be computed via an elliptic integral.

In order to approximate the affine arc length along an arbitrary curve, we use the formula (15) to determine the interpolating conic section, and then (20) to compute the approximation to the arc length between two successive mesh points. Although the resulting formula is affine invariant by construction, it is not so easy to re-express it directly in terms of the triangular areas (7). Indeed, we strongly suspect that the explicit formula (guaranteed by the general theory) is a highly complicated transcendental function, and not amenable to explicit computation in this form, although of course, one can readily use (20) in conjunction with (15) to directly compute the approximate affine arc length in an affine-invariant manner. Another approach is, in the elliptical case, to replace the area of the elliptical sector indicated in Corollary 12 by the area of an approximating triangle whose vertices are the two mesh points and the center of the interpolating ellipse. This

does have an explicit formula in terms of the ten triangular areas of the affine pentagram. (Indeed, we are employing a similar device in the Euclidean case, since we should be approximating the Euclidean arc length of the curve by the arc length along the interpolating circle, but, at least to the order of approximation of interest, this can be more simply computed by just using the Euclidean distance between the two mesh points.)

For example, to determine a fully equi-affine invariant finite difference approximation to the derivative κ_s of affine curvature with respect to affine arc length at a mesh point P_i, we can use the finite difference quotient

$$\widetilde{\kappa}_s(P_{i-3}, P_{i-2}, P_{i-1}, P_i, P_{i+1}, P_{i+2}) = \\ = \frac{\widetilde{\kappa}(P_{i-2}, P_{i-1}, P_i, P_{i+1}, P_{i+2}) - \widetilde{\kappa}(P_{i-3}, P_{i-2}, P_{i-1}, P_i, P_{i+1})}{\mathbf{L}(P_{i-1}, P_i; P_{i-2}, P_{i-1}, P_i, P_{i+1}, P_{i+2})}. \tag{22}$$

Here $\widetilde{\kappa}(P_{i-2}, P_{i-1}, P_i, P_{i+1}, P_{i+2})$ denotes the affine curvature of the conic passing through the points $P_{i-2}, P_{i-1}, P_i, P_{i+1}, P_{i+2}$, as given by (12) in conjunction with (17), (16), while

$$\mathbf{L}(P_{i-1}, P_i; P_{i-2}, P_{i-1}, P_i, P_{i+1}, P_{i+2}) = \int_{P_{i-1}}^{P_i} ds$$

denotes the affine arc length from P_{i-1} to P_i as measured along the same conic, and so given by (20) (or its triangular approximation). Again, as with the Euclidean case, the six point formula (22) is not symmetrical with respect to to the central mesh point P_i, and thus subject to numerical bias. Therefore, we propose a more symmetrical centered difference seven point version:

$$\widetilde{\kappa}_s(P_{i-3}, P_{i-2}, P_{i-1}, P_i, P_{i+1}, P_{i+2}, P_{i+3}) = \\ = \frac{\widetilde{\kappa}(P_{i-1}, P_i, P_{i+1}, P_{i+2}, P_{i+3}) - \widetilde{\kappa}(P_{i-3}, P_{i-2}, P_{i-1}, P_i, P_{i+1})}{\mathbf{L}(P_{i-1}, P_{i+1}; P_{i-2}, P_{i-1}, P_i, P_{i+1}, P_{i+2})}. \tag{23}$$

to approximate $\kappa_s(P_i)$. Again, we can use these formula to provide a fully equi-affine invariant numerical approximation to the equi-affine signature curve associated with an arbitrary curve in the plane, and thus perform a fully equi-affine invariant object recognition. Numerical experimentation in this case remains to be done.

4. Differential Invariants and Joint Invariant Signatures.

The Euclidean affine and groups are certainly not the only ones of interest in computer vision, and so we now indicate the general methods lying behind the constructions in the last two sections. This indicates how (at least in principle) to find invariant finite difference approximations to differential invariants of general submanifolds for arbitrary transformation groups acting on arbitrary spaces. For simplicity of exposition, though, we restrict our attention to the case of planar curves, and their differential invariants under a finite-dimensional, connected Lie group acting transitively on $E \simeq \mathbb{R}^2$, with coordinates x, y. Details and generalizations can be found in [**21**].

Remark: Lie, [**16**], completely classified all possible finite-dimensional transformation groups on the plane, up to change of coordinates, and their associated differential invariants. The corresponding classification in three-dimensional space remains incomplete, although many examples are known. See [**21**] for details.

Let G be an r-dimensional Lie group acting transitively on E. We are interested in the differential invariants of curves $\mathcal{C} \subset E$ under the group G. We represent the curve $\mathcal{C}$ (locally) as a function $y = u(x)$. Let $\mathrm{J}^n \simeq \mathbb{R}^{n+2}$ denote the n^{th} jet space of E — the coordinates of J^n are provided by the independent variable x, the dependent variable u and the derivatives of u with respect to x up to order n, denoted $(x, u^{(n)}) = (x, u, u_x, u_{xx}, \dots, u_n)$. A function $F(x, u^{(n)})$ depending on the jet space coordinates is said to have *order* n provided F does explicitly depend on n^{th} order derivatives. For example, the Euclidean curvature $\kappa = u_{xx}/(1 + u_x^2)$ is a second order function.

The group G acts on curves by transforming them pointwise, and hence induces a prolonged action $G^{(n)}$ on the jet space, which is found by determining how the derivative coordinates are transformed under the group elements. (In practical terms, this just amounts to the chain rule for derivatives.) We make the technical assumption, for simplicity, that G is an *ordinary* r-dimensional transformation group, which means that $G^{(n)}$ acts transitively on (an open subset of) J^n for each $0 \leq n \leq r - 2$. Most groups are ordinary. Indeed, Lie's classification shows that the only transitive group actions which fail to be ordinary are the elementary similarity group $(x, u) \mapsto (\lambda x + c, \lambda u + d)$ and some minor variants thereof. (These "non-ordinary" groups can also be analyzed, cf. [**21**], but the results are slightly different.) The theorems quoted above concerning the differential invariants of the Euclidean and equi-affine groups are both special cases of general theorems on the differential invariants of ordinary transformation groups. (Again, these in turn are special cases of general theorems on the differential invariants of curves and even more general submanifolds of higher dimensional Euclidean spaces, cf. [**21**].)

Theorem 13. *Let G be an ordinary r-dimensional transformation group acting on $E \simeq \mathbb{R}^2$. Then there is, up to constant multiple, a unique G-invariant one-form of lowest order, $ds = P(x, u^{(n)})\, dx$, which we call the G-invariant arc length element. The order n of ds is at most $n \leq r - 2$.*

For example, the Euclidean group has dimension $r = 3$, and admits an arc length element of order $n = 1 = r - 2$. On the other hand, the equi-affine group has dimension $r = 5$, but its arc length element has order $n = 2 < 3 = r - 2$. An interesting and unresolved problem is to geometrically characterize those groups whose arc length element has order strictly less than $r - 2$.

Theorem 14. *Let G be an ordinary r-dimensional transformation group acting on $E \simeq \mathbb{R}^2$. Then there is, up to constant multiple, a unique differential invariant of lowest order, $\kappa(x, u^{(r-1)})$, having order exactly $r - 1$, which we call the G-invariant curvature. Moreover, the derivatives of the G-invariant curvature with respect to the G-invariant arc length, $d^m\kappa/ds^m$, $m \geq 0$, provide a complete list of differential invariants of G, meaning that any other differential invariant is a function of these: $I = I(\kappa, \kappa_s, \kappa_{ss}, \dots)$.*

Again, the Euclidean curvature has order $r-1=2$, and the equi-affine curvature has order $r-1=4$. See [**21**] for the complete list of all possible transformation groups in the plane, and their associated invariant arc length element and curvature. Of course, once one has determined the formulas for κ and ds for curves given by graphs $y=u(x)$, one can recompute them for arbitrary parametrized curves $\mathcal{C}=(x(t),y(t))$. The result as stated still holds true.

Thus, according to Theorem 14, a complete list of differential invariant signatures associated with a plane curve is provided by the group-invariant curvature and its successive derivatives with respect to arc length. How many of these are required to uniquely characterize the curve up to a group transformation? The answer is that we only need to know the first two, namely κ and κ_s.

Definition 15. Let G be an ordinary transformation group acting on $E\simeq\mathbb{R}^2$. Then the *signature curve* associated with a parametrized plane curve $\mathcal{C}=\{(x(t),y(t))\}\subset E$ is the curve $\mathcal{S}\subset\widehat{E}\simeq\mathbb{R}^2$ parametrized by the G-invariant curvature and its first derivative with respect to arc length: $\mathcal{S}=\{(\kappa(t),\kappa_s(t))\}\subset\widehat{E}$.

Note that since κ has order $r-1$, then κ_s has order $r=\dim G$, and so the signature curve requires the computation of r^{th} order derivatives of the parametrizing functions. Generically, the signature curve is given by specifying κ_s as a function of κ, so that $\kappa_s=H(\kappa)$. Therefore, a curve parametrized by a graph $y=u(x)$ will be a solution to the r^{th} order ordinary differential equation

$$\kappa_s(x,u^{(r)})=H(\kappa(x,u^{(r-1)})) \tag{24}$$

determined by its signature curve. Thus, the curve will be uniquely recovered from the signature curve and the initial conditions

$$u(x_0)=y_0,\qquad u_x(x_0)=y_1,\qquad\dots\qquad u_{r-1}(x_0)=y_{r-1}. \tag{25}$$

The importance of the signature curve lies in the fact that it uniquely characterizes the original curve up to a group transformation.

Theorem 16. *Let G be an ordinary transformation group acting on $E\simeq\mathbb{R}^2$. Two smooth (C^r) curves $\mathcal{C}$ and $\overline{\mathcal{C}}$ are equivalent up to a group transformation, $\overline{\mathcal{C}}=g\cdot\mathcal{C}$, if and only if their signature curves are identical: $\overline{\mathcal{S}}=\mathcal{S}$.*

Proof: The proof of Theorem 16 just relies on the uniqueness theorem for ordinary differential equations. Indeed, since κ and κ_s are differential invariants, their values are identical for two curves related by a group transformation. Conversely, if the signature curves are identical, the two curves are solutions to the signature equation (24) corresponding to two different sets of initial conditions (25). Transitivity of $G^{(r-1)}$ on J^{r-1} implies that it acts transitively on the set of possible initial conditions $(x_0,u_0^{(r-1)})$. Let $g\in G$ map the initial conditions for $\mathcal{C}$ to those of $\overline{\mathcal{C}}$. Uniqueness of solutions to the differential equation (24) implies that g maps $\mathcal{C}$ to $\overline{\mathcal{C}}$, completing the proof. *Q.E.D.*

Remark: An interesting question is which signature curves $\mathcal{S}$ correspond to closed curves $\mathcal{C}\subset E$.

Theorem 16 is a special case of a general theorem of É. Cartan characterizing equivalent submanifolds of a homogeneous space. Indeed, a homogeneous space $M = G/H$, where $H \subset G$ is a Lie subgroup of the Lie group G, can be identified with a global transitive action of G on M, and conversely; the subgroup H is just the isotropy subgroup of the point $x_0 \in M$ which gets identified with the identity element of G. For example, the Euclidean plane is identified as the homogeneous space $E = \mathrm{E}(2)/\mathrm{O}(2)$, where the rotation subgroup $\mathrm{O}(2)$ is the isotropy subgroup of the origin in E. Cartan's Theorem states that the signature submanifold corresponding to a submanifold $S \subset G/H$ is parametrized by its $(n+1)^{\mathrm{st}}$ order differential invariants, where n denotes the order of a Frenet frame on S. Two submanifolds are equivalent under a group transformation if and only if their signature submanifolds are identical. See [**8**], [**13**] for details. The signature curve (or manifold) plays the same role in the theory of transformation groups that the classifying curve (or manifold) does in the Cartan equivalence method, [**21**].

Of particular importance are the curves whose G-invariant curvature is a constant. Such curves play the same role for general transformation groups that the circles and straight lines play for the Euclidean group, and the conic sections play for the equi-affine group. If $\kappa = c$ is constant, then $\kappa_s \equiv 0$, and hence the associated signature curve degenerates to a single point $\mathcal{S} = \{(c, 0)\}$. Thus, such curves are found, not by solving an r^{th} order equation (24), but rather as solutions to the $(r-1)^{\mathrm{st}}$ order ordinary differential equation

$$\kappa(x, u^{(r-1)}) = c. \tag{26}$$

In fact, one does not need to integrate the ordinary differential equation (26), since these curves can be found directly from the group action.

Theorem 17. *Let G be an ordinary transformation group acting on $\mathbb{R}^2$. A curve $\mathcal{C} \subset M$ has constant G-invariant curvature if and only if it is an orbit of a one-parameter subgroup of G, i.e., $\mathcal{C} = \{\exp(t\mathbf{v})P_0\}$ for some infinitesimal generator $\mathbf{v}$ of the group action. Two curves have the same constant curvature if and only if they are related by a group transformation, $\overline{\mathcal{C}} = g \cdot \mathcal{C}$.*

Theorem 17 is a special case of a result of É. Cartan which states that a submanifold of a homogeneous space is homogeneous if and only if all its differential invariants are constant; see [**8**], [**13**] for details.

Thus, for the Euclidean group, we recover the circles and straight lines as the constant curvature curves, while for the special affine group, the curves with constant affine curvature are the conic sections. For the projective group in the plane, the curves of constant projective curvature are the so-called "W-curves" which were investigated in detail by Klein and Lie in the last century; see [**14**], [**33**; §III.8].

Since (26) has order $r-1$, given r points $P_1, \dots, P_r \in E$ in "general position", there exists a unique constant curvature curve $\mathcal{C}_0(P_1, \dots, P_r)$ passing through them. Let $\widetilde{\kappa}(P_1, \dots, P_r)$ denote its curvature. Since (26) is a G-invariant ordinary differential equation, $\widetilde{\kappa}(P_1, \dots, P_r)$ is a joint invariant of the r points.

Let $\mathcal{C} \subset M$ be an arbitrary curve in the plane. We are interested in constructing a G-invariant finite difference approximation to its G-invariant curvature $\kappa(P_1)$ at a given

point $P_1 \in \mathcal{C}$ in the curve. Choose $r-1$ nearby points $P_2, \dots, P_r \in \mathcal{C}$. Then the curvature

$$\widetilde{\kappa} = \widetilde{\kappa}(P_1, \dots, P_r) \approx \kappa(P_1) \tag{27}$$

of the constant curvature curve $\mathcal{C}_0 = \mathcal{C}_0(P_1, \dots, P_r)$ passing through the points determines our approximation to $\kappa(P_1)$. This provides a general method for constructing G-invariant finite difference approximations to the G-invariant curvature of a curve.

Conjecture: The following series expansion holds:

$$\widetilde{\kappa} = \kappa + \frac{1}{r}\left(\sum_{i=1}^{r} L_i\right)\frac{d\kappa}{ds} + \frac{1}{r(r+1)}\left(\sum_{1 \le i \le j \le r} L_i L_j\right)\frac{d^2\kappa}{ds^2} + \cdots, \tag{28}$$

where κ, $d\kappa/ds$, etc. are evaluated at P_1, and

$$L_i = \int_{P_1}^{P_j} ds, \tag{29}$$

denotes the G-invariant "distance" from the point P_1 to P_j, measured as the G-invariant arc length along the constant curvature curve $\mathcal{C}_0$. The expansion assumes that all the arc lengths L_i are small.

Example 18. Consider the translation group $(x, u) \mapsto (x + c, u + d)$. In this case, $\kappa = du/dx$, and the constant curvature curves are the straight lines. Then $\widetilde{\kappa}(P_1, P_2) = (u_2 - u_1)/(x_2 - x_1)$. Therefore, the expansion (28) is merely the Taylor series, and so is valid to general order! (Note that since dx is the translation-invariant arc length, the "length" of a straight line segment is $\int_{P_1}^{P_2} dx = x_2 - x_1$.)

Thus, the conjectured series expansion (28) is valid up to order 2 for the translation group, the Euclidean group, and the special affine group. Direct verification for other planar groups appears to be problematic because the formulas for the finite difference approximation $\widetilde{\kappa}$ are not so easy to come by, because the constant curvature curves involve transcendental functions. Moreover, the Euclidean series (3) shows that the natural generalization of (28) is not valid to order 3. The proof of the second order expansion (28) and the determination of its higher order terms, remains an important open problem.

Approximations to the G-invariant arc length from P_1 to P_2, say, are determined by computing the arc length $\mathbf{L}(P_1, P_2) = \int_{P_1}^{P_2} ds$ from P_1 to P_2 along the constant curvature curve $\mathcal{C}_0(P_1, \dots, P_r)$ passing through them; again this is a joint invariant of the r mesh points, and so invariant under G. Thus, one obtains a G-invariant finite difference approximation to the derivative κ_s at a point P_1 by choosing r nearby points $P_2, \dots, P_{r+1}$ and using the G-invariant difference quotient

$$\widetilde{\kappa}_s = \frac{\widetilde{\kappa}(P_1, \dots, P_{r-1}, P_r) - \widetilde{\kappa}(P_1, \dots, P_{r-1}, P_{r+1})}{\mathbf{L}(P_1, P_2)} \approx \kappa_s(P_1), \tag{30}$$

where $\mathbf{L}(P_1, P_2)$ is computed along one of the two possible constant coefficient curves $\mathcal{C}_0(P_1, \dots, P_{r-1}, P_r)$ or $\mathcal{C}_0(P_1, \dots, P_{r-1}, P_{r+1})$, or, more symmetrically, their average. As

in the Euclidean and affine cases, a centered difference approach would be more accurate. The details of this construction remain to be fully explored.

Formulas (27), (30) thus provide a completely G-invariant finite difference approximation to the signature curve $\mathcal{S}$ associated with a curve $\mathcal{C} \subset E$, guaranteeing that two curves related by a group transformation have *identical* discrete signatures. Thus our approach provides a fully group-invariant method for numerically approximating the differential invariant signature curves of arbitrary transformation groups. Extensions to surfaces are straightforward in principle, although the precise numerical implementation remains to be fully explored.

Acknowledgments: We would like to thank Peter Giblin for reverifying the Euclidean curvature expansion (3), Boris Komrakov for guiding us to the work of Cartan, and Guillermo Sapiro for inspiring comments on signature curves.

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