# SOME APPLICATIONS OF SPLINE FUNCTIONS TO PROBLEMS IN COMPUTER GRAPHICS

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## 1. Introduction

Since their introduction by I.J. Schoenberg in 1946, splines have played an increasingly greater role in many problems involving interpolation and approximation. The application of cubic and perhaps even more general splines to the graphing of functions by computer, for which splines are ideally suited, has not been so thoroughly investigated. A few of the many areas in this field where splines can be effectively utilized are the graphing of smooth curves and surfaces and the solution of boundary value problems in ordinary and partial differential equations. The great advantage of spline approximations over linear and other interpolation schemes is the convergence not only of the spline functions themselves to the function being interpolated, but also the first three derivatives converge in a uniform manner. Thus, for example, in the area of curve plotting, the splines will give a $C^2$ curve sequence which converges to the original smooth curve along with the first three derivatives, whereas a linear interpolation on the curve will only give a piecewise smooth approximation.

In this thesis I will present the basic theory of cubic splines needed to solve some of these graphic problems and illustrate this theory with the solution of two such problems: the drawing of closed plane curves through a given set of points in $\mathbb{R}^2$ and the approximation of surfaces by bicubic splines on star-shaped planar regions bounded by Jordan curves. The first part of the thesis will derive the relevant spline theory for these problems. The second part contains a discussion of the graphic results when these problems were programmed on a computer and numerous figures that serve to illustrate different aspects of the theory. The third part gives a short description and a listing of each subroutine that was programmed to obtain the figures in the second part, as well as some general hints for the use of the programs. It is hoped that the simplicity and power of the spline methods demonstrated by the examples will spark further interest in applications of splines to further areas in computer graphics.

# I. Cubic Spline Theory

## 2. Basic Theory of One Dimensional Cubic Splines

Consider the closed interval $[a,b]$ where $a,b\in\mathbb{R}$ subdivided by a mesh $\Delta=\{x_j: j=0,1,\ldots,n\}$ such that $a=x_0<x_1<\ldots<x_n=b$. We shall usually be concerned only with uniform meshes, namely those where

$$|x_j-x_{j-1}|=h \quad , \qquad j=1,2,\ldots,n$$

but for the time being, we shall let $\Delta$ be quite arbitrary. We are further given a set of ordinates $Y=\{y_0, y_1,\ldots,y_n\}$ with each $y_j$ corresponding to the value of the function $f(x_j)$ that is to be interpolated.

Definition A cubic spline on $[a,b]$ with respect to the mesh $\Delta$ and the ordinates Y, denoted by $S_\Delta(x;Y)$ or just $S_\Delta(x)$ (we will also sometimes omit the subscript $\Delta$), is a function that satisfies the conditions

i) $S_\Delta\in C^2[a,b]$

ii) $S_\Delta$ is a cubic polynomial on each interval $[x_{j-1}, x_j]$, $j=1,2,\ldots,n$

iii) $S_\Delta(x_j)=y_j$.

It can be shown, [ANW], that given any mesh and corresponding set of ordinates Y, then there always exists a cubic spline $S_\Delta(x;Y)$. In fact, in general there will be an infinite number of cubic splines satisfying these conditions, depending on their behavior near the ends of the interval $[a,b]$. (For example see the section on cardinal splines.)

As a typical example of the convergence properties of cubic splines, we cite the following theorem from [ANW].

Definition The norm of a mesh $\Delta$ is defined by

$$\|\Delta\| = \sup\{|x_j-x_{j-1}| : j=1,2,\ldots,n\}$$

Theorem 2.1 Let $f\in C^3[a,b]$ and let $\{\Delta_k: k=1,2,\ldots\}$ be a sequence of meshes on [a,b] such that

$$\lim_{k\to\infty} \|\Delta_k\| = 0$$

and

$$R = \sup\left\{\frac{\|\Delta_k\|}{|x_{j,k}-x_{j-1,k}|} ;\ k=1,2,\ldots,\ j=1,2,\ldots,n_k\right\} < \infty.$$

Let $Y_k = \{ f(x) : x \in \Delta_k \}$. Then if $S_{\Delta_k}(x;Y_k)$ is periodic if f is periodic or satisfies certain elementary nonperiodic end conditions (see [ANW], page 29) then

$$f^{(p)}(x) - S^{(p)}(x;Y_k) = o(\|\Delta_k\|^{3-p}) \ , \ p=0,1,2,3$$

uniformly with respect to x in [a,b]. If in addition $f'''(x)$ satisfies a Hölder condition on [a,b] of order $\alpha$, $0<\alpha\leq 1$, then

$$f^{(p)}(x) - S^{(p)}(x;Y_k) = O(\|\Delta_k\|^{3+\alpha-p}) \ , \ p=0,1,2,3$$

uniformly with respect to x in [a,b].

For the proof of this theorem and further theorems of this type consult [ANW].

Since this thesis will be primarily concerned with splines defined over uniform meshes, it is useful to determine what the error is in replacing a nonuniform mesh $\Delta$ by a uniform mesh $\tilde{\Delta}$ and evaluating the spline over $\tilde{\Delta}$ with respect to the same set of ordinates Y. By the analysis of [A1] it can be seen that if the mesh is locally approximately uniform, then the error incurred by replacing $S_\Delta(x;Y)$ by $S_{\tilde{\Delta}}(x;Y)$ will be governed by the local modulus of continuity of the function f. In other words, the more rapidly f varies near a point, the closer $\tilde{\Delta}$ must approximate $\Delta$ near that point.

## 3. Representation of a Cubic Spline on a Uniform Mesh

Let n be a positive integer, and consider the interval $[a,b]$ along with the uniform mesh

$$\Delta = \left\{ a + \frac{j}{n}(b-a):\ j=0,1,\dots,n \right\}.$$

We can transform the interval $[a,b]$ into $[0,n]$ by means of the linear map that takes $x \in \mathbb{R}$ into $n(x-a)/(b-a)$, such that $\Delta$ becomes the mesh $\{0,1,\dots,n\}$. We need therefore to only be concerned with the interval $[0,n]$ and the mesh $\{0,1,\dots,n\}$ when solving for a spline over a uniform mesh. The construction of a spline S on this interval with respect to the ordinates $Y=\{y_0,y_1,\dots,y_n\}$ is as follows. Since S must be a cubic polynomial in each interval between mesh points, we have

$$S''(x) = M_{j-1}(j-x) + M_j(x-j+1), \quad x \in [j-1,j] \tag{3.1}$$

where the $M_j$, $j=0,1,\dots,n$, are the moments or second derivatives of S at the point j. Integrating this expression twice gives

$$S(x) = M_{j-1}\frac{(j-x)^3}{6} + M_j\frac{(x-j+1)^3}{6} + C_1 x + C_2, \quad x \in [j-1,j].$$

Setting $S(j-1)=y_{j-1}$, $S(j)=y_j$ yields

$$\begin{aligned} S(x) = M_{j-1}\frac{(j-x)^3}{6} + M_j\frac{(x-j+1)^3}{6} &+ \left(y_{j-1} - \frac{M_{j-1}}{6}\right)(j-x) + \\ &+ \left(y_j - \frac{M_j}{6}\right)(x-j+1), \quad x \in [j-1,j] \end{aligned} \tag{3.2}$$

$$S'(x) = -M_{j-1}\frac{(j-x)^2}{2} + M_j\frac{(x-j+1)^2}{2} + (y_j - y_{j-1}) - \frac{M_j - M_{j-1}}{6}. \tag{3.3}$$

The moments must satisfy equations such that the derivative $S'$ is continuous at the mesh points,

$$\begin{aligned} S'(j-) &= \frac{M_{j-1}}{6} + \frac{M_j}{3} + (y_j - y_{j-1}) \\ S'(j+) &= -\frac{M_j}{3} - \frac{M_{j+1}}{6} + (y_{j+1} - y_j) \end{aligned} \qquad j=1,2,\dots,n-1.$$

Equating these two gives the n-1 equations

$$\tfrac{1}{2}M_{j-1} + 2M_j + \tfrac{1}{2}M_{j+1} = 3(y_{j-1} - 2y_j + y_{j+1}), \quad j=1,\ldots,n-1 \tag{3.4}$$

in the n+1 unknowns $M_0,\ldots,M_n$. To obtain the remaining two equations, it is necessary to impose some kind of "end conditions" on the spline. The nonperiodic case will be treated in a different manner in the section on cardinal splines. (A derivation in the spirit of the following can be found in[ANW].) We will here derive expressions for the $M_j$ in the case of a periodic spline, namely one that satisfies

$$S^{(p)}(0+) = S^{(p)}(n-), \qquad p=0,1,2.$$

In particular, this implies that $y_0=y_n$ and $M_0=M_n$. Equating the first derivative terms yields the remaining equation

$$\tfrac{1}{2}M_{n-1} + 2M_n + \tfrac{1}{2}M_1 = 6(y_{n-1} - 2y_n + y_1).$$

The system of equations in $M_0,\ldots,M_n$ can be written compactly in matrix form as

$$A \cdot M = d \tag{3.5}$$

where

$$A = \begin{bmatrix} 2 & \frac{1}{2} & . & . & . & \frac{1}{2} \\ \frac{1}{2} & 2 & \frac{1}{2} & . & . & . \\ . & \frac{1}{2} & . & & & . \\ . & & & . & & . \\ . & & & \frac{1}{2} & 2 & \frac{1}{2} \\ \frac{1}{2} & . & . & . & \frac{1}{2} & 2 \end{bmatrix}, \quad M = \begin{bmatrix} M_0 \\ M_1 \\ . \\ . \\ . \\ M_n \end{bmatrix}, \quad d = \begin{bmatrix} d_0 \\ d_1 \\ . \\ . \\ . \\ d_n \end{bmatrix} \tag{3.6}$$ *

with $d_j = 3(y_{j-1} - 2y_j + y_{j+1})$. $(y_{n+1} = y_1)$. Note that by the criteria of Gerschgorin's Theorem, A is obviously invertible, so the periodic spline on [0,n] exists and is uniquely determined. To explicitly calculate A we define the $n \times n$ determinant

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*In this and subsequent diagramming of matrices, only the nonzero elements will usually be displayed.

$$D_n = \det \begin{vmatrix} 2 & \frac{1}{2} & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{2} & 2 & \frac{1}{2} & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2} & \cdot & & & \cdot \\ \cdot & & & \ddots & \frac{1}{2} & \cdot \\ \cdot & & & \frac{1}{2} & 2 & \frac{1}{2} \\ \cdot & \cdot & \cdot & \cdot & \frac{1}{2} & 2 \end{vmatrix} \qquad (3.7)$$

and let $D_{-1}=0$, $D_0=1$. The $D_n$ satisfies the difference equation

$$D_n - 2D_{n-1} + \tfrac{1}{4}D_{n-2} = 0 \qquad (3.8)$$

as can be seen by expanding $D_n$ by the first row. The roots of the characteristic equation of this difference equation,

$$x^2-2x+\tfrac{1}{4} = 0$$

are

$$x_1 = \frac{2+\sqrt{3}}{2}, \qquad x_2 = \frac{2-\sqrt{3}}{2}$$

The general solution is then, [IK],

$$D_n = \alpha_1 x_1^{n+1} + \alpha_2 x_2^{n+1} .$$

Evaluating the constants $\alpha_1$ and $\alpha_2$ gives

$$\alpha_1 + \alpha_2 = 0$$

$$\alpha_1\left(\frac{2+\sqrt{3}}{2}\right) + \alpha_2\left(\frac{2-\sqrt{3}}{2}\right) = 1,$$

hence

$$\alpha_1 = \frac{1}{\sqrt{3}}, \qquad \alpha_2 = \frac{-1}{\sqrt{3}} .$$

Thus we obtain

$$D_n = \frac{(1+\sqrt{3}/2)^{n+1} - (1-\sqrt{3}/2)^{n+1}}{\sqrt{3}}. \qquad (3.9)$$

Now to compute the determinant of A,

$$\det A = \det \begin{vmatrix} 2 & \frac{1}{2} & \cdot & \cdot & \cdot & \frac{1}{2} \\ \frac{1}{2} & 2 & \frac{1}{2} & & & \cdot \\ \cdot & \frac{1}{2} & \cdot & & & \cdot \\ \cdot & & & \ddots & \frac{1}{2} & \cdot \\ \cdot & & & \frac{1}{2} & 2 & \frac{1}{2} \\ \frac{1}{2} & \cdot & \cdot & \cdot & \frac{1}{2} & 2 \end{vmatrix}$$

$$= 2\det\begin{vmatrix} 2 & \frac12 & \cdot & \cdot & \cdot \\ \frac12 & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \frac12 \\ \cdot & \cdot & \cdot & \frac12 & 2 \end{vmatrix} - \tfrac12\det\begin{vmatrix} \frac12 & \frac12 & \cdot & \cdot & \cdot \\ 0 & 2 & \frac12 & & \cdot \\ \cdot & \frac12 & \cdot & & \cdot \\ \cdot & & & \cdot & \frac12 \\ \frac12 & \cdot & \cdot & \frac12 & 2 \end{vmatrix} + \tfrac12(-1)^{n-1}\det\begin{vmatrix} \frac12 & 2 & \frac12 & \cdot & \cdot \\ 0 & \frac12 & 2 & \cdot & \cdot \\ \cdot & & \cdot & & \frac12 \\ \cdot & & & \frac12 & 2 \\ \frac12 & \cdot & \cdot & 0 & \frac12 \end{vmatrix}$$

$$= 2D_{n-1} - \tfrac12\left[\tfrac12\det\begin{vmatrix} 2 & \frac12 & \cdot & \cdot & \cdot \\ \frac12 & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \frac12 \\ \cdot & \cdot & \cdot & \frac12 & 2 \end{vmatrix} + (-1)^{n-2}\tfrac12\det\begin{vmatrix} \frac12 & \cdot & \cdot & \cdot & \cdot \\ 2 & \frac12 & & & \cdot \\ \frac12 & & \cdot & & \cdot \\ \cdot & & & \frac12 & \cdot \\ \cdot & \cdot & \frac12 & 2 & \frac12 \end{vmatrix}\right] +$$

$$+ (-1)^{n-1}\tfrac12\left[\tfrac12\det\begin{vmatrix} \frac12 & 2 & \frac12 & \cdot & \cdot \\ \cdot & \frac12 & & & \cdot \\ \cdot & & \cdot & & \frac12 \\ \cdot & & & \frac12 & 2 \\ \cdot & \cdot & \cdot & \cdot & \frac12 \end{vmatrix} + (-1)^{n-2}\tfrac12\det\begin{vmatrix} 2 & \frac12 & \cdot & \cdot & \cdot \\ \frac12 & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \frac12 \\ \cdot & \cdot & \cdot & \frac12 & 2 \end{vmatrix}\right]$$

$$= 2D_{n-1} - \tfrac14 D_{n-2} + (-1)^{n-1}\tfrac14(\tfrac12)^{n-2} + (-1)^{n-1}\tfrac14(\tfrac12)^{n-2} - \tfrac14 D_{n-2}.$$

$$\det A = 2D_{n-1} - \tfrac12 D_{n-2} + (-2)^{1-n}. \qquad (3.10)$$

These equations follow from expansion by rows and columns and the fact that the second and third n-2Xn-2 determinants in the trird line are lower and upper triangular, respectively. Now let $\delta_0, \delta_1, \ldots, \delta_{n-1}$ be the elements of the first row of $A^{-1}$. Then, since A is a circulant matrix,

$$A^{-1}_{i,j} = \delta_{|i-j|}. \qquad (3.11)$$

To compute

$$(-1)^i\delta_j\cdot\det A = \det\begin{vmatrix} \frac12 & 2 & & & & & & & & & \\ & \frac12 & & & & & & & & & \\ & & \cdot & & & & & & & & \\ & & & \frac12 & 2 & & & & & & \\ & & & & \frac12 & & & & & & \\ & & & & & \frac12 & 2 & 0 & 0 & & \\ & & & & & 0 & \frac12 & \frac12 & 0 & & \\ & & & & & 0 & 0 & 2 & \frac12 & & \\ & & & & & & & \frac12 & 2 & \frac12 & \\ & & & & & & & & \frac12 & \cdot & \\ & & & & & & & & & \cdot & \frac12 \\ (\frac12) & & & & & & & & & \frac12 & 2 \end{vmatrix} \mathrm{j}$$

$$\mathrm{j}$$

(expanding by the j'th row).

$$= \tfrac{1}{2}\det \left|\begin{array}{cccc:cccc} \frac12 & 2 & \frac12 & & & & & \\ & \frac12 & \ddots & & & & & \\ & & \ddots & 2 & & & & \\ & & & \frac12 & & & & \\ \hdashline & & & & 2 & \frac12 & & \\ & & & & \frac12 & \ddots & & \\ & & & & & \ddots & \ddots & \frac12 \\ (\frac12) & & & & & & \frac12 & 2 \end{array}\right|_{\,j-1} - \tfrac{1}{2}\det \left|\begin{array}{ccccc:cccc} \frac12 & 2 & \frac12 & & & & & & \\ & \frac12 & \ddots & & & & & & \\ & & \ddots & & & & & & \\ & & & \frac12 & 2 & & & & \\ & & & 0 & 0 & \frac12 & & & \\ \hdashline & & & & & 2 & \frac12 & & \\ & & & & & & \ddots & & \\ & & & & & & & \ddots & \frac12 \\ (\frac12) & & & & & & & \frac12 & 2 \end{array}\right|_{\,j}$$

$$= \tfrac12(\tfrac12)^{j-1} D_{n-2-(j-1)} - \tfrac12 \left[ \tfrac12\det \left|\begin{array}{ccccc:cccc} \frac12 & 2 & \frac12 & & & & & & \\ & \frac12 & \ddots & & & & & & \\ & & \ddots & & & & & & \\ & & & \frac12 & 2 & & & & \\ 0 & \ldots & \ldots & 0 & 0 & \frac12 & & & \\ \hdashline & & & & & 2 & \frac12 & & \\ & & & & & \frac12 & \ddots & & \\ & & & & & & \ddots & \ddots & \frac12 \\ & & & & & & & \frac12 & 2 \end{array}\right|_{\,j-2} + (-1)^{n-3}\tfrac12\det \left|\begin{array}{cccc:cccc} 2 & \frac12 & & & & & & \\ \frac12 & \ddots & & & & & & \\ & \ddots & \ddots & \frac12 & & & & \\ & & \frac12 & 2 & & & & \\ \hdashline & & & & \frac12 & & & \\ & & & & 2 & \frac12 & & \\ & & & & & \ddots & & \\ & & & & & & \ddots & \frac12 \\ & & & & & & 2 & \frac12 \end{array}\right|_{\,j-1} \right]$$

(where the last term is omitted in case j=0)

$$= 2^{-j} D_{n-j-1} + 0 - (-1)^{n-3}\tfrac12 \cdot D_{j-1} \cdot (\tfrac12)^{n-2-(j-1)}$$

$$= 2^{-j} D_{n-j-1} + (-1)^{n} 2^{j-n} D_{j-1}.$$

Thus,

$$\delta_j = \left[ (-2)^{-j} D_{n-j-1} + (-2)^{j-n} D_{j-1} \right] / \det A \qquad (3.12)$$

Finally by use of (3.6) we can express the moments $M_i$ in terms of the $A^{-1}_{i,j}$ and the ordinates $y_j$ by

$$\begin{aligned} M_i &= \sum_{j=1}^{n} A^{-1}_{i,j}\, d_j = \sum_{j=1}^{n} 3A^{-1}_{i,j}\,(y_{j-1} - 2y_j + y_{j+1}) \\ &= 3\sum_{j=1}^{n} (A^{-1}_{i,j-1} - 2A^{-1}_{i,j} + A^{-1}_{i,j+1})\, y_j \;. \end{aligned} \qquad (3.\approx3)$$

Now since

$$\begin{aligned} \delta_{k-1} - 2\delta_k + \delta_{k+1} &= \left[ (-2)^{-k+1} D_{n-k} + (-2)^{k-1-n} D_{k-2} - 2(-2)^{-k} D_{n-k-1} - \right. \\ &\quad \left. - 2(-2)^{k-n} D_{k-1} + (-2)^{-k-1} D_{n-k-2} + (-2)^{k+1-n} D_k \right] / \det A \\ &= \left[ (-2)^{1-k} (D_{n-k} - 2D_{n-k-1} + \tfrac14 D_{n-k-2}) - 3(-2)^{-k} D_{n-k-1} + \right. \\ &\quad \left. + (-2)^{k+1-n} (D_k - 2D_{k-1} + \tfrac14 D_{k-2}) - 3(-2)^{k-n} D_{k-1} \right] / \det A \\ &= -6\,\delta_k, \qquad k=1,\ldots,n-1 \end{aligned} \qquad (3.14a)$$

by (3.8), and

$$\begin{aligned}\delta_{-1}-2\delta_0+\delta_1 &= 2\delta_1-2\delta_0\\ &= 2[(-2)^{-1}D_{n-2}+(-2)^{1-n}D_0-D_{n-1}]/\det A\\ &= 2[2D_{n-1}-\tfrac{1}{2}D_{n-2}+(-2)^{1-n}-3D_{n-1}]/\det A\\ &= 2-6\delta_0 \end{aligned} \qquad (3.14b)$$

by (3.10), combining (3.13), (3.14a and b) and (3.11) gives rise to the formula

$$M_i = -18\sum_{j=1}^{n} A^{-1}_{i,j}\, y_j + 6y_i \ . \qquad (3.15)$$

We now summarize the relevant formulae of this section for the computation of the periodic spline $S(x;Y)$ on $[0,n]$:

$$S(x;Y) = M_{j-1}\frac{(i-x)^3}{6} + M_j\frac{(x-i+1)^3}{6} + \left(y_{j-1}-\frac{M_{j-1}}{6}\right)(i-x) + \left(y_j-\frac{M_j}{6}\right)(x-i+1), \qquad x\in[i-1,i]$$

$$\left.\begin{aligned} M_i &= -18\sum_{j=1}^{n} A^{-1}_{i,j}\, y_j + 6y_i\\ A^{-1}_{i,j} &= \delta_{|i-j|}\\ \delta_k &= [(-2)^{-k}D_{n-k-1} + (-2)^{k-n}D_{k-1}]/(\det A), \quad k=0,\dots,n-1\\ \det A &= 2D_{n-1} - \tfrac{1}{2}D_{n-2} + (-2)^{1-n}\\ D_p &= \frac{(1+\sqrt{3}/2)^{p+1} - (1-\sqrt{3}/2)^{p+1}}{\sqrt{3}} \qquad p=-1,0,1,\dots \end{aligned}\right\} \quad (3.16)$$

## 4. Cardinal Spline Representations

While the construction of the periodic spline on the interval $[0,n]$ given in the previous section eventually gives a reasonably simple set of equations, (3.16), the intervening algebra is overly complex and tedious. In fact, a similar derivation can be devised for the nonperiodic spline (see [ANW]); it is even more complicated. Another method based on the "linearity" of splines, which yields the same result, although perhaps requiring more computing time and storage (see [AW] for some estimates) has a much simpler derivation. We first quote a result from [ANW]:

Lemma 4.1: Let $\Delta=\{x_0,x_1,\dots,x_n\}$ be a mesh on the interval $[a,b]$ and let Y be a corresponding set of ordinates. Given $s,t\in R$ there exists a unique spline $S_\Delta(x;Y)$ such that

$$S_\Delta(x_j;Y) = y_j\ , \qquad j=0,1,\dots,n$$
$$S'_\Delta(x_0;Y) = s$$
$$S'_\Delta(x_n;Y) = t.$$

The basic observation for the cardinal spline representation is that given two spline functions $S(x;Y)$ and $S(x;\tilde{Y})$ on the same mesh $\Delta$, then the function

$$\alpha S(x;Y) + \beta S(x;\tilde{Y})$$

where $\alpha,\beta\in R$ is also a spline on $\Delta$ corresponding to the ordinates $\alpha Y + \beta\tilde{Y}$. Thus, in view of the lemma, it seems natural to consider the set of all splines on $\Delta$ as a vector space with basis of the form

$$\{\hat{C}_0, C_0, C_1, \dots, C_{n-1}, C_n, \hat{C}_n\}$$

where

$$\left.\begin{aligned} \hat{C}_0(x_j) &= 0 \qquad j=0,\dots,n \\ \hat{C}'_0(x_0) &= 1 \\ \hat{C}'_0(x_n) &= 0 \end{aligned}\right\} \quad (4.1a)$$

$$\left.\begin{aligned} C_i(x_j) &= \delta_{ij} \qquad i,j=0,\dots,n \\ C'_i(x_0) &= C'_i(x_n) = 0 \end{aligned}\right\} \quad (4.1b)$$

$$\left.\begin{aligned} \hat{C}_n(x_j) &= 0 \qquad j=0,\dots,n \\ \hat{C}'_n(x_0) &= 0 \\ \hat{C}'_n(x_n) &= 1. \end{aligned}\right\} \quad (4.1c)$$

Then given any set of ordinates Y and values for the derivatives at a and b - $y_0'$ and $y_n'$

$$S(x;Y,y_0',y_n') = \sum_{j=0}^{n} y_j C_j(x) + y_0'\hat{C}_0(x) + y_n'\hat{C}_n(x) \qquad (4.2)$$

will be the spline passing through the ordinates Y and having the specified derivatives at a and b. It can easily be seen that the right hand expression satisfies all the conditions for S, hence by the lemma they must be equal. In particular, if $\Delta=\{0,1,...,n\}$, we can view $\Delta$ as a subset of $\mathbb{Z}$, the integers, and then the cardinal spline basis for will be a subset of $\{C_i: C_i(j)=\delta_{ij},\ i,j\in\mathbb{Z}\}$, the cardinal spline basis of $\mathbb{Z}$, namely the subset $\{C_{-1},C_0,...,C_n\}$ (We shall show below that $\hat{C}_0$ and $\hat{C}_n$ are multiples of $C_{-1}$ and $C_{n+1}$.) It can obviously be seen that all the cardinal splines must be translates of, say, $C_0$:

$$C_i(x) = C_0(x-i)\ , \qquad i\in\mathbb{Z}. \qquad (4.3)$$

Let $\lambda=-2+\sqrt{3}$. Then, as may be verified by direct computation, the primary cardinal spline $C_0$ is given by

$$C_0(x) = \begin{cases} (3\lambda+2)x^3 - 3(\lambda+1)x^2 + 1, & 0\leq x\leq 1 \\ 3\lambda^j\left[(\lambda+1)(x-j)^3 - (\lambda+2)(x-j)^2 + (x-j)\right], & j\leq x\leq j+1 \\ C_0(-x), & x<0 \end{cases} \qquad (4.4)$$

For later reference, we compute

$$C_0'(j) = \begin{cases} 0\ , & j=0 \\ 3\lambda^j\ , & j=1,2,... \\ -3\lambda^{-j}\ , & j=-1,-2,... \end{cases} \qquad (4.5)$$

$$C_0''(j) = \begin{cases} -6(\lambda+1)\ , & j=0 \\ -6\lambda^j(\lambda+2)\ , & j=1,2,... \\ -6\lambda^{-j}(\lambda+2)\ , & j=-1,-2,... \end{cases} \qquad (4.6)$$

## 5. End Conditions

With the cardinal spline representation of a given spline, we are now in a position to derive specific formulae for various types of splines. Since the number of splines on a mesh $\Delta$ satisfying a given set of ordinates is infinite by lemma 4.1, it is necessary to specify the behavior of the spline near the ends of the interval. In this section various possibilities for these end conditions are investigated. We will again work with the mesh $\Delta=\{0,1,\ldots,n\}$, although the generalization to an arbitrary mesh is immediate. The basic equation is

$$S(x;Y) = \sum_{j=0}^{n} y_j C_j(x) + \alpha C_{-1}(x) + \beta C_{n+1}(x)$$

or

$$S(x;Y) = \sum_{j=0}^{n} y_j C_0(x-j) + \alpha C_0(x+1) + \beta C_0(x-n-1) \qquad (5.1)$$

where $\alpha$ and $\beta$ are variable, depending on the end conditions. Note that by lemma 4.1 and (4.5) all splines satisfying the ordinates Y are expressed by (5.1).

a) Specified Derivatives

The derivatives $S'(0)=y'_0$ and $S'(n)=y'_n$ are specified in advance in these end conditions. Then solving for $y'_0$ and $y'_n$ we have

$$\sum_{j=0}^{n} C'_0(-j)y_j + \alpha C'_0(1) + \beta C'_0(-n-1) = y'_0$$
$$\sum_{j=0}^{n} C'_0(n-j)y_j + \alpha C'(n+1) + \beta C'(1) = y'_n$$

or, on substitution from (4.5),

$$3\lambda\alpha - 3\lambda^{n+1}\beta = y'_0 + 3\sum_{j=0}^{n}\lambda^j y_j$$
$$3\lambda^{n+1}\alpha + 3\lambda\beta = y'_n - 3\sum_{j=0}^{n}\lambda^{n-j} y_j$$

Thus we obtain

$$\alpha = \frac{\left[\frac{1}{3}(y'_0 + \lambda^n y'_n) + \sum_{j=0}^{n}(\lambda^j - \lambda^{2n-j})y_j\right]}{\lambda(1+\lambda^{2n})}$$
$$\beta = \frac{\left[\frac{1}{3}(y'_n - \lambda^n y'_0) - \sum_{j=0}^{n}(\lambda^{n-j} + \lambda^{n+j})y_j\right]}{\lambda(1+\lambda^{2n})} \qquad (5.2)$$

b) Parabolic Run-out

With these end conditions, the spline is required to be parabolic on the two end intervals $[0,1]$ and $[n-1,n]$. Then for $0\leq x\leq 1$, $-j\leq x-j\leq -j+1$, so $j-1\leq j-x\leq j$ and $j-x-(j-1)=1-x$. Computing, we obtain through use of (5.1) and (4.4)

$$\begin{aligned} S(x) = {} & y_0[(3\lambda+2)x^3-3(\lambda+1)x^2+1] + \\ & + y_1[(3\lambda+2)(1-x)^3-3(\lambda+1)(1-x)^2+1] + \\ & + \sum_{j=2}^{n} 3\lambda^{j-1}y_j[(\lambda+1)(1-x)^3-(\lambda+2)(1-x)^2+(1-x)] + \\ & + 3\lambda\alpha[(\lambda+1)x^3-(\lambda+2)x^2+x] + \\ & + 3\lambda^n\beta[(\lambda+1)(1-x)^3-(\lambda+2)(1-x)^2+(1-x)], \quad 0\leq x\leq 1. \end{aligned}$$

Also for $n-1\leq x\leq n$, $n-j-1\leq x-j\leq n-j$ and $x-j-(n-j-1)=x-n+1$. Thus

$$\begin{aligned} S(x) = {} & \sum_{j=0}^{n-2} 3\lambda^{n-j-1}y_j[(\lambda+1)(x-n+1)^3-(\lambda+2)(x-n+1)^2+(x-n+1)] + \\ & + y_{n-1}[(3\lambda+2)(x-n+1)^3-3(\lambda+1)(x-n+1)^2+1] + \\ & + y_n[(3\lambda+2)(n-x)^3-3(\lambda+1)(n-x)^2+1] + \\ & + 3\lambda^n\alpha[(\lambda+1)(x-n+1)^3-(\lambda+2)(x-n+1)^2+(x-n+1)] + \\ & + 3\lambda\beta[(\lambda+1)(n-x)^3-(\lambda+2)(n-x)^2+(n-x)], \quad n-1\leq x\leq n. \end{aligned}$$

Setting the coefficient of $x^3$ equal to zero in these two expressions yields, upon dividing by $3\lambda(\lambda+1)$ the equations

$$\begin{aligned} \alpha - \lambda^{n-1}\beta &= \frac{3\lambda+2}{3\lambda(\lambda+1)}(y_1-y_0) - \sum_{j=2}^{n}\lambda^{j-2}y_j \\ -\lambda^{n-1}\alpha + \beta &= \frac{3\lambda+2}{3\lambda(\lambda+1)}(y_{n-1}-y_n) + \sum_{j=0}^{n-2}\lambda^{n-j-2}y_j \end{aligned} \tag{5.3}$$

Solving for $\alpha$ and $\beta$

$$\begin{aligned} \alpha &= \frac{(3\lambda+2)[y_1-y_0+\lambda^{n-1}(y_{n-1}-y_n)]}{3\lambda(\lambda+1)(1-\lambda^{2n-2})} + \frac{\sum_{j=0}^{n-2}\lambda^{2n-j-3}y_j - \sum_{j=2}^{n}\lambda^{j-2}y_j}{(1-\lambda^{2n-2})} \\ \beta &= \frac{(3\lambda+2)[y_{n-1}-y_n+\lambda^{n-1}(y_1-y_0)]}{3\lambda(\lambda+1)(1-\lambda^{2n-2})} + \frac{\sum_{j=0}^{n-2}\lambda^{n-j-2}y_j - \sum_{j=2}^{n}\lambda^{n+j-3}y_j}{(1-\lambda^{2n-2})} \end{aligned} \tag{5.4}$$

c) Rabinowitz End Conditions

Another possibility for end conditions, originally suggested by P. Rabinowitz, is to prescribe that the discontinuity in the third derivative of the spline over each of the end two intervals be removed. Since S is cubic,

$$S'''(x) = S''(j+1) - S''(j) , \quad j \leqslant x \leqslant j+1.$$

Therefore this condition is ensured by the equations

$$\begin{aligned} S''(2) - 2S''(1) + S''(0) &= 0 \\ S''(n-2) - 2S''(n-1) + S''(n) &= 0 \end{aligned} \qquad (5.5)$$

Substituting (5.1) and (4.6) these become

$$\begin{aligned} 0 = {} & [\lambda^2(\lambda+2)-2\lambda(\lambda+2)+(\lambda+1)]y_0 + [\lambda(\lambda+2)-2\cdot(\lambda+1)+\lambda(\lambda+2)]y_1 + \\ & +[(\lambda+1)-2\lambda(\lambda+2)+\lambda^2(\lambda+2)]y_2 + \sum_{j=3}^{n}(\lambda^2-2\lambda+1)(\lambda+2)\lambda^{j-2}y_j + \\ & + (\lambda^2-2\lambda+1)(\lambda+2)\lambda\alpha + (\lambda^2-2\lambda+1)(\lambda+2)\lambda^{n-1}\beta \\ 0 = {} & \sum_{j=0}^{n-3}(\lambda^2-2\lambda+1)(\lambda+2)\lambda^{n-j-2}y_j + [(\lambda+1)-2\lambda(\lambda+2)+\lambda^2(\lambda+2)]y_{n-2} + \\ & + [\lambda(\lambda+2)-2(\lambda+1)+\lambda(\lambda+2)]y_{n-1} + [\lambda^2(\lambda+2)-2\lambda(\lambda+2)+(\lambda+1)]y_n + \\ & + (\lambda^2-2\lambda+1)(\lambda+2)\lambda^{n-1}\alpha + (\lambda^2-2\lambda+1)(\lambda+2)\lambda\beta. \end{aligned}$$

Simplifying, these become

$$\begin{aligned} \alpha + \lambda^{n-2}\beta &= -\sum_{j=3}^{n}\lambda^{j-3}y_j - \frac{(\lambda^3-3\lambda+1)(y_0+y_2) + (\lambda^2+2\lambda-2)y_1}{\lambda(\lambda-1)^2(\lambda+2)} \\ \lambda^{n-2}\alpha + \beta &= -\sum_{j=0}^{n-3}\lambda^{n-j-3}y_j - \frac{(\lambda^3-3\lambda+1)(y_{n-2}+y_n) + (\lambda^2+2\lambda-2)y_{n-1}}{\lambda(\lambda-1)^2(\lambda+2)} \end{aligned} \qquad (5.6)$$

from which $\alpha$ and $\beta$ can easily be computed.

d) <u>Periodic End Conditions</u>

Rather than using the derivation of section 3, an alternate approach to obtain a periodic spline on 0,n is to impose the periodic end conditions

$$\begin{aligned} S'(0) &= S'(n) \\ S''(0) &= S''(n) \end{aligned} \qquad (5.7)$$

(It is assumed that $y_0 = y_n$.) Expanding these equations gives

$$\begin{aligned} -\sum_{j=1}^{n}3\lambda^j y_j + 3\lambda\alpha - 3\lambda^{n+1}\beta &= \sum_{j=0}^{n-1}3\lambda^{n-j}y_j + 3\lambda^{n+1}\alpha - 3\lambda\beta \\ -6(\lambda+1)y_0 - \sum_{j=1}^{n}6\lambda^j(\lambda+2)y_j - 6\lambda(\lambda+2)\alpha - 6\lambda^{n+1}(\lambda+2)\beta &= \\ = -\sum_{j=0}^{n-1}6\lambda^{n-j}(\lambda+2)y_j - 6(\lambda+1)y_n - 6\lambda^{n+1}(\lambda+2)\alpha - 6\lambda(\lambda+2)\beta & \end{aligned}$$

or, since $y_0 = y_n$,

$$\alpha + \beta = \frac{\sum_{j=1}^{n} \lambda^{j-1}(y_{n-j} + y_j)}{1-\lambda^n}$$
$$\alpha - \beta = \frac{\sum_{j=1}^{n} \lambda^{j-1}(y_{n-j} - y_j)}{1-\lambda^n} \tag{5.8}$$

hence

$$\alpha = \frac{\sum_{j=1}^{n} \lambda^{j-1} y_{n-j}}{1-\lambda^n}$$
$$\beta = \frac{\sum_{j=1}^{n} \lambda^{j-1} y_j}{1-\lambda^n} \tag{5.9}$$

A further observation can be made, namely, that given the interval $[0,n]$ we can form a basis of periodic splines

$$C_{p,i}(x,n) \;:\; i=0,1,\ldots,n-1$$

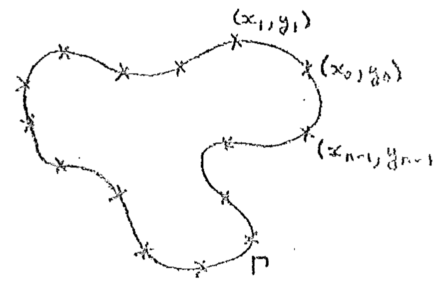
where $C_{p,i}(j,n)=\delta_{ij}$. It is apparent that these cardinal periodic splines are all translates, modulo n, of the fundamental cardinal periodic spline on $[0,n]$ $C_p(x,n)$ given by

$$C_p(i,n) = \begin{cases} 1 & i=0,n \\ 0 & i=2,3,\ldots,n-1 \end{cases} \tag{5.10}$$

In particular, substitution in equation (5.9) yields $\alpha=\beta=\lambda^{n-1}/(1-\lambda^n)$, hence

$$C_p(x,n) = C_0(x) + C_0(x-n) + \frac{\lambda^{n-1}}{1-\lambda^n}[C_0(x+1) + C_0(x-n-1)], \quad 0 \leq x \leq n. \tag{5.11}$$

## 6. Plane Curve Fitting




Let $\Gamma$ be a closed curve in the x-y plane and suppose we are given a set of points

$$\{(x_i,y_i):\ i=0,1,\ldots,n\}$$

where $(x_0,y_0)=(x_n,y_n)$, numbered in consecutive order along the curve $\Gamma$. Using point number as a parametric variable to describe $\Gamma$, we can compute the two periodic splines $S_p(t;X)$ and $S_p(t;Y)$ where $X=\{x_0,x_1,\ldots,x_n\}$ and $Y=\{y_0,y_1,\ldots,y_n\}$ and consider the curve $\Gamma_s=\{(S_p(t;X),S_p(t;Y))\ :\ 0\leq t\leq n\}$ as a spline approximation to $\Gamma$. The properties of the convergence of the various $\Gamma_s$'s to $\Gamma$ will be similar to those given in theorem 2.1 and its analogs in [ANW]. In other words, besides giving a good approximation to $\Gamma$, the curves $\Gamma_s$ also yield good approximations to the first three derivatives of the parametric representation of $\Gamma$.

Actually, the number of points needed so that the spline approximates $\Gamma$ . . close is quite small, especially if these points are chosen judiciously. Using arc length along $\Gamma$ from $(x_0,y_0)$ as another parametrization of $\Gamma$ with parametric variable s, we have each point $(x_i,y_i)$ located at a distance $s_i$ along $\Gamma$ from $(x_0,y_0)$. The mesh $\{0,1,\ldots,n\}$ used in the previous parametrization of the splines can be thought of as a uniform approximation to the nonuniform mesh $S=\{s_0,s_1,\ldots,s_n\}$ (or, more acurately, some multiple of S). From the remark in section 2 on the imposition of a nonuniform mesh, it can be seen that the spacing of the points along $\Gamma$ should be locally nearly uniform with respect to arc length, although this of course does not imply that this should be done globally. Also it can be shown that the convergence properties of theorem 2.1 vary as the local magnitude of the second derivative of the function approximated, so in this case a general rule for the choice of points on $\Gamma$ would be to take the spacing between points small where the curvature of $\Gamma$ is large. Examples of the concepts in this section can be found in section 11. Further discussion and illustrations can be found in [AW].

## 7. Spline Surfaces

The extension of the concept of spline functions to twc or even more dimensions is quite natural on rectangular regions. For the purposes of this thesis it will suffice to consider only the two dimensional case with the splines defined on uniform meshes. Consider the rectangular mesh $\Delta=\{(i,j):\ i=0,1,\dots,n,\ j=0,1,\dots,m\}$ and the set of ordinates $Z=\{z_{ij}:\ (i,j)\in\Delta\}$. A basis of doubly cubic splines on $\mathbb{Z}\times\mathbb{Z}$ is given by

$$\{C_0(x-i)C_0(y-j)\ :\ (i,j)\in\mathbb{Z}\}.$$

We then express the general spline on the rectangle $[0,n]\times[0,m]$ by the formula

$$\begin{aligned} S(x,y) = {} & \sum_{i=0}^{n}\sum_{j=0}^{m} z_{ij}\, C_0(x-i)C_0(y-j) + \\ & + C_0(y+1)\sum_{i=0}^{n}\alpha_i C_0(x-i) + C_0(y-m-1)\sum_{i=0}^{n}\beta_i C_0(x-i) + \\ & + C_0(x+1)\sum_{j=0}^{m}\gamma_j C_0(y-j) + C_0(x-n-1)\sum_{j=0}^{m}\delta_j C_0(y-j) + \\ & + \eta_{00}C_0(x+1)C_0(y+1) + \eta_{n0}C_0(x-n-1)C_0(y+1) + \\ & + \eta_{0m}C_0(x+1)C_0(y-m-1) + \eta_{nm}C_0(x-n-1)C_0(y-m-1) \end{aligned} \tag{7.1}$$

where $\alpha_0,\dots,\alpha_n,\beta_0,\dots,\beta_n,\gamma_0,\dots,\gamma_m,\delta_0,\dots,\delta_m,\eta_{00},\eta_{n0},\eta_{0m},\eta_{nm}$ are variable depending on end conditions. The substitution of $x=i$ (or $y=j$) will yield a one dimensional cubic spline so the imposition of end conditions along these lines and use of the equations in section 5 gives values for the $\alpha$'s, $\beta$'s, $\gamma$'s, and $\delta$'s. To determine $\eta_{00},\eta_{n0},\eta_{0m},\eta_{nm}$ it is necessary to impose further restrictions at the corners of the rectangle, which will be related to the value of $\partial^2 S(x,y)/\partial x\partial y$ at the corner points. Rather than try to find realistic conditions to use, $\eta_{00}$, $\eta_{n0}$, $\eta_{0m}$, $\eta_{nm}$ will hereafter be set to zero.

## 8. Splines on Polar Regions

In this section we derive a formula for a spline surface defined over the unit disk. Consider the polar mesh on the disk, given in polar coordinates,

$$\Delta = \left\{ \left(\frac{i}{n}, \frac{2\pi j}{m}\right) \ : \ i=0,\ldots,n, \ \ j=0,\ldots,m \right\}$$

and a corresponding set of ordinates Z defined over $\Delta$. To find a cardinal spline basis on $\Delta$, let

$$u=nr \qquad \text{and} \quad v=m\theta/2\pi. \tag{8.1}$$

Under this transformation the unit circle goes onto the rectangle $[0,n]\times[0,m]$. Now since the splines in the direction must match up at $\theta=0$ and $\theta=2\pi$, it seems natural rather than use the cardinal splines $C_\sigma(v-i)$ in the v direction, to use the basis of periodic splines on $[0,m]$, namely (see section 5d)

$$\{ C_p(v-j,m) \ : \ j=1,2,\ldots,m \}.$$

We then obtain the representation

$$\begin{aligned} S(u,v) = {} & \sum_{i=0}^{n}\sum_{j=1}^{m} z_{ij}\, C_\sigma(u-i)C_p(v-j,m) + \\ & + \sum_{j=1}^{m} [\alpha_j C_\sigma(u+1) + \beta_j C_\sigma(u-n-1)] C_p(v-j,m) \end{aligned} \tag{8.2}$$

where the $\gamma_j$'s and $\delta_j$'s of formula (7.1) have been implicitly determined by the use of periodic splines in the v direction. Furthermore, since $f(x)\equiv 1$ is a periodic spline on $[o,m]$ and by Lemma 4.1,

$$\sum_{j=1}^{m} C_p(v-j,m) = 1 \ , \qquad 0\leq v\leq m. \tag{8.3}$$

Combining (8.3) and the fact that $z_{00}=z_{01}=\ldots=z_{0m}$, (8.2) simplifies to

$$\begin{aligned} S(u,v) = {} & z_{00}C_\sigma(u) + \sum_{i=1}^{n}\sum_{j=1}^{m} z_{ij}\, C_\sigma(u-i)C_p(v-j,m) + \\ & + \sum_{j=0}^{m} [\alpha_j C_\sigma(u+1) + \beta_j C_\sigma(u-n-1)] C_p(v-j,m) . \end{aligned} \tag{8.4}$$

We now assume that $m=2\mu$ for some $\mu\in\mathbb{Z}$. In order to ensure radial continuity of the first and second derivatives of

S at the origin of the disk, we must have

$$\begin{aligned} S'(0,k) &= -S'(0,k+\mu) \\ S''(0,k) &= S''(0,k+\mu) \end{aligned} \quad , \quad k=0,1,\ldots,\mu. \qquad (8.5)$$

Fix a k between 0 and $\mu$, then

$$\begin{aligned} S(u,k) = {} & z_{00}C\ (u) + \sum_{i=1}^{n} z_{ik}C\ (u-i) + \\ & + \alpha_k C_0(u+1) + \beta_k C_0(u-n-1). \end{aligned} \qquad (8.6)$$

Substituting (8.6) into (8.5) and referring to (4.5) and (4.6),

$$\begin{aligned} & \sum_{i=1}^{n} -3\lambda^i z_{ik} + 3\lambda\alpha_k - 3\lambda^{n+1}\beta_k = \\ & \qquad = -\left[\sum_{i=1}^{n} -3\lambda^i z_{i\,k+\mu} + 3\lambda\alpha_{k+\mu} - 3\lambda^{n+1}\beta_{k+\mu}\right] \end{aligned}$$

or

$$\alpha_k + \alpha_{k+\mu} - \lambda^n\beta_k - \lambda^n\beta_{k+\mu} = \sum_{i=1}^{n}(z_{ik} + z_{i\,k+\mu})\lambda^{i-1}.$$

We also have

$$\begin{aligned} & -6(\lambda+1)z_{00} + \sum_{i=1}^{n} -6\lambda^i(\lambda+2)z_{ik} - 6\lambda(\lambda+2)\alpha_k - 6\lambda^{n+1}(\lambda+2)\beta_k = \\ & \quad = -6(\lambda+1)z_{00} + \sum_{i=1}^{n} -6\lambda^i(\lambda+2)z_{i\,k+\mu} - 6\lambda(\lambda+2)\alpha_{k+\mu} - 6\lambda^{n+1}(\lambda+2)\beta_{k+\mu} \end{aligned}$$

or

$$\alpha_k - \alpha_{k+\mu} + \lambda^n\beta_k - \lambda^n\beta_{k+\mu} = \sum_{i=1}^{n}(z_{i\,k+\mu} - z_{ik})\lambda^{i-1}.$$

Therefore radial continuity of the first two derivatives at the origin is met when the equations

$$\begin{aligned} \alpha_k + \alpha_{k+\mu} - \lambda^n\beta_k - \lambda^n\beta_{k+\mu} &= \omega_1 \\ \alpha_k - \alpha_{k+\mu} + \lambda^n\beta_k - \lambda^n\beta_{k+\mu} &= \omega_2 \end{aligned} \quad k=0,1,\ldots,\mu \qquad (8.7)$$

where

$$\begin{aligned} \omega_1 &= \sum_{i=1}^{n}(z_{i\,k+\mu} + z_{ik})\lambda^{i-1} \\ \omega_2 &= \sum_{i=1}^{n}(z_{i\,k+\mu} - z_{ik})\lambda^{i-1} \end{aligned} \qquad (8.8)$$

are satisfied. To get two further equations for $\alpha_k, \alpha_{k+\mu}, \beta_k, \beta_{k+\mu}$ to solve along with (8.7), we must prescribe end conditions for each radius vector $\{(r,\theta)\colon 0 \leq r \leq 1,\ \theta = 2\pi k/m\}$ near the edge of the unit disk. Again there are a number of possibilities:

a) Zero End Conditions

This prescribes that $\beta_k=0$ for all k., which might be viewed as a kind of "natural" end condition. Thus (8.7) reduces to

$$\alpha_k + \alpha_{k+\mu} = \omega_1$$
$$\alpha_k - \alpha_{k+\mu} = \omega_2$$

hence

$$\alpha_k = \tfrac{1}{2}(\omega_1 + \omega_2)$$
$$\alpha_{k+\mu} = \tfrac{1}{2}(\omega_1 - \omega_2) \qquad (8.9)$$

b) Parabolic Run-out

We again want S(u,k) to be parabolic on the interval n-1,n . Thus the second equation of (5.3) is applicable and we obtain

$$a\,\alpha_k + b\,\beta_k = \omega_3$$
$$a\,\alpha_{k+\mu} + b\,\beta_{k+\mu} = \omega_4 \qquad (8.10a)$$

where

$$a = \lambda^{n-1}, \quad b = -1$$

$$\omega_3 = -\lambda^{n-2} z_{00} - \sum_{i=1}^{n-2} \lambda^{n-i-2} z_{ik} + \frac{3\lambda+2}{3\lambda(\lambda+1)} (z_{nk} - z_{n-1,k}) \qquad (8.11a)$$

$$\omega_4 = -\lambda^{n-2} z_{00} - \sum_{i=1}^{n-2} \lambda^{n-i-2} z_{i,k+\mu} + \frac{3\lambda+2}{3\lambda(\lambda+1)} (z_{n,k+\mu} - z_{n-1,k+\mu}).$$

c) Rabinowitz End Conditions

In this case we again require that the discontinuity in the third derivative of S(u,k) with respect to u be removed at u=n-1. Then equation (5.6b) is applicable giving

$$a\,\alpha_k + b\,\beta_k = \omega_3$$
$$a\,\alpha_{k+\mu} + b\,\beta_{k+\mu} = \omega_4 \qquad (8.10b)$$

where

$$a = \lambda^{n-2}\ , \quad b = 1 \quad \text{and}$$

$$\omega_3 = -\sum_{i=0}^{n-3} \lambda^{n-i-3} z_{ik} - \frac{(\lambda^3-3\lambda+1)(z_{n-2\,k}+z_{nk}) + (\lambda^2+2\lambda-2)z_{n-1\,k}}{\lambda(\lambda-1)^2(\lambda+2)} \tag{8.11b}$$

$$\omega_4 = -\sum_{i=0}^{n-3} \lambda^{n-i-3} z_{i\,k+\mu} - \frac{(\lambda^3-3\lambda+1)(z_{n-2\,k+\mu}+z_{n\,k+\mu}) + (\lambda^2+2\lambda-2)z_{n-1\,k+\mu}}{\lambda(\lambda-1)^2(\lambda+2)}$$

Combining equations (8.7) and (8.10a) or (8.10b) we have a system of the form

$$\begin{aligned} x + y - \ell z - \ell w &= \omega_1 \\ x - y + \ell z - \ell w &= \omega_2 \\ ax \quad + bz \quad &= \omega_3 \\ ay \quad + bw &= \omega_4 \end{aligned} \tag{8.12}$$

where $\ell = \lambda^n$, $x = \alpha_k$, $y = \alpha_{k+\mu}$, $z = \beta_k$, $w = \beta_{k+\mu}$. The explicit solution to (8.12), which may be verified by direct substitution, is

$$\begin{aligned} z &= \frac{\frac{1}{2}a^2\ell(\omega_1-\omega_2) - \frac{1}{2}ab(\omega_1+\omega_2) + b\,\omega_3 - a\,\ell\omega_4}{(b^2 - a^2\ell^2)} \\ y &= \tfrac{1}{2}(\omega_1-\omega_2) + \ell z \\ w &= (s - ay)/b \\ x &= \tfrac{1}{2}(\omega_1+\omega_2) + \ell w. \end{aligned} \tag{8.13}$$

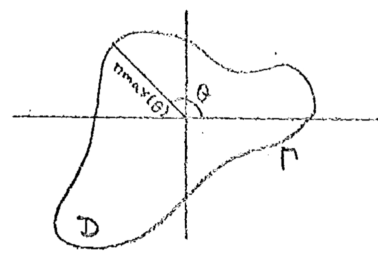
Note finally that

$$\begin{aligned} \tfrac{1}{2}(\omega_1+\omega_2) &= \sum_{i=1}^{n} z_{i\,k+\mu}\,\lambda^{i-1} \\ \tfrac{1}{2}(\omega_1-\omega_2) &= \sum_{i=1}^{n} z_{ik}\,\lambda^{i-1} \end{aligned} \tag{8.14}$$

Therefore the equations for a polar spline on the unit disk are given by (8.4) and (8.11), (8.13) and (8.14), or (8.9) and (8.14) in the case of zero end conditions.

## 9. General Star-Shaped Regions and Surface Singularities

The extension of the preceding analysis for polar splines to more general planar regions is immediate. Suppose, for example, that D is a star-shaped region with respect to the origin bounded by a Jordan curve $\Gamma$. Given $0<\theta<2\pi$ define $rmax(\theta)$ to be the distance from the origin to $\Gamma$ in the $\theta$-direction. Since D is star-shaped, this definition is unambiguous. Then introduce the coordinate change




$$\bar{r} = \frac{r}{rmax(\theta)}, \quad \bar{\theta} = \theta \qquad (9.1)$$

which transforms D into the unit disk, where the previous analysis and formulae apply. Although this will require giving the ordinates along curves "similar" to $\Gamma$, with respect to r, this requirement is not as stringent as it may sound, since the results on the imposition of a uniform mesh apply again, so only approximations to these curves need to be used with minimal loss in accuracy.

A more serious drawback to this method occurs when we inspect the behavior of the derivatives of the resulting surface spline, since these will now involve derivatives of $rmax(\theta)$. In particular, the partial derivative of $S(\bar{r},\bar{\theta})$ with respect to $\theta$ is

$$\frac{\partial}{\partial\theta} S(\bar{r},\bar{\theta}) = \frac{\partial}{\partial\bar{\theta}} S(\bar{r},\bar{\theta}) + \frac{r}{rmax'(\theta)} \frac{\partial}{\partial\bar{r}} S(\bar{r},\bar{\theta}) \qquad (9.2)$$

so a discontinuity in the first derivative of rmax at some $\theta_0$ will result in a discontinuity of the derivative of S with respect to $\theta$ along the ray $\theta=\theta_0$.

Even if $\Gamma$ is smooth, there may still be a singularity of $S(\bar{r},\bar{\theta})$ at the origin of D. We have

$$\frac{\partial}{\partial r} S(\tilde{r},\tilde{\theta}) = \frac{\partial}{\partial \tilde{r}} S(\tilde{r},\tilde{\theta}) \frac{1}{rmax(\theta)}$$

which does not in general equal

$$-\frac{\partial}{\partial r} S(\tilde{r},\tilde{\theta}+\pi)\Big|_{r=0} = -\frac{\partial}{\partial \tilde{r}} S(\tilde{r},\tilde{\theta}+\pi) \frac{1}{rmax(\theta+\pi)}\Big|_{r=0}$$

$$= -\frac{\partial}{\partial \tilde{r}} S(\tilde{r},\tilde{\theta}) \frac{1}{rmax(\theta+\pi)}\Big|_{r=0}$$

unless $rmax(\theta) = rmax(\theta+\pi)$. Several approaches to partially resolve this problem have been suggested, (see [ANW]), but these sacrifice the continuity of other derivatives of S or impose special restrictions on the ordinates, hence are not entirely satisfactory.

## 10. Approximation and Truncation

Many of the formulae for cardinal spline representations can be simplified if a certain degree of approximation is allowed for a saving in computer time and storage. We note that $\lambda \approx -.267949$ and thus $|\lambda| < 1$. Thus terms in the equations involving $\lambda^n$ for $n > n_0$, for some relatively large $n_0$ can be omitted with little loss in accuracy. As an example, the equation for the periodic cardinal spline (5.11) can be simplified by ignoring the last two terms for $n > n_0 + 1$. A practical value of $n_0$ would be around 10, since $|\lambda^{10}| < 2\times10^{-6}$, so the loss in accuracy for most purposes, especially in applications to computer graphics, will be acceptable. In fact the derivation of many of the equations becomes much simpler for large n. For example, the equations in the $\alpha$'s and $\beta$'s will uncouple using this approach; the derivation in [A2] follows this idea. However, since many of the splines used in the applications in this thesis involved meshes of less than 8 points in any one direction, it was deemed necessary for the accuracy desired to derive all the equations in exact form. The reader is referred to [A2] for their approximate derivation.

## II. Discussion of Computer Results

## 11. Plane Curves and Lettering

To investigate the convergence properties of spline approximations to plane curves, the coordinates of sixty-four points on a standard French curve were found and spline approximations based on sixteen, twenty-one, thirty-two and all sixty-four of these points were calculated and drawn by the computer. The results are displayed in figures one through four. The curve based on sixty-four points is an almost exact duplicate of the original French curve, failing only the original curve had a cusp. As can be seen, even the spline curves based on a comparatively small number of points provide a reasonably good fit to the curve, the major regions of disagreement being where the curvature is large. This confirms the remarks in section 6 - where the curvature is large, the mesh size should be small. The lower half of the curve, where hardly any curvature exists, remains fairly constant in each of the four spline fits.

An interesting application of the curve drawing properties of splines is to construct a font of letters based on the spline approximations to the original font. For this purpose, a standard block capital font was used. The coordinates of an average of ten points on the outer contour of each letter were chosen, the small number of points being used so that the spline would add some extra "flourishes" and other interesting features to the letters. A program was written to draw any message of alphabetic characters on the CALCOMP, from which the title page of this thesis was drawn. The small number of points needed to give a recognizable letter makes this method ideally suited to applying the computer to many drawing tasks.

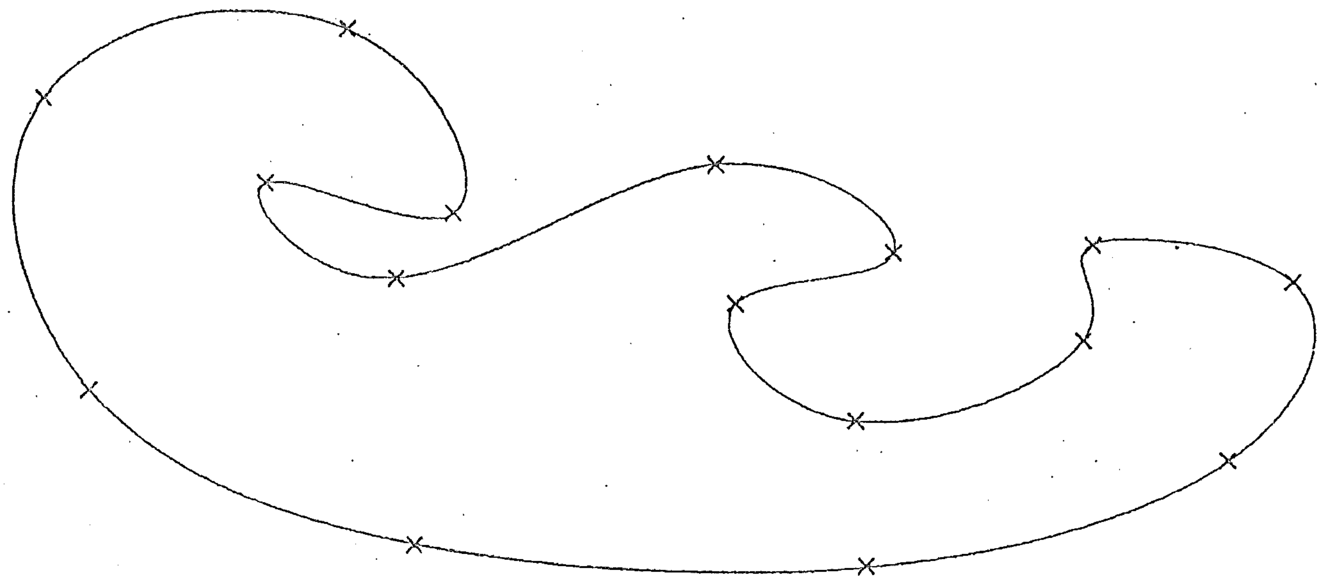


Figure 1: Sixteen point spline approximation to a French curve

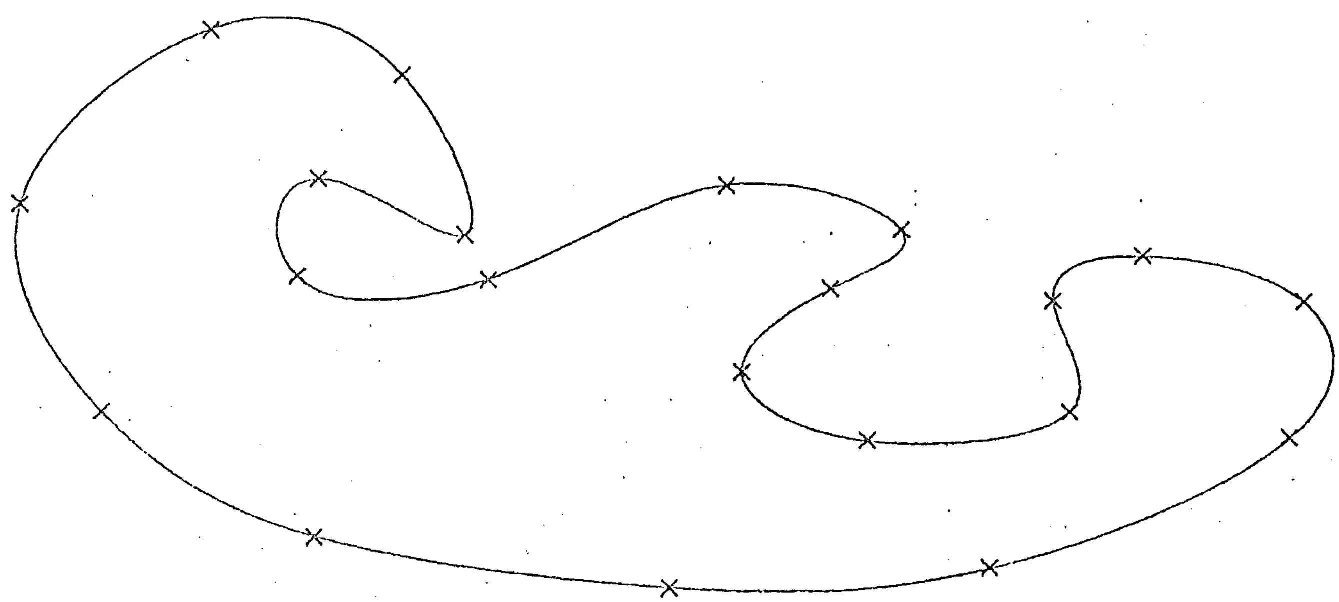


Figure 2: Twenty-one point spline approximation to a French curve

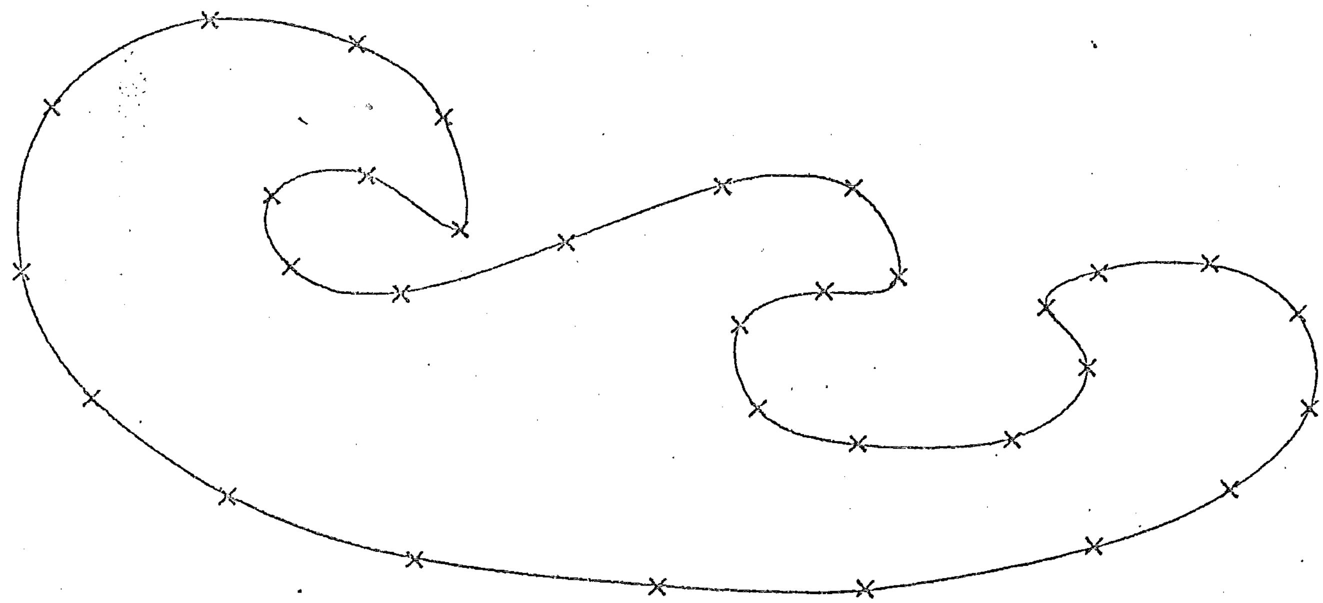


Figure 3: Thirty-two point spline approximation to a French curve.

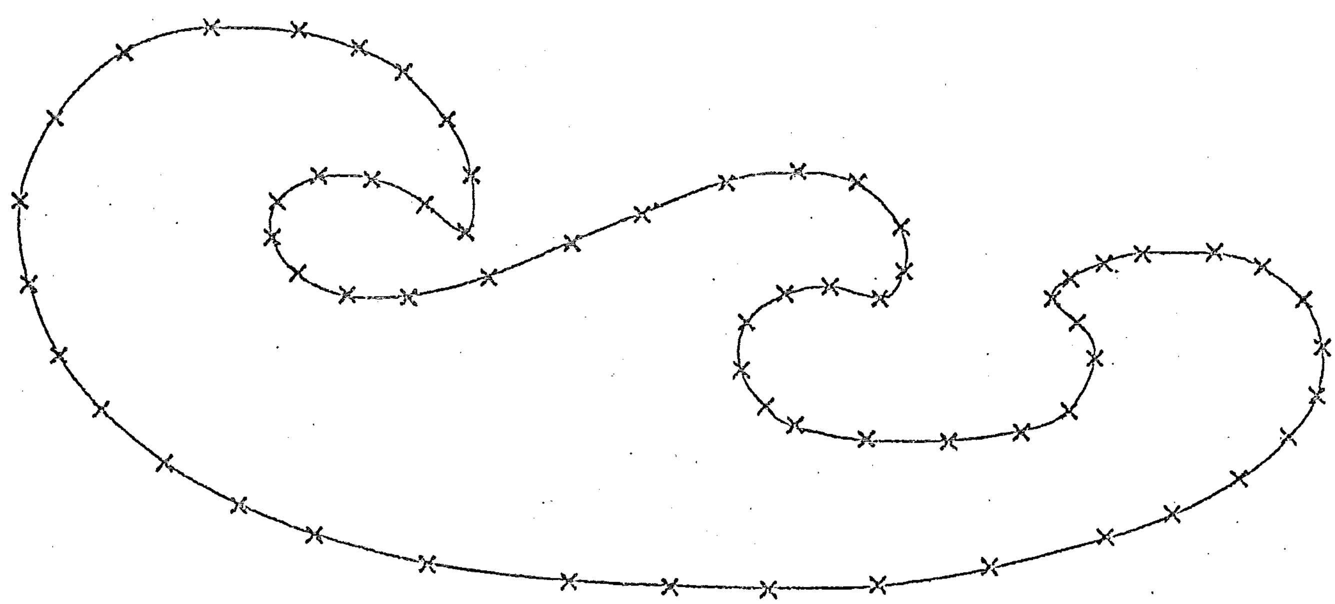


Figure 4: Sixty-four point spline approximation to a French curve.

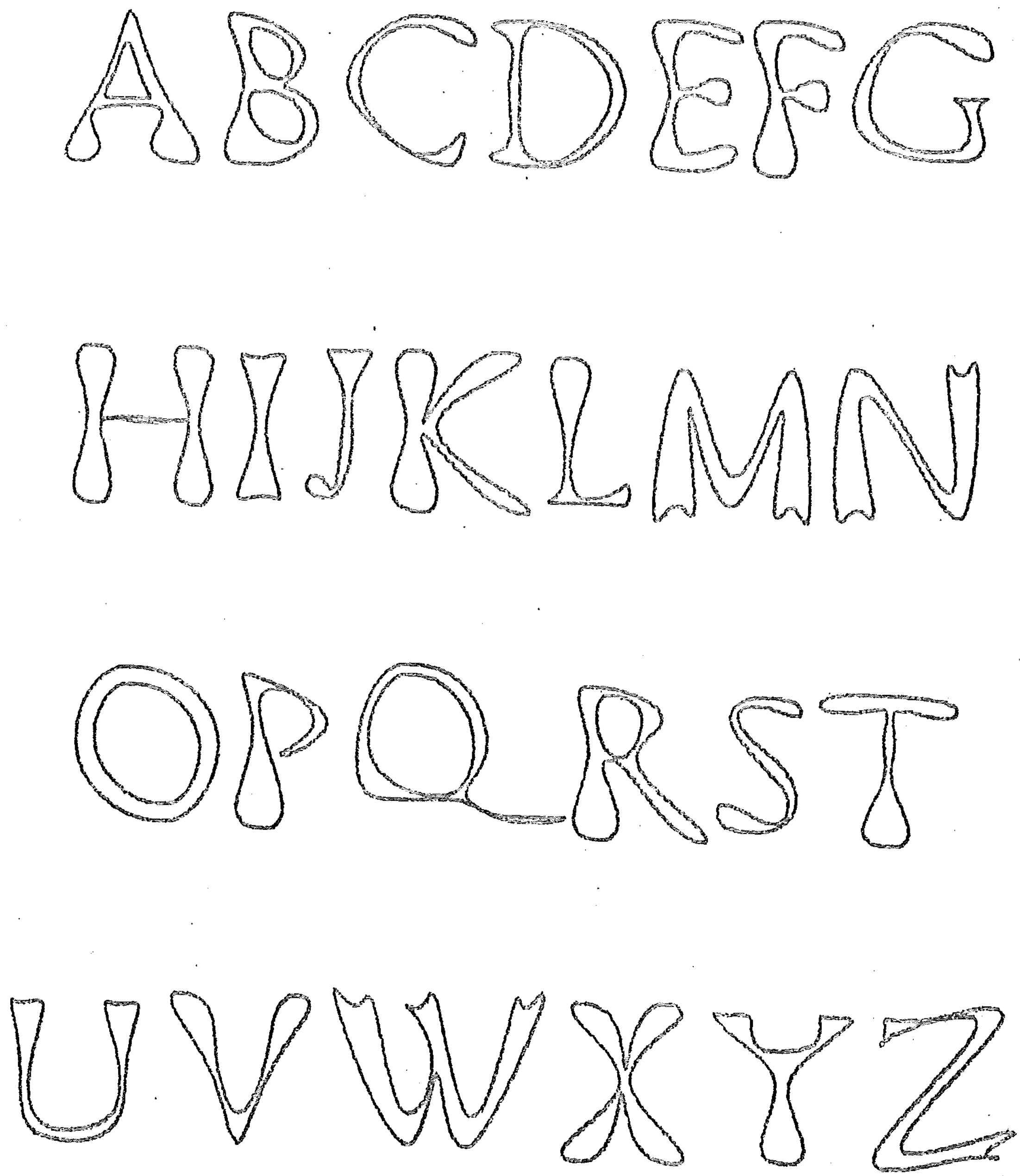



Figure 5: A spline font alphabet.

## 12. Polar Spline Examples

The remaining figures illustrate several aspects of the theory of polar spline on star-shaped regions. They are labelled as to the function $z=f(x,y)$ approximated by splines, the polar mesh size - number of divisions in the r-direction by the number in the $\theta$-direction, the region the spline is computed over - circle means the unit disk and triangle is the equilateral triangle of altitude 3 centered at the origin, the end conditions of the spline and the side length of the square mesh over which the surface is drawn, all square meshes being 15 by 15 points. The functions plotted were the plane $z=x+1.3y$ and the saddle $z=x^2-y^2$, chosen for their simplicity so that the effects of the spline approximation could be easily noticed. The spline surfaces were drawn by the Brown University scope operating programs and photographed directly from the scope.

Figures six through eight show the effects of the different end conditions on the polar spline approximations to the planar function. As can be seen, the parabolic end conditions yield the most true approximation followed by the Rabinowitz and finally the zero end conditions. On the basis of this observation, all further spline functions are to be done with parabolic end conditions. It is surprising how close the spline approximation using parabolic end conditions is, especially considering the small number of mesh points used - 4 by 4. A numerical comparison of the values of the spline and the values of the function over the mesh reveals an overall error of less than 4%, the greatest error being near the corners of the square, i.e. near the edges of the disk. In fact, most mesh points have an error of less than 2%. This also holds for the parabolic spline approximation to $x^2-y^2$

using the same polar mesh, as shown in figure nine.

The next four figures investigate the convergence of the spline approximation to the plane over the triangle as the number of mesh points increases. The crinkles in the surfaces result from equation (9.2) and are along the rays through the corners of the triangle. The crinkles are smoothed out as the number of mesh points increases - the most noticeable smoothing occuring when the number of divisions in the $\theta$-direction is increased. There is hardly any change in the spline surfaces when the number of divisions in the r-direction is increased from figure 11 to figure 12, indicating that an optimum mesh for polar splines would be one that has relatively more divisions in the $\theta$-direction.

The final four figures investigate the singularity of splines at the origin, the splines being computed over a triangular region. Figures 14 and 16 display the the spline surface approximation to $z=x^2-y^2$ over a square of side length 1, while figures 15 and 17 show an enlarged view of 14 and 16 respectively near the origin on a square of side length .1. To emphasize the singularity, the function values were multiplied by a factor of 20 in the latter two figures. A distinct discontinuity in the first derivative can be noticed in these diagrams, whereas it is not so noticeable in the larger surfaces. A smoothing out of the singularity corresponding to an increase in the number of mesh points from figure 15 to figure 17 can be noticed.

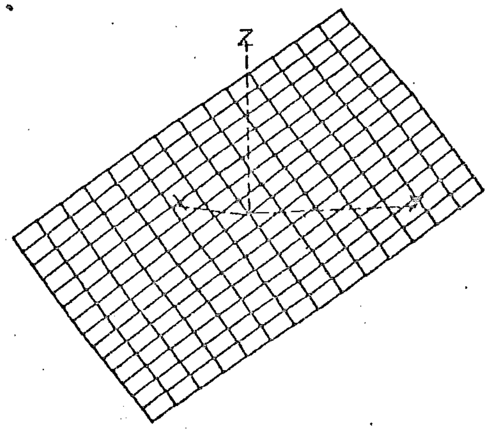



Figure 6: Polar Spline

| Function | Mesh Size | Region | End Conditions | Square Size |
|---|---|---|---|---|
| z=x+1.3y | 4 by 4 | Circle | Parabolic | 1.414 |

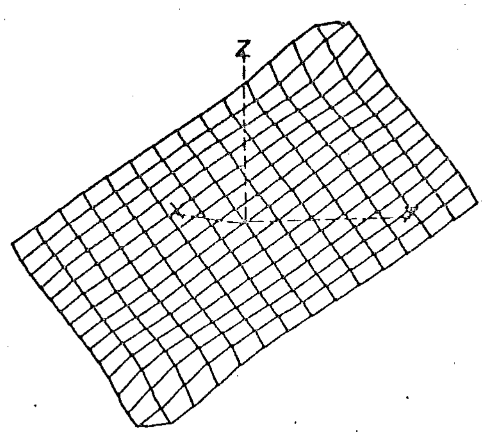



Figure 7: Polar Spline

| Function | Mesh Size | Region | End Conditions | Square Size |
|---|---|---|---|---|
| z=x+1.3y | 4 by 4 | Circle | Rabinowitz | 1.414 |

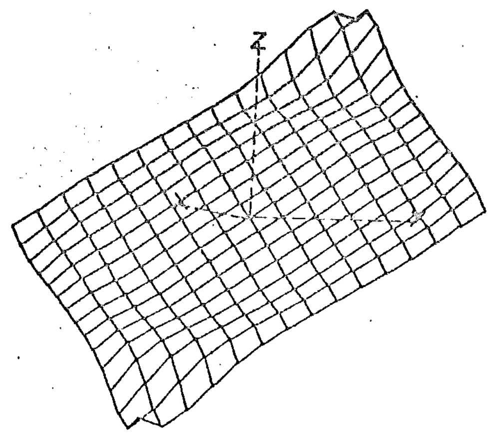



Figure 8: Polar Spline

| Function | Mesh Size | Region | End Conditions | Square Size |
|---|---|---|---|---|
| z=x+1.3y | 4 by 4 | Circle | Zero | 1.414 |

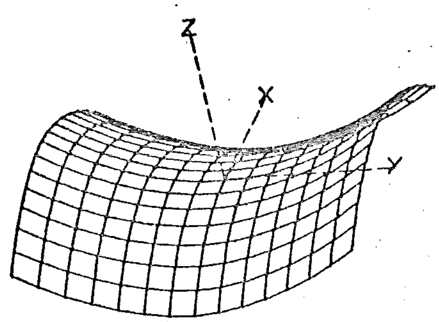



Figure 9: Polar Spline

| Function | Mesh Size | Region | End Conditions | Square Size |
|---|---|---|---|---|
| $z=x^2-y^2$ | 4 by 4 | Circle | Parabolic | 1.414 |

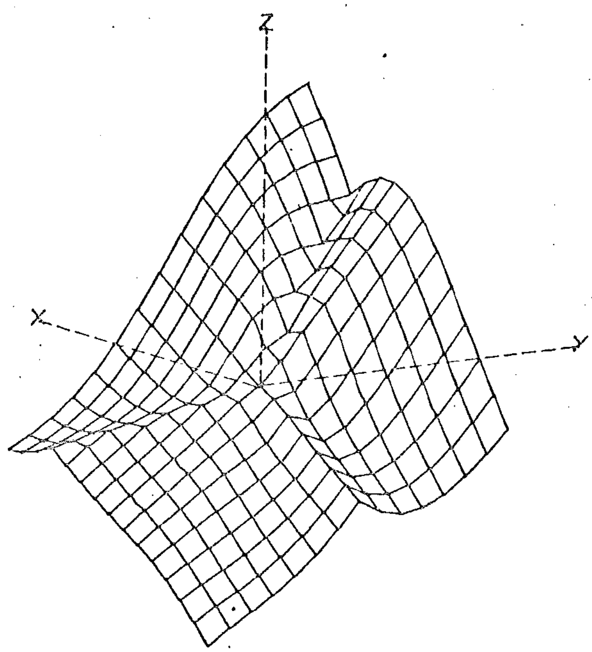



Figure 10: Polar Spline

| Function | Mesh Size | Region | End Conditions | Square Size |
|---|---|---|---|---|
| z=x+1.3y | 8 by 6 | Triangle | Parabolic | 1.0 |

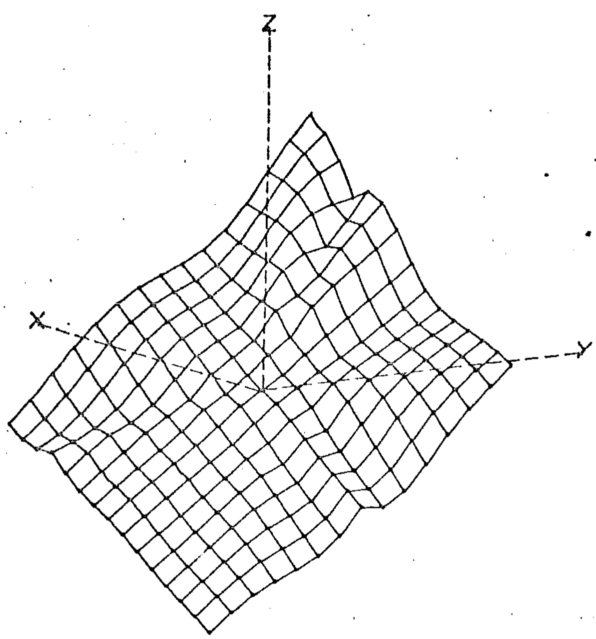



Figure 11: Polar Spline

| Function | Mesh Size | Region | End Conditions | Square Size |
|---|---|---|---|---|
| z=x+1.3y | 8 by 12 | Triangle | Parabolic | 1.0 |

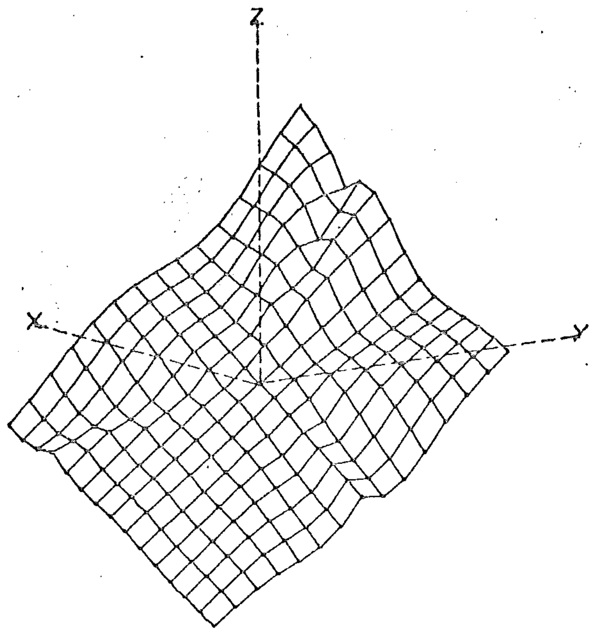



Figure 12: Polar Spline

| Function | Mesh Size | Region | End Conditions | Square Size |
|---|---|---|---|---|
| z=x+1.3y | 12 by 12 | Triangle | Parabolic | 1.0 |

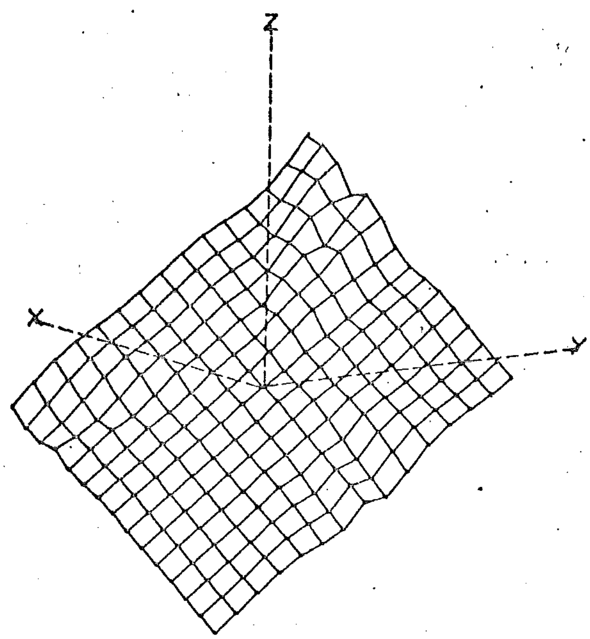



Figure 13: Polar Spline

| Function | Mesh Size | Region | End Conditions | Square Size |
|---|---|---|---|---|
| z=x+1.3y | 12 by 18 | Triangle | Parabolic | 1.0 |

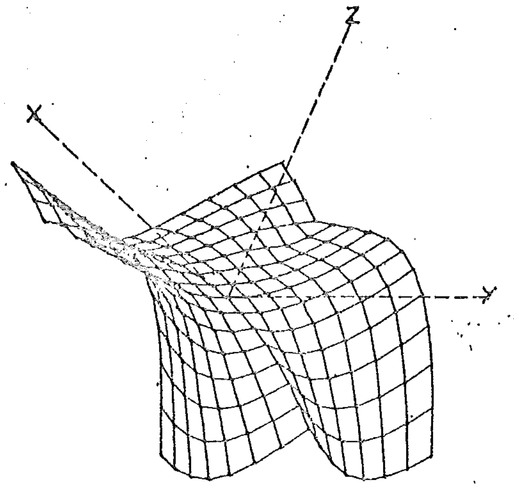



Figure 14: Polar Spline

| Function | Mesh Size | Region | End Conditions | Square Size |
|---|---|---|---|---|
| $z=x^2-y^2$ | 8 by 6 | Triangle | Parabolic | 1.0 |

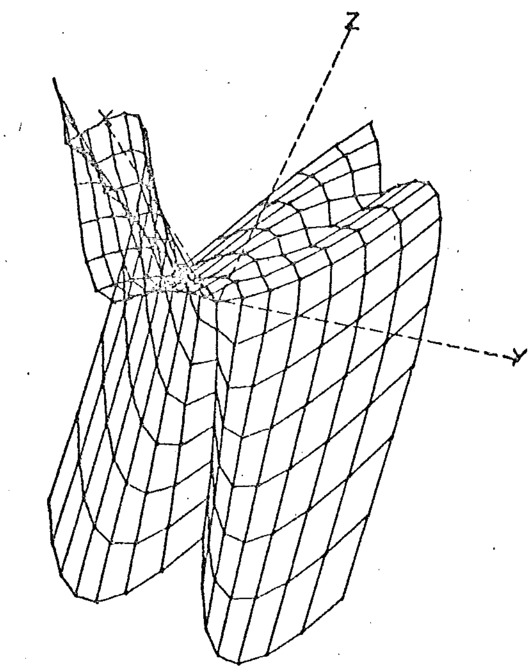



Figure 15: Polar Spline

| Function | Mesh Size | Region | End Conditions | Square Size |
|---|---|---|---|---|
| $z=x^2-y^2$ | 8 by 6 | Triangle | Parabolic | 0.1 |

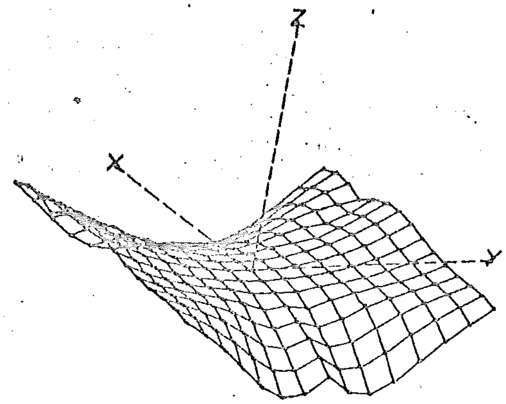



Figure 16: Polar Spline

| Function | Mesh Size | Region | End Conditions | Square Size |
|---|---|---|---|---|
| $z=x^2-y^2$ | 8 by 12 | Triangle | Parabolic | 1.0 |

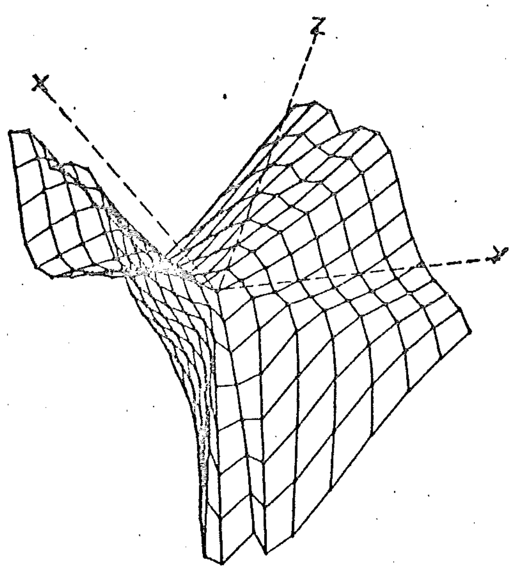



Figure 17: Polar Spline

| Function | Mesh Size | Region | End Conditions | Square Size |
|---|---|---|---|---|
| $z=x^2-y^2$ | 8 by 12 | Triangle | Parabolic | 0.1 |

References

[A1] - Ahlberg, J.H., "The Parametric Representation of Cubic Splines and the Imposition of a Uniform Mesh," Brown University Report, September, 1971.

[A2] - Ahlberg, J.H., "Spline Surfaces," Brown University Report, October, 1972.

[ANW] - Ahlberg, J.H., Nilson, E.N., and Walsh, J.L., <u>The Theory of Splines and Their Applications</u>, Academic Press, New York, 1967.

[AW] - Ahlberg, J.H., "Cardinal Splines and Plane Curves," Brown University Report, August, 1969.

[IK] - Isaacson, Eugene and Keller, Herbert Bishop, <u>Analysis of Numerical Methods</u>, John Wiley and Sons, Inc., New York, 1966.