Quasi-Exact Solvability in the Real Domain

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Abstract. We summarize the classification of finite-dimensional quasi-exactly solvable real Lie algebras of first-order differential operators in the real plane. These results are applied to the construction of new quasi-exactly solvable Schrödinger operators.

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1. Introduction.

A Schrödinger operator H is said to be quasi-exactly solvable, [**11**], [**12**], [**13**], if there exists a finite-dimensional Lie algebra $\mathfrak{g}$ of first-order differential operators, admitting a finite-dimensional module, or representation space, $\mathfrak{N}$ of smooth functions, such that H can be written as a bilinear combination

$$H = \sum_{a,b=1}^{r} C_{ab} T^a T^b + \sum_{a=1}^{r} C_a T^a \tag{1}$$

of the generators T^a of $\mathfrak{g}$. Here $r = \dim \mathfrak{g}$, the C_{ab}'s and C_a's are real constants, and we have omitted an irrelevant constant term that can be absorbed in the energy. The Hamiltonian H thus admits $\mathfrak{N}$ as a finite-dimensional invariant space, $H(\mathfrak{N}) \subset \mathfrak{N}$, and assuming that the functions in $\mathfrak{N}$ are normalizable, the spectrum of a quasi-exactly solvable Schrödinger operator H has an algebraic sector, which can be computed using linear algebra. In the decomposition (1), even though H is required to be a real differential operator, the generators T^a of $\mathfrak{g}$ could conceivably be complex-valued, with complex coefficients C_{ab}, C_a also. In practice, however, if the operators T_a are complex-valued the conditions on the complex constants C_{ab} and C_a arising from the fact that H must be a *real* differential operator — i.e., the potential V must be a real-valued function when the coordinates are real — are virtually impossible to satisfy. Therefore, the primary objects of interest for the construction of real quasi-exactly solvable Schrödinger operators are finite-dimensional real Lie algebras of real-valued first-order differential operators on an open subset M of m-dimensional Euclidean space, which admits a finite-dimensional module of (smooth) *complex-valued* wave functions.

Complete results are known in one dimension. In this case, the real and complex classifications are identical, since, up to equivalence, there is essentially just one family of one-dimensional quasi-exactly solvable Lie algebras of first-order differential operators, indexed by a single quantum number $n \in \mathbb{N}$; the symmetry algebra can be identified with the unimodular Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ (or $\mathfrak{sl}(2, \mathbb{R})$) corresponding to the projective group action, having its standard representation on the space of polynomials of degree at most n. The complete list of one-dimensional quasi-exactly solvable Schrödinger operators was found in [**12**]; further, a complete solution to the normalizability problem for these operators was recently determined, [**5**], [**7**]. The higher-dimensional case is much more challenging, owing notably to the fact that, already in the case of planar vector fields, there are infinitely many distinct finite-dimensional Lie algebras of vector fields, of arbitrarily large dimension. Moreover, some of the complex Lie algebras have several different inequivalent real forms, and so the classification of complex quasi-exactly solvable Lie algebras or differential operators does not fully resolve the corresponding real problem. In the two-dimensional case, a complete list of Lie algebras of first-order differential operators in two complex variables was found by us in [**1**], [**2**]. Our starting point was Lie's complete classification of the finite-dimensional Lie algebras of vector fields in two complex variables, [**9**], [**10**], and then applying methods based on Lie algebra cohomology to determine the associated Lie algebras of differential operators. In [**6**], this classification was applied to construct several new families of normalizable quasi-exactly solvable Schrödinger operators in two dimensions, on both flat and curved spaces. The complete classification of (normalizable) quasi-exactly solvable Schrödinger operators remains to be done, although this appears to be an extremely difficult problem. In this paper, we summarize our recently completed classification of Lie algebras of first-order differential operators in two real variables, [4]. Our starting point will be the classification of Lie algebras of vector fields in $\mathbb{R}^2$ that was rigorously established in [3]. Interestingly, for the five additional real forms not appearing in Lie's complex classification of Lie algebras of vector fields, every associated Lie algebra of differential operators is a subalgebra of a discrete family of algebras isomorphic to $\mathfrak{so}(3,1)$. We will obtain a few interesting new examples of real quasi-exactly solvable Schrödinger operators in two-dimensions.

2. Lie Algebras of Differential Operators.

Let us briefly review the results underlying the classification of Lie algebras of differential operators in the complex domain, referring the reader to [7] for details. Let M be an open subset of m-dimesional complex Euclidean space. Let F(M)denote the space of complex-valued analytic functions on M. Let $\mathcal{V}(M)$ denote the space of analytic vector fields on M, which forms an infinite-dimensional Lie algebra based on the standard Lie bracket operation $[\mathbf{v},\mathbf{w}]$. The Lie algebra of first-order differential operators on M can be identified with the semidirect product of these two Lie algebras, $\mathcal{D}^1(M) = \mathcal{V}(M) \ltimes \mathcal{F}(M)$.

In local coordinates $z = (z^1, \ldots, z^m)$, a first-order differential operator has the form

$$T = \mathbf{v} + f = \sum_{i=1}^{m} \xi^i(z)\,\frac{\partial}{\partial z^i} + f(z), \tag{2}$$

where the coefficients ξ^i and f are analytic functions of z. We let $\pi\colon \mathcal{D}^1(M) \to \mathcal{V}(M)$, with $\pi(\mathbf{v}+f) = \mathbf{v}$, denote the natural projection of a first-order differential operator onto its vector field part.

We are interested in studying Lie subalgebras $\mathfrak{g} \subset \mathcal{D}^1(M)$. Our classification of finite-dimensional subalgebras will be local so that from now on we will usually avoid explicit use of the term "local". The vector field part of the algebra, defined as $\mathfrak{h} = \pi(\mathfrak{g}) \subset \mathcal{V}(M)$, forms a Lie algebra of vector fields on M. Let $\mathfrak{M} = \mathfrak{g} \cap \mathcal{F}(M)$ denote the subspace of $\mathfrak{g}$ consisting of all the multiplication operators in $\mathfrak{g}$. It is immediately clear that $\mathfrak{M}$ must be an $\mathfrak{h}$-module, meaning that if $\mathbf{v} \in \mathfrak{h}$ and $h \in \mathfrak{M}$, then $\mathbf{v}(h) \in \mathfrak{M}$. Consider the quotient space $\mathcal{Q} = \mathcal{F}(M)/\mathfrak{M}$, which also forms an $\mathfrak{h}$-module, and define $F : \mathfrak{h} \to \mathcal{Q}$ by

$$\langle F\,; \mathbf{v}\rangle = [f] \in \mathcal{Q}, \qquad \text{if} \quad \mathbf{v} \in \mathfrak{h} \quad \text{and} \quad \mathbf{v} + f \in \mathfrak{g}. \tag{3}$$

The fact that the differential operators in $\mathfrak{g}$ form a Lie algebra implies that F must satisfy the bilinear identity

$$\mathbf{v}\langle F\,;\mathbf{w}\rangle - \mathbf{w}\langle F\,;\mathbf{v}\rangle - \langle F\,;[\mathbf{v},\mathbf{w}]\rangle = 0, \qquad \mathbf{v},\mathbf{w} \in \mathfrak{h}. \tag{4}$$

In the language of Lie algebra cohomology, [8], the left hand side of (4) defines the differential δF of the linear map F; the fact that it vanishes implies that F defines a 1-cocycle on $\mathfrak{h}$ with values in the $\mathfrak{h}$-module $\mathcal{Q}$. The basic classification theorem, [7], can be stated as follows.

Proposition 1. *Let* $\mathfrak{g} \subset \mathcal{D}^1(M)$ *be a Lie algebra of first-order differential operators. Then* $\mathfrak{g}$ *can be represented by a triple* $(\mathfrak{h}, \mathfrak{M}, F)$, *where:*

(*i*) $\mathfrak{h} = \mathfrak{g} \cap \mathcal{V}(M)$ *is a Lie algebra of vector fields on* M,

(*ii*) $\mathfrak{M} = \mathfrak{g} \cap \mathcal{F}(M)$ *is an* $\mathfrak{h}$*-module of scalar-valued functions,*

(*iii*) $F \in Z^1(\mathfrak{h}, \mathcal{Q})$ *is a* $\mathcal{Q} = \mathcal{F}(M)/\mathfrak{M}$*–valued 1-cocycle on* $\mathfrak{h}$.

There are two classes of equivalence maps that preserve the basic Lie algebra structure of the space $\mathcal{D}^1(M)$. The first are the *changes of variables*, provided by local diffeomorphisms $\varphi: M \to M$, which act naturally on $\mathcal{D}^1(M)$ via

$$\varphi_*(T) = \varphi_*(\mathbf{v} + f) = d\varphi(\mathbf{v}) + f \circ \varphi^{-1}, \tag{5}$$

where $d\varphi$ is the usual differential (push-forward) map on vector fields. The second are the *gauge transformations*, obtained by multiplying the functions in $\mathcal{F}(M)$ by a fixed non-vanishing function[†] $\eta(x) = e^{\sigma(x)}$. The corresponding gauge action on a differential operator is given by

$$\mathcal{G}_\sigma(T) = e^{-\sigma} \cdot T \cdot e^{\sigma} \qquad \text{so that} \qquad \mathcal{G}_\sigma(\mathbf{v} + f) = \mathbf{v} + f + \mathbf{v}(\sigma). \tag{6}$$

Thus, a gauge transformation has the effect of modifying the cocycle F by the coboundary $\delta\sigma = \delta \log \eta$.

Definition 2. Two Lie algebras of differential operators $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$ are *equivalent* if and only if there is an *equivalence map* $\Psi = (\varphi, \mathcal{G}_\sigma)$, consisting of a change of variables and a gauge transformation, that maps one to the other, so $\widetilde{\mathfrak{g}} = \mathcal{G}_\sigma \circ \varphi_*(\mathfrak{g})$.

Theorem 3. *There is a one-to-one correspondence between equivalence classes of Lie algebras* $\mathfrak{g}$ *of first-order differential operators on* M *and equivalence classes of triples* $(\mathfrak{h}, \mathfrak{M}, [F])$, *where*

(*i*) $\mathfrak{h}$ *is a Lie algebra of vector fields,*

(*ii*) $\mathfrak{M} \subset \mathcal{F}(M)$ *is an* $\mathfrak{h}$*-module of functions,*

(*iii*) $[F]$ *is a cohomology class in* $H^1(\mathfrak{h}, \mathcal{Q})$, *where* $\mathcal{Q} = \mathcal{F}(M)/\mathfrak{M}$.

The classification of Lie algebras of first-order differential operators therefore reduces to the problem of classifying triples $(\mathfrak{h}, \mathfrak{M}, [F])$ under local changes of variables. We should note that most of the known results on Lie algebra cohomology, [**8**], are *not* directly applicable since they apply to cohomology classes having values in finite-dimensional modules, whereas in our case the relevant module $\mathcal{Q}$ has finite *co-dimension* in $\mathcal{F}(M)$.

† Here, in contrast to the usual physics version, the gauge factor η is not assumed to be unitary, i.e., σ is not necessarily purely imaginary.

3. Real Forms of Complex Algebras.

We now turn to the main topic of this paper, which is the extension of the known classification results for complex Lie algebras of differential operators to the real domain. Assuming analyticity, there is an important connection between the real and complex objects, provided by the inverse procedures of restriction to the real domain and analytic continuation. Since we are dealing with local issues, we can assume, for simplicity, that our differential operators are defined on open subsets of the appropriate real or complex Euclidean space. First, suppose $\widehat{M} \subset \mathbb{C}^m$ is an open domain in complex m-dimensional Euclidean space, and $\widehat{f}: \widehat{M} \to \mathbb{C}$ a complex-analytic function. Then the restriction of $\widehat{f}$ to[‡] $M = \widehat{M} \cap \mathbb{R}^m$ defines a (in general) complex-valued real-analytic function $f: M \to \mathbb{C}$. We let $\mathcal{R}: \mathcal{F}(\widehat{M}) \to \mathcal{F}(M, \mathbb{C})$ denote this restriction map. Conversely, given a complex-valued analytic function $f \in \mathcal{F}(M, \mathbb{C})$ defined on an open subset $M \subset \mathbb{R}^m$, its analytic continuation to the complex domain $\mathbb{C}^m$ defines a complex-analytic function $\widehat{f}: \widehat{M} \to \mathbb{C}$ defined on a subdomain $\widehat{M} \subset \mathbb{C}^m$ such that $\widehat{M} \cap \mathbb{R}^m = M$. We will always assume that, by suitably restricting the domain $\widehat{M}$, the analytic continuation $\widehat{f}$ is a single-valued function. In our applications, the functions considered are, by and large, combinations of rational and exponential functions, and so the more technical issues associated with the process of analytic continuation do not arise. We let $\mathcal{C}: \mathcal{F}(M, \mathbb{C}) \to \mathcal{F}(\widehat{M})$ denote the process of analytic continuation, so $\mathcal{C}(f) = \widehat{f}$. Note that the restriction and analytic continuation operators are inverses of each other, meaning that $\mathcal{R} \circ \mathcal{C} = \mathbb{1}$ and $\mathcal{C} \circ \mathcal{R} = \mathbb{1}$, provided M and $\widehat{M}$ are suitably related.

The restriction of a subspace $\widehat{\mathfrak{M}} \subset \mathcal{F}(\widehat{M})$ to the real axis will be a subspace $\mathcal{R}(\widehat{\mathfrak{M}})$ of the space $\mathcal{F}(M, \mathbb{C})$ of complex-valued functions. However, in general $\mathcal{R}(\widehat{\mathfrak{M}})$ is not the complexification $\mathfrak{M} \otimes \mathbb{C}$ of a subspace $\mathfrak{M}$ of real-valued functions. This happens if and only if the subspace $\mathcal{R}(\widehat{\mathfrak{M}})$ equals its own complex conjugate.

Similar considerations provide a correspondence between real-analytic complex-valued vector fields and differential operators defined on $M \subset \mathbb{R}^m$ and their complex-analytic counterparts, defined on $\widehat{M} \subset \mathbb{C}^m$.

Proposition 4. *The analytic continuation of a real-valued finite-dimensional Lie algebra of real-analytic differential operators defines a finite-dimensional Lie algebra of complex-analytic differential operators. Conversely, a Lie algebra of complex-analytic differential operators determines a real Lie algebra of differential operators via restriction if and only if its restriction is a complexified Lie algebra.*

Two complex Lie algebras of differential operators are *equivalent* if and only if there is a complex-analytic change of variables and gauge transformation mapping one to the other. Similarly, two real Lie algebras of differential operators are *equivalent* if and only if there is a real-analytic change of variables and gauge transformation mapping one to the other. Clearly, analytically continuing the complexifications of two equivalent real Lie algebras produces (locally) equivalent complex Lie algebras. The converse, though, is false

‡ We always assume that $\widehat{M} \cap \mathbb{R}^m \neq \varnothing$.

in general: two real Lie algebras on M whose complexifications have equivalent analytic continuations are not necessarily equivalent.

In general, by a *real form* of a complex Lie algebra of differential operators $\widetilde{\mathfrak{g}}$, we mean any real Lie algebra of differential operators $\mathfrak{g}$ which is obtained by first applying a complex change of variables and gauge transformation, leading to the complex-equivalent algebra $\widehat{\mathfrak{g}} = \Psi(\widetilde{\mathfrak{g}})$, and then restricting to the real axis. Proposition 4 requires that the resulting complex-valued algebra $\mathfrak{g}_{\mathbb{C}} = \mathcal{R}(\widehat{\mathfrak{g}})$ be a complexified algebra, whereby $\mathfrak{g} = \operatorname{Re} \mathfrak{g}_{\mathbb{C}}$, and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$. Two different real forms of a given complex Lie algebra will always be analytically continuable to complex-equivalent Lie algebras, although the real forms may not be real-equivalent (even if they are isomorphic as abstract Lie algebras). For example, the real Lie algebras generated by $\partial_x + \partial_y$, $x\partial_x + y\partial_y$, $x^2\partial_x + y^2\partial_y$, and by ∂_x, $x\partial_x + y\partial_y$, $(x^2 - y^2)\partial_x + 2xy\partial_y$ are both isomorphic to $\mathfrak{sl}(2,\mathbb{R})$, and are real forms of the complex Lie algebra with generators $\partial_z + \partial_w$, $z\partial_z + w\partial_w$, $z^2\partial_z + w^2\partial_w$. However, the two real forms are not equivalent — there is no real change of variables mapping one to the other; see [**3**]. A key problem, then, is to determine the different possible (real-)inequivalent real forms of a given complex Lie algebra of differential operators.

We have the basic complexification result:

Theorem 5. *Let $\widehat{\mathfrak{h}} \subset \mathcal{V}(\widehat{M})$ be a complex Lie algebra of complex-analytic vector fields, and let $\widehat{\mathfrak{M}} \subset \mathcal{F}(\widehat{M})$ be a complex $\widehat{\mathfrak{h}}$-module of complex-analytic functions. Suppose the restrictions $\mathcal{R}(\widehat{\mathfrak{h}}) \subset \mathcal{V}(M,\mathbb{C})$ and $\mathcal{R}(\widehat{\mathfrak{M}}) \subset \mathcal{F}(M,\mathbb{C})$ are complexified spaces:*

$$\mathcal{R}(\widehat{\mathfrak{h}}) = \mathfrak{h} \otimes \mathbb{C} \equiv \mathfrak{h}_{\mathbb{C}}, \qquad \mathcal{R}(\widehat{\mathfrak{M}}) = \mathfrak{M} \otimes \mathbb{C} \equiv \mathfrak{M}_{\mathbb{C}}.$$

The quotient module $\widehat{\mathcal{Q}} = \mathcal{F}(\widehat{M})/\widehat{\mathfrak{M}}$ restricts to $\mathcal{R}(\widehat{\mathcal{Q}}) = \mathcal{Q}_{\mathbb{C}} = \mathcal{Q} \otimes \mathbb{C} = \mathcal{F}(M,\mathbb{C})/\mathfrak{M}_{\mathbb{C}}$, with real counterpart $\mathcal{Q} = \mathcal{F}(M)/\mathfrak{M}$. Moreover, the restriction $\mathcal{R} \circ H^1(\widehat{\mathfrak{h}}, \widehat{\mathcal{Q}}) \circ \mathcal{C}$ of the associated cohomology space $H^1(\widehat{\mathfrak{h}}, \widehat{\mathcal{Q}})$ is also a complexified space, with

$$\mathcal{R} \circ H^1(\widehat{\mathfrak{h}}, \widehat{\mathcal{Q}}) \circ \mathcal{C} = H^1(\mathfrak{h}_{\mathbb{C}}, \mathcal{Q}_{\mathbb{C}}) = H^1(\mathfrak{h}, \mathcal{Q}) \otimes \mathbb{C} \equiv H^1_{\mathbb{C}}.$$

In particular, Theorem 5 implies that the real and complex cohomology spaces of a complex Lie algebra of vector fields and any of its real forms have the same dimension:

$$\dim_{\mathbb{C}} H^1(\widehat{\mathfrak{h}}, \widehat{\mathcal{Q}}) = \dim_{\mathbb{C}} H^1(\mathfrak{h}_{\mathbb{C}}, \mathcal{Q}_{\mathbb{C}}) = \dim_{\mathbb{R}} H^1(\mathfrak{h}, \mathcal{Q}). \tag{7}$$

Therefore, the space of complex-inequivalent Lie algebras of differential operators corresponding to a given $\widehat{\mathfrak{h}} \subset \mathcal{V}(\widehat{M})$ and $\widehat{\mathfrak{M}} \subset \mathcal{F}(\widehat{M})$ has the same dimension as the space of real-inequivalent Lie algebras of differential operators corresponding to the real forms $\mathfrak{h} \subset \mathcal{V}(M)$ and $\mathfrak{M} \subset \mathcal{F}(M)$. If $\mathcal{B} = \{[F_1], \ldots, [F_n]\}$ forms a basis for the complex cohomology space $H^1(\mathfrak{h}_{\mathbb{C}}, \mathcal{Q}_{\mathbb{C}})$, then a basis for the real form $H^1(\mathfrak{h}, \mathcal{Q})$ can be found among $\operatorname{Re} \mathcal{B} \cup \operatorname{Im} \mathcal{B} = \{[\operatorname{Re} F_1], \ldots, [\operatorname{Re} F_n], [\operatorname{Im} F_1], \ldots [\operatorname{Im} F_n]\}$. In other words, exactly n of the real and imaginary parts of the complex cocycles $F_1, \ldots, F_n$ will be linearly independent modulo coboundaries.

Theorem 5 implies that the problem of classifying real Lie algebras of differential operators can be tackled directly as follows. Let $\mathfrak{h}$ be a real form of a complex Lie algebra of vector fields $\widetilde{\mathfrak{h}}$. Let $\varphi: \widetilde{M} \to \widehat{M}$ be the change of variables mapping $\widetilde{\mathfrak{h}}$ to a complex-equivalent Lie algebra $\widehat{\mathfrak{h}} = \varphi(\widetilde{\mathfrak{h}})$, whose restriction coincides with the complexification of our chosen real form: $\mathfrak{h}_{\mathbb{C}} = \mathcal{R}(\widehat{\mathfrak{h}}) = \mathfrak{h} \otimes \mathbb{C}$. If $\widetilde{\mathfrak{M}} \subset \mathcal{F}(\widetilde{M})$ is a finite-dimensional $\widetilde{\mathfrak{h}}$-module of complex-analytic functions, then $\widehat{\mathfrak{M}} = \varphi_*(\widetilde{\mathfrak{M}})$ is a finite-dimensional $\widehat{\mathfrak{h}}$-module obtained by applying the change of variables. We assume that its restriction $\mathcal{R}(\widehat{\mathfrak{M}})$ is a complexified module: $\mathfrak{M}_{\mathbb{C}} = \mathcal{R}(\widehat{\mathfrak{M}}) = \mathfrak{M} \otimes \mathbb{C}$, with $\mathfrak{M} \subset \mathcal{F}(M)$ a real $\mathfrak{h}$-module. Set $\mathcal{Q}_{\mathbb{C}} = \mathcal{F}(M, \mathbb{C})/\mathfrak{M}_{\mathbb{C}} = \mathcal{Q} \otimes \mathbb{C}$, with $\mathcal{Q} = \mathcal{F}(M)/\mathfrak{M}$. According to Theorem 5, the cohomology space $H^1_{\mathbb{C}} = H^1(\mathfrak{h}_{\mathbb{C}}, \mathcal{Q}_{\mathbb{C}})$ is complexified: $H^1_{\mathbb{C}} = H^1 \otimes \mathbb{C}$, where $H^1 = H^1(\mathfrak{h}, \mathcal{Q})$. Any complex Lie algebra of differential operators with vector field part $\widehat{\mathfrak{h}}$, represented by a cohomology class in the space $H^1(\widehat{\mathfrak{h}}, \widehat{\mathcal{Q}}) \cong H^1_{\mathbb{C}}$, is found by applying φ_* to a Lie algebra of differential operators represented by an element of $\widetilde{H}^1 = H^1(\widetilde{\mathfrak{h}}, \widetilde{\mathcal{Q}})$, where $\widetilde{\mathcal{Q}} = \mathcal{F}(\widetilde{M})/\widetilde{\mathfrak{M}}$. Theorem 5 implies that a real basis for H^1 can be constructed by taking the real and imaginary parts of the elements of a basis for $H^1(\widehat{\mathfrak{h}}, \widehat{\mathcal{Q}})$ and restricting ourselves to the real subspace.

4. Quasi-Exact Solvability.

A finite-dimensional Lie algebra $\mathfrak{g} \subset \mathcal{D}^1(M)$ is called *quasi-exactly solvable* if it admits a non-zero finite-dimensional module (or representation space) $\mathfrak{N} \subset \mathcal{F}(M)$ consisting of functions on M. The condition of quasi-exact solvability has one elementary consequence that simplifies our classification procedure. We state this result for complex Lie algebras, although the real version is identical.

Proposition 6. *If $\mathfrak{g}$ is a quasi-exactly solvable Lie algebra represented by the triple $(\mathfrak{h}, \mathfrak{M}, [F])$ as in Theorem 3, then the module $\mathfrak{M}$ of multiplication operators is either trivial, $\mathfrak{M} = 0$, or consists of constants, $\mathfrak{M} = \mathbb{C}$. Moreover, if $\mathfrak{g} = (\mathfrak{h}, 0, [F])$, then $\mathfrak{g} \subset \widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}$, where $\widetilde{\mathfrak{g}} = (\mathfrak{h}, \mathbb{C}, [F])$ is a quasi-exactly solvable central extension of $\mathfrak{g}$.*

Therefore, we can assume, without loss of generality, that the only multiplication operators in our Lie algebra are the constant functions, i.e., $\mathfrak{g} \cap \mathcal{F}(M) = \mathbb{C}$. The quotient space $\mathcal{Q} = \mathcal{F}(M)/\mathbb{C}$ is also fixed in this case.

A basic fact is that the quasi-exact solvability condition is respected by the process of complexification.

Proposition 7. *Let $\widehat{M} \subset \mathbb{C}^m$ be an open subset, and let $\widehat{\mathfrak{g}} \subset \mathcal{D}^1(\widehat{M})$ be a Lie algebra of complex-analytic first-order differential operators. Let $M = \widehat{M} \cap \mathbb{R}^m$, and assume that the restriction $\mathcal{R}(\widehat{\mathfrak{g}})$ is a complexified Lie algebra of differential operators, so that $\mathcal{R}(\widehat{\mathfrak{g}}) = \mathfrak{g} \otimes \mathbb{C} \equiv \mathfrak{g}_{\mathbb{C}}$ for some real Lie algebra $\mathfrak{g} \subset \mathcal{D}^1(M)$. Then $\widehat{\mathfrak{g}}$ is quasi-exactly solvable if and only if $\mathfrak{g}_{\mathbb{C}}$ is quasi-exactly solvable if and only if $\mathfrak{g}$ is quasi-exactly solvable.*

5. The Planar Case.

In the two-dimensional complex case, the Lie algebras of vector fields were first classified by Lie, [**10**]. In their canonical forms, each of the Lie algebras appearing in the complex classification is a complexified algebra, and hence has an obvious real counterpart, obtained by restricting the coordinates to be real. Moreover, according to Theorem 5, in such cases the associated real Lie algebras of differential operators and finite-dimensional modules are readily obtained by restriction. Interestingly, every imprimitive real Lie algebra is obtained by this simple procedure. In addition to these real Lie algebras, there are precisely five additional primitive real Lie algebras of vector fields in two dimensions that are not equivalent to any of the Lie algebras obtained by straightforward restriction of the complex normal forms. However, the complexification (or analytic continuation) of each of these five additional Lie algebras will, of course, be equivalent, under a complex diffeomorphism, to one of the complex normal forms on our list. The complete list of these additional real forms along with their canonical complexification appear in Table 1. Therefore, to complete the classification of all real Lie algebras of first-order differential operators, we need only determine the real cohomology spaces associated with these five additional real forms, and, further, to determine what values of the cohomology parameters will produce quasi-exactly solvable algebras. Remarkably, only one of the additional real forms admits non-trivial cohomology under the assumption of quasi-exact solvability. We have the following result, [4].

Theorem 8. *Among the five additional real Lie algebras of planar vector fields in* $\mathbb{R}^2$*, the only one admitting a nonzero real-valued quasi-exactly solvable cohomology class is (a central extension of)* $\mathfrak{so}(3,1)$*, for which the corresponding Lie algebra of first-order differential operators is spanned by*

$$\begin{gathered} T^0 = 1, \qquad T^1 = \partial_x, \qquad T^2 = \partial_y, \qquad T^3 = x\partial_x + y\partial_y, \qquad T^4 = y\partial_x - x\partial_y, \\ T^5 = (x^2 - y^2)\partial_x + 2xy\partial_y - 2nx, \qquad T^6 = 2xy\partial_x + (y^2 - x^2)\partial_y - 2ny, \end{gathered} \tag{8}$$

where n *is a non-negative integer. Every finite-dimensional Lie algebra of first order differential operators with vector field part given by one of the five additional primitive real forms listed in Table 4 is a subalgebra of one of the Lie algebras given by* (8).

6. New Quasi-Exactly Solvable $\mathfrak{so}(3,1)$ Potentials.

We now use the results of the preceding section to construct new examples of quasi-exactly solvable Schrödinger operators. By a Schrödinger operator we mean of course a second order differential operator of the form

$$H = -\Delta + V(x); \tag{9}$$

here

$$\Delta = \sum_{i,j=1}^{m} g^{ij}(x^1, \ldots, x^m) \nabla_i \nabla_j,$$

is the Laplacian (kinetic energy) operator on the finite-dimensional real Riemannian manifold M with contravariant metric (g^{ij}), and ∇_i is the covariant derivative associated to this metric. Let $\mathfrak{g}$ be one of our real quasi-exactly solvable Lie algebras of first order differential operators written in canonical form. The most general (real) second-order differential operator L which is quasi-exactly solvable with hidden symmetry algebra $\mathfrak{g}$ takes the form

$$L = \sum_{a,b=1}^{r} C_{ab} T^a T^b + \sum_{a=1}^{r} C_a T^a, \tag{10}$$

where the T^a are the generators of $\mathfrak{g}$, and C_{ab}, C_a are real constants. Now, in general L need not be a Schrödinger operator (9); however, L can sometimes be transformed into a Schrödinger operator by applying a suitable change of variables φ and gauge transformation $\mathcal{G}_\sigma$. The transformed operator $L = e^{-\sigma} \circ \varphi_*(L) \circ e^{\sigma}$ is then a quasi-exactly solvable operator with respect to the transformed algebra $\mathcal{G}_\sigma \circ \varphi_*(\mathfrak{g}) \cong \mathfrak{g}$, with generators $\widetilde{T}^a = \mathcal{G}_\sigma \circ \varphi_*(T^a)$.

We recall now the necessary and sufficient conditions under which the operator (10) can be transformed into a Schrödinger operator H, cf. for example [**6**]. First of all, we rewrite L in the form

$$L = -\sum_{i,j=1}^{m} g^{ij}(x) \frac{\partial^2}{\partial x^i \, \partial x^j} + \sum_{i=1}^{m} b^i(x) \frac{\partial}{\partial x^i} + c(x).$$

We must first require that L be *elliptic*, meaning that the quadratic form associated to the symmetric matrix $(g^{ij}(x))$ be positive definite everywhere. We may thus interpret the functions $g^{ij}(x)$ as the contravariant components of a Riemannian metric

$$ds^2 = \sum_{i,j=1}^{m} g_{ij}(x) \, dx^i \, dx^j. \tag{11}$$

It is convenient to express L in covariant form as follows:

$$L = -\sum_{i,j=1}^{m} g^{ij} (\nabla_i - A_i)(\nabla_j - A_j) + V,$$

where ∇_i is the covariant derivative associated to the metric (11), and

$$A_j = \sum_{j=1}^{m} g_{ij} \left[\frac{b^i}{2} + \frac{1}{2\sqrt{g}} \sum_{k=1}^{m} \frac{\partial}{\partial x^k} (\sqrt{g}\, g^{ik}) \right], \qquad A^i = \sum_{j=1}^{m} g^{ij} A_j,$$

$$V = c + \sum_{i=1}^{m} \left[A_i A^i - \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g} A^i) \right],$$

with $g = \det(g_{ij})$. We define the *magnetic one-form* associated with such an operator to be

$$\omega = \sum_{i=1}^{m} A_i \, dx^i.$$

Theorem 9. *The necessary and sufficient condition for an elliptic second order differential operator L to be equivalent to a Schrödinger operator is that its magnetic one-form be closed:*

$$d\omega = 0. \tag{12}$$

For a given Lie algebra of differential operators $\mathfrak{g}$, equation (12) is equivalent to a set of algebraic equations in the coefficients C_{ab} and C_a, which are called the *closure conditions*. In the complex case, these conditions were extensively analyzed in [**6**], although their complete solution, and hence the complete classification of quasi-exactly solvable Schrödinger operators, remains problematic.

We now proceed to construct a few new examples of quasi-exactly solvable Hamiltonians. According to Theorem 8, there is no loss of generality in working exclusively with $\mathfrak{so}(3,1)$. Interestingly, we are not aware of any quasi-exactly solvable potential that has been *explicitly* linked to $\mathfrak{so}(3,1)$ in the literature. The only example of quasi-exactly solvable $\mathfrak{so}(3,1)$ potential that we know of is the remarkable multiparameter family recently constructed by Zaslavskii, [**14**; eq. (31–33)]. Although the Hamiltonians in this family were obtained without explicitly using Lie algebraic techniques, it can be shown (cf. [**15**]) that they arise from Hamiltonians that are quasi-exactly solvable with respect to the Lie algebra $\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)$. In fact, these Hamiltonians could have been obtained much more directly by starting with the most general Lie-algebraic second-order differential operator by taking an arbitrary polynomial of degree two in the generators of $\mathfrak{so}(3,1)$, and imposing that *i*) the coordinates for the induced metric be isothermal, and *ii*) the closure conditions be satisfied. It can be shown that the most general Hamiltonian satisfying these two conditions depends on 15 real parameters satisfying 9 algebraic constraints. Thus, the set of all such Hamiltonians is parametrized by a 6-dimensional algebraic variety. Although the number of essential parameters in Zavslaskii's multiparameter family is 6, we shall now show that the latter family is only one of several components of the variety. Indeed, we now present a different 6-parameter family of Hamiltonians satisfying the above two conditions.

Indeed, consider the family of second-order differential operators (10) defined by the following choice of the coefficients C_{ab} and C_a :

$$(C_{ab}) = \begin{pmatrix} \alpha & 0 & 0 & 0 & \mu/2 & -\beta \\ 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & \gamma & \lambda \\ 0 & 0 & \beta & \mu & -\lambda & \gamma \\ \mu/2 & 0 & \gamma & -\lambda & \nu & 0 \\ -\beta & 0 & \lambda & \gamma & 0 & \nu \end{pmatrix}, \qquad (C_a) = -2\,n(0,0,0,\beta,\gamma,\lambda),$$

where $T^1,\dots,T^6$ are the $\mathfrak{so}(3,1)$ generators given by (8). (We omit $T^0=1$ without loss of generality.) A long but straightforward calculation shows that the closure conditions are satisfied. The associated metric is

$$ds^2 = A^{-1}\,(dx^2+dy^2), \tag{13}$$

with

$$A = \alpha + \mu\,x^2 - 2\,\beta\,x\,y + 2\,\gamma\,x\,(x^2+y^2) + 2\,\lambda\,y\,(x^2+y^2) + \nu\,(x^2+y^2)^2,$$

so that the (x,y) coordinates are isothermal. The Gaussian curvature is given by

$$\kappa = A^{-1}\left[a+b\,x+c\,y+d\,(x^2+y^2)+e\,xy+f\,x^2+g\,x\,(x^2+y^2)+h\,y\,(x^2+y^2)+k\,(x^2+y^2)^2\right],$$

with

$$\begin{aligned} a &= \alpha\,\mu, & b &= 8\,\alpha\,\gamma, & c &= 8\,\alpha\,\lambda, \\ d &= 2\,(4\,\alpha\,\nu - \beta^2), & e &= 2\,\beta\,\mu, & f &= -\mu^2, \\ g &= 2\,(2\,\beta\,\lambda - \gamma\,\mu), & h &= 2\,(2\,\beta\,\gamma + \lambda\,\mu), & k &= \mu\,\nu - 2\,\gamma^2 - 2\,\lambda^2. \end{aligned}$$

Since the closure conditions are satisfied, we know that L is equivalent under a gauge transformation to a Schrödinger operator (9) on the open subset of $\mathbb{R}^2$ where A is positive, with metric given by (13). In fact, we have

$$H = \eta\,T\,\eta^{-1} = -\Delta + V(x,y), \qquad \text{where} \qquad \eta = A^{-n/2},$$

and the potential is

$$\begin{aligned} V = \frac{n\,(n+2)}{A}\Big[&-4\,\alpha\,(\gamma\,x+\lambda\,y) + x^2\,\mu^2 - 2\,\beta\,\mu\,x\,y + (\beta^2 - 4\,\alpha\,\nu)\,(x^2+y^2) \\ &+ 2\,x\,(\gamma\,\mu - \beta\,\lambda)\,(x^2+y^2) - 2\,\beta\,\gamma\,y\,(x^2+y^2) + (\gamma^2+\lambda^2)\,(x^2+y^2)^2\Big]. \end{aligned}$$

It should be noted that for generic values of the parameters neither the above solution of the closure conditions (nor the one given by Zaslavskii), satisfy the additional condition that the associated metric be positive definite in all of $\mathbb{R}^2$.

One can obtain many other multiparameter families of $\mathfrak{so}(3,1)$ potentials, for instance by dropping the condition that the (x,y) coordinates be isothermal for the metric. We shall content ourselves with the following example, in which

$$(C_{ab}) = \operatorname{diag}(\alpha,\alpha,\beta,\gamma,\lambda,\lambda), \qquad (C_a) = 0.$$

Again, the closure conditions are satisfied by the above choice of coefficients. The associated contravariant metric tensor (g^{ij}) is given by

$$\begin{aligned} g^{11} &= \alpha + \beta\,x^2 + \gamma\,y^2 + \lambda\,(x^2+y^2)^2, \\ g^{12} &= (\beta-\gamma)\,x\,y, \\ g^{22} &= \alpha + \gamma\,x^2 + \beta\,y^2 + \lambda\,(x^2+y^2)^2. \end{aligned}$$

The Gaussian curvature is

$$\kappa = \frac{(-\beta+3\gamma)(\alpha^2+\lambda^2r^8) + 2(\beta\gamma+4\alpha\lambda)r^2(\alpha+\lambda r^4) + 2\alpha\lambda(5\beta+\gamma)r^4}{(\alpha+\gamma\,r^2+\lambda\,r^4)^2}$$

with $r^2 = x^2+y^2$. In contrast with the previous case, if the parameters α, β, γ and λ are positive, then the metric is positive definite for (x,y) ranging over all of $\mathbb{R}^2$.

As before, the fact that the closure conditions are satisfied guarantees the existence of a gauge factor η such that $H = \eta\, L\, \eta^{-1}$ is a Schrödinger operator. If

$$\rho = 4\,\alpha\,\lambda - \beta^2,$$

the gauge factor is given by

$$\eta = \begin{cases} \exp\left(\dfrac{n\beta}{\sqrt{\rho}} \arctan \dfrac{\beta + 2\,\lambda\, r^2}{\sqrt{\rho}}\right) \left(\alpha + \beta\, r^2 + \lambda\, r^4\right)^{-\frac{1}{4}-\frac{n}{2}} \left(\alpha + \gamma\, r^2 + \lambda\, r^4\right)^{\frac{1}{4}}, & \rho > 0; \\ \exp\left(-\dfrac{2\,n\,\alpha}{2\,\alpha + \beta\, r^2}\right) \left(2\,\alpha + \beta\, r^2\right)^{-\frac{1}{2}-n} \left(4\,\alpha^2 + 4\,\alpha\,\gamma\, r^2 + \beta^2\, r^4\right)^{\frac{1}{4}}, & \rho = 0; \\ \left(\alpha + \beta\, r^2 + \lambda\, r^4\right)^{-\frac{3}{4}-n} \left(\alpha + \gamma\, r^2 + \lambda\, r^4\right)^{\frac{1}{4}} \left(\dfrac{2\,\lambda\, r^2 + \beta - \sqrt{-\rho}}{2\,\lambda\, r^2 + \beta + \sqrt{-\rho}}\right)^{\frac{n\,\beta}{\sqrt{-\rho}}}, & \rho < 0. \end{cases}$$

In all cases, the expression for the potential V is

$$\begin{aligned} 4V = {} & \frac{16\,\alpha\,\beta\, n\,(1+n) + r^2\ \left[\beta^2\,(3 + 16\,n + 16\,n^2) - 4\,\alpha\,\lambda\,(3 + 8\,n + 4\,n^2)\right]}{\alpha + \beta\, r^2 + \lambda\, r^4} \\ & + \frac{5\,(\beta-\gamma)\,(4\,\alpha\,\lambda - \gamma^2) + 3\,\lambda\,(2\,\beta\,\gamma - 3\,\gamma^2 + 4\,\alpha\,\lambda)\, r^2}{\lambda\,(\alpha + \gamma\, r^2 + \lambda\, r^4)} \\ & - \frac{5\,(\beta-\gamma)\,(4\,\alpha\,\lambda - \gamma^2)\,(\alpha + \gamma\, r^2)}{\lambda\,(\alpha + \gamma\, r^2 + \lambda\, r^4)^2} + 4\gamma - 4\beta\,(1 + 2\,n)^2, \end{aligned}$$

with $r^2 = x^2 + y^2$. Since the potential is a function of r only, it is natural to look for eigenfunctions of H which depend on r only. When this is done, it can be shown that one ends up with an effective Hamiltonian on the line which is an element of the enveloping algebra of the standard realization of $\mathfrak{sl}(2,\mathbb{R})$ in one dimension. Thus, no new quasi-exactly solvable one-dimensional potentials are obtained by reduction of the above quasi-exactly solvable $\mathfrak{so}(3,1)$ potential. This lends additional support to the observation of [**6**] that reduction of two-dimensional quasi-exactly solvable Schrödinger operators does not lead to any new one-dimensional quasi-exactly solvable operators. However, a full explanation of this fact remains obscure.

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Table 1

Additional Primitive Lie Algebras of Vector Fields in $\mathbb{R}^2$

	Generators	Dim	Structure
1.	$\partial_x, x\partial_x + y\partial_y, (x^2 - y^2)\partial_x + 2xy\partial_y$	3	$\mathfrak{sl}(2)$
2.	$y\partial_x - x\partial_y, (1 + x^2 - y^2)\partial_x + 2xy\partial_y,$ $2xy\partial_x + (1 - x^2 + y^2)\partial_y$	3	$\mathfrak{so}(3)$
3.	$\partial_x, \partial_y, \beta(x\partial_x + y\partial_y) + y\partial_x - x\partial_y$	3	$\mathbb{R} \ltimes \mathbb{R}^2$
4.	$\partial_x, \partial_y, x\partial_x + y\partial_y, y\partial_x - x\partial_y$	4	$\mathbb{R}^2 \ltimes \mathbb{R}^2$
5.	$\partial_x, \partial_y, x\partial_x + y\partial_y, y\partial_x - x\partial_y,$ $(x^2 - y^2)\partial_x + 2xy\partial_y, 2xy\partial_x + (y^2 - x^2)\partial_y$	6	$\mathfrak{so}(3,1)$