Classification of Invariant Wave Equations

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Abstract. In this paper we characterize the possible symmetry groups of wave equations and certain evolutionary generalizations, in a single time variable and one or more spatial variables. Furthermore, we describe a complete classification of two-dimensional wave equations $u_{tt} = F[u]$ and potential evolutionary equations $u_{xt} = F[u]$ having a point or contact symmetry group. The results rely on Lie's classification of planar transformation groups and their relative differential invariants.

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1. Introduction.

One of the most basic constructions of modern physics is the formulation of field equations (or variational principles) admitting a known symmetry group. It has been known since the days of Sophus Lie that this can be readily done, in the regular case, by assembling suitable combinations of differential invariants of the transformation group. Although Lie's general theorem would appear to completely resolve the issue of classifying differential equations admitting prescribed symmetry groups, a more subtle question has recently been of importance, and cannot be quite so immediately answered. The problem is to classify invariant differential equations of a specified form admitting a prescribed symmetry group. For example, the classification of geometric diffusion equations admitting symmetry groups of visual significance is a problem of importance in computer vision and image processing. In [**12**], [**13**], a complete classification for subgroups of the projective group was determined. More generally, one can ask for a complete list of invariant evolution equations admitting a prescribed symmetry group, and the latter problem was completely solved in [**14**], using the theory of relative invariants. It was found that any transformation group in the field and spatial variables (but fixing the time variable) always admits an infinite collection of invariant evolution equations; see Theorem 4.11 below and also the work of Sokolov, [**17**]. The reason why this problem is not an immediate consequence of the classification of differential invariants for the transformation group in question is that it may not be so evident which combinations of differential invariants, *if any*, can be used to produce the equation having the specified form. In the case of evolution equations, the fact that the time variable introduces an additional coordinate into the picture implies that one needs to compute a new basis of fundamental differential invariants, even when the purely spatial derivative invariants are known.

In this paper, we shall consider the classification of wave equations in both one and several space variables, and a single time variable admitting a prescribed finite-dimensional symmetry group. This problem is of interest in computer vision and other applications, first since one might desire to use hyperbolic, rather than parabolic, processing on an image. A second reason arises in image enhancement, in which one uses a hyperbolic regularization to effect a backwards (and hence ill-posed) parabolic equation, cf. [**16**]. (See also [**6**] for equations of Euler–Poisson–Darboux type.) Since the image smoothing was done in a a group-invariant manner, one might reasonably ask for similarly invariant hyperbolic enhancers. Surprisingly, the above-mentioned result for evolution equations is no longer valid — *not* every spatial transformation group admits an invariant hyperbolic wave equation. We determine a complete set of conditions that a transformation group admit an invariant evolutionary or wave equation. Further, using the differential invariants for the groups, completely characterize all possible invariant equations admitted by a symmetry group of the prescribed type. In the planar case (one independent spatial variable and one dependent variable), we then use Lie's complete classification of groups of point and contact transformations in the plane to find a complete list of invariant wave equations.

We shall assume that the reader is familiar with the basic theory of symmetry groups of differential equations, as presented in [**9**], [**10**]. We shall make extensive use of the theory of differential invariants, as presented in the latter book, as well as [**11**], [**15**]. Since we are relying on Lie's classification of finite-dimensional transformation groups acting on a two-dimensional complex manifold, cf. [**8**], [**10**], we shall assume that the variables are, in general, complex-valued. In the case of point transformation groups, the extension of these results to real differential equations is not difficult. Unfortunately, there is, as far as we know, no complete classification of real groups of contact transformations acting in two dimensions. The present paper can be viewed as a start towards the classification of differential invariants for surfaces under transformation groups in three-dimensional space, where the group acts completely trivially on the time coordinate. In [**7**] Lie describes a partial classification of three-dimensional transformation groups, and claims that he has completed it but these results never appeared in print. An important task awaiting completion is the complete classification of the differential invariants of Lie's three-dimensional transformation groups.

2. Jet spaces and Prolongations.

Before proceeding to a detailed discussion of our results, we need to first review the theory of differential invariants and, more generally, relative differential invariants. Since all our considerations are local, we will not lose any generality by working in Euclidean space. We will consider the total space $E \simeq X \times U$, where, in the cases considered in this paper, $U \simeq \mathbb{R}$ has coordinate u, the scalar dependent variable, whereas $X \simeq \mathbb{R}^p$, has coordinates $x = (x^1, \ldots, x^p)$, representing the spatial independent variables. The n^{th} jet space $\mathrm{J}^n E$ thus has coordinates $(x, u^{(n)})$, where $u^{(n)}$ stands for all partial derivatives

$$u_K = \frac{\partial^l u}{(\partial x^1)^{k_1} \cdots (\partial x^p)^{k_p}}, \qquad \text{where} \qquad \begin{array}{l} K = (k_1, \ldots, k_p), \\ l = \#K = k_1 + \cdots + k_p \leq n. \end{array} \tag{2.1}$$

We will use the basic multi-indices e_i, $i = 0, \ldots, p$, which has a single 1 in the i^{th} position and zeros elsewhere. Thus $u_i = u_{e_i} = \partial u / \partial x^i$. Moreover, we write $L \subseteq K$ if all entries of L are less than or equal to those of K, so $0 \leq l_j \leq k_j$, $j = 0, \ldots, p$. Similarly, we write $L \subsetneq K$ if $L \subseteq K$, but $L \neq K$. Note that the difference $K - L$ is a well-defined multi-index if and only if $L \subseteq K$. Finally, we write $\binom{K}{L}$ for the standard multi-nomial coefficient, which is non-zero provided $L \subseteq K$. A function $F(x, u^{(n)})$ depending on independent and dependent variables and their derivatives is known as a *differential function*, and $n = \operatorname{ord} F$ is its order, which means that F really does depend on n^{th} order derivatives of u.

We shall consider both point transformation groups, which are local transformation groups $G = G^{(0)}$ acting on the space $E = X \times U$, and contact transformation groups, which, by Bäcklund's Theorem, [**1**], [**10**], are at most first order, and thus realized as a local transformation group $G^{(1)}$ on the first jet space $\mathrm{J}^1 E$ preserving the contact ideal generated by the contact form $\theta = du - \sum_{i=1}^{p} u_i \, dx^i$. In both cases, the group induces a corresponding transformation group $G^{(n)}$ on the n^{th} jet space $\mathrm{J}^n E$, called the n^{th} prolongation of G, which can be uniquely characterized as the n^{th} order contact transformation group projecting back to the original group action.

We let $\mathfrak{g}$ denote the Lie algebra of G. Consider an infinitesimal generator of the group action

$$\mathbf{v}^{(0)} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \varphi \frac{\partial}{\partial u}, \tag{2.2}$$

corresponding to the Lie algebra element $\mathbf{v} \in \mathfrak{g}$. In the case of point transformations, the coefficients ξ^i, φ of $\mathbf{v}^{(0)}$ depend on x, u. The group is fiber-preserving (or projectable, [**9**]) if $\xi^i = \xi^i(x)$ only depend on the independent variables. Finally, the group consists of *affine bundle maps* if it consists of transformations $(x, u) \mapsto (\Phi(x), A(x)u + B(x))$ which are fiber-preserving and affine in the dependent variable u at each point; the infinitesimal generators have $\varphi = \alpha(x)u + \beta(x)$. For instance, most linear partial differential equations have affine bundle symmetry groups. The infinitesimal generators of a contact transformation group have the same form (2.2), but the coefficients ξ^i, φ, are allowed to depend on the first order derivatives of u provided they satisfy the *contact conditions*

$$\frac{\partial \varphi}{\partial u_k} = \sum_{i=1}^{p} u_i \frac{\partial \xi^i}{\partial u_k}, \qquad k = 1, \ldots, p. \tag{2.3}$$

In all cases, the *characteristic* of the vector field (2.2) is defined to be the first order differential function

$$Q(x, u^{(1)}) = \varphi - \sum_{i=1}^{p} \xi^i u_i. \tag{2.4}$$

The vector field can be recovered from its characteristic by solving (2.4) for φ and using

$$\xi^i(x, u^{(1)}) = -\frac{\partial Q(x, u^{(1)})}{\partial u_i}, \tag{2.5}$$

which is a consequence of the contact conditions (2.3), to construct the coefficients ξ^i. The corresponding infinitesimal generator

$$\mathbf{v}^{(n)} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\#K \leq n} \varphi^K \frac{\partial}{\partial u_K}, \tag{2.6}$$

of $G^{(n)}$ defines the n^{th} prolongation of $\mathbf{v}$, whose coefficients are given by the standard prolongation formula

$$\varphi^K = D_K Q + \sum_{i=1}^{p} \xi^i u_{Ki}. \tag{2.7}$$

Here D_K is the total derivative corresponding to the multi-index K, and we use the notation $u_{Ki} = u_{K+e_i} = D_i u_K$.

For later reference, we require some elementary formulas for higher order total derivatives of certain types of differential functions. Both results are easy to prove by induction using Leibniz' rule.

Lemma 2.1. *Suppose $\varphi(x, u)$ is a zeroth order differential function. Then its first and second order total derivatives have the form*

$$\begin{aligned} D_i \varphi &= \frac{\partial \varphi}{\partial u} u_i + \frac{\partial \varphi}{\partial x^i}, \\ D_i D_j \varphi &= \frac{\partial \varphi}{\partial u} u_{ij} + \frac{\partial^2 \varphi}{\partial u^2} u_i u_j + \frac{\partial^2 \varphi}{\partial u \partial x^i} u_j + \frac{\partial^2 \varphi}{\partial u \partial x^j} u_i + \frac{\partial^2 \varphi}{\partial x^i \partial x^j}. \end{aligned} \tag{2.8}$$

Furthermore, for any multi-index K with $k = \#K \geq 3$,

$$D_K \varphi = \varphi_u u_K + \sum_{i=1}^{p} k_i (D_i \varphi_u) u_{K-e_i} + F(x, u^{(k-2)}), \tag{2.9}$$

where F is a differential function depending on at most $(k-2)^{\text{nd}}$ order derivatives of u.

Lemma 2.2. *Suppose $Q(x, u^{(1)})$ is a first order differential function. Then its first and second order total derivatives have the form*

$$\begin{aligned} D_i Q &= \sum_{l=1}^{p} \frac{\partial Q}{\partial u_l} u_{il} + \frac{\partial Q}{\partial u} u_i + \frac{\partial Q}{\partial x^i}, \\ D_i D_j Q &= \sum_{l=1}^{p} \frac{\partial Q}{\partial u_l} u_{ijl} + \sum_{l,m=1}^{p} \frac{\partial^2 Q}{\partial u_l \partial u_m} u_{il} u_{jm} + \frac{\partial Q}{\partial u} u_{ij} + \\ &\quad + \sum_{l=1}^{p} \left(\frac{\partial^2 Q}{\partial u_l \partial u} (u_j u_{il} + u_i u_{jl}) + \frac{\partial^2 Q}{\partial u_l \partial x^i} u_{jl} + \frac{\partial^2 Q}{\partial u_l \partial x^j} u_{il} \right) + R(x, u^{(1)}), \end{aligned} \tag{2.10}$$

where R is a first order differential function. Furthermore, for any multi-index K with $k = \#K \geq 3$,

$$D_K Q = \sum_{l=1}^{p} \frac{\partial Q}{\partial u_l} u_{Kl} + Q_u u_K + \sum_{i,l=1}^{p} k_i D_i \left(\frac{\partial Q}{\partial u_l} \right) u_{K-e_i+e_l} + R(x, u^{(k-1)}), \tag{2.11}$$

where R is a differential function depending on at most $(k-1)^{\text{st}}$ order derivatives of u.

Note that in both sets of formulae, the multi-indices of orders 1 and 2 do not fit into the general higher order pattern.

3. Relative Differential Invariants.

Throughout this section, we let G be a transformation group (either point or contact) acting via prolongation on the jet spaces $\mathrm{J}^n E$ over a bundle $E = X \times U$. Recall that an (absolute) differential invariant is an invariant differential function for a prolonged group action. A differential operator is said to be G-invariant if it maps differential invariants to higher order differential invariants, and thus, by iteration, produces hierarchies of differential invariants of arbitrarily large order. A general theorem guarantees the existence of sufficiently many such differential operators so as to completely generate all the higher order independent differential invariants of the group by successively differentiating lower order differential invariants. Thus, a complete description of all the differential invariants is provided by a collection of low order "fundamental" differential invariants along with the requisite invariant differential operators.

Theorem 3.1. *Suppose that G is a group of point or contact transformations. Then there exist $p = \dim X$ invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$, and a system of fundamental differential invariants $J_1, \ldots, J_m$, such that, locally, every differential invariant can be written as a function of the iterated derivatives $\mathcal{D}_{j_1} \cdots \mathcal{D}_{j_\kappa} J_\nu$.*

A relative differential invariant is, roughly speaking, a differential function which is invariant, up to a factor, under the prolonged group action. The theory of relative differential invariants is a particular case of the general theory of relative invariants of transformation group actions, in which the group acts by prolongation on a suitable jet space, cf. [**10**]. See [4] for a detailed development and recent results describing general classification of relative invariants.

Definition 3.2. A *differential multiplier* of order n is a linear map $\mathbf{v} \mapsto H_{\mathbf{v}}(x, u^{(n)})$ that maps each Lie algebra element $\mathbf{v} \in \mathfrak{g}$ to a differential function $H_{\mathbf{v}}(x, u^{(n)})$, and satisfies the *cocycle condition*

$$\mathbf{v}^{(n)}(H_{\mathbf{w}}(x, u^{(n)})) - \mathbf{w}^{(n)}(H_{\mathbf{v}}(x, u^{(n)})) = H_{[\mathbf{v},\mathbf{w}]}(x, u^{(n)}). \tag{3.1}$$

The cocycle condition (3.1) implies that the associated infinitesimal generators of the differential multiplier representation

$$\mathcal{D}_{\mathbf{v}} = \mathbf{v}^{(n)} - H_{\mathbf{v}}(x, u^{(n)}), \tag{3.2}$$

form a Lie algebra of first order differential operators on $\mathrm{J}^n E$ having the same commutation relations as the Lie algebra $\mathfrak{g}$. In this paper, we only need consider scalar multipliers, although extensions to matrix-valued multipliers are straightforward, [4], [**10**].

Definition 3.3. A differential function $R(x, u^{(n)})$ is called a *relative differential invariant* for the differential multiplier $H_{\mathbf{v}}$ if it satisfies

$$\mathcal{D}_{\mathbf{v}}(R) = \mathbf{v}^{(n)}(R) - H_{\mathbf{v}} \cdot R = 0, \qquad \text{for all} \qquad \mathbf{v} \in \mathfrak{g}. \tag{3.3}$$

Thus ordinary (or absolute) differential invariants are relative differential invariants for the trivial differential multiplier $H_{\mathbf{v}} \equiv 0$. Note that if R is a relative differential invariant for the multiplier $H_{\mathbf{v}}$ and S is a relative differential invariant for the multiplier $K_{\mathbf{v}}$, then the product $R \cdot S$ is a relative differential invariant for the sum $H_{\mathbf{v}} + K_{\mathbf{v}}$. If R_0 is one particular relative differential invariant of weight $H_{\mathbf{v}}$, then every other such relative differential invariant has the form $R = I R_0$, where I is an arbitrary absolute differential invariant.

Relative differential invariants can be used to construct invariant differential equations. The following result is standard; see [4] for a proof and [**3**] for additional applications.

Theorem 3.4. *A regular partial differential equation* $\Delta(x, u^{(n)}) = 0$ *admits* G *as a symmetry group if and only if* Δ *is a relative differential invariant for some differential multiplier of* G.

Proposition 3.5. *Suppose* $R(x, u^{(n)})$ *is a relative differential invariant of weight* $H_{\mathbf{v}}$. *Then the partial differential equation* $R(x, u^{(n)}) = S(x, u^{(n)})$ *admits* G *as a symmetry group if and only if* $S(x, u^{(n)})$ *is also a relative differential invariant of weight* $H_{\mathbf{v}}$.

The existence of relative differential invariants of sufficiently high order is a consequence of general results in [**10**], [**15**]; see also [4] for generalizations. Recall that a group is said to act *effectively freely* on the manifold M if the quotient group G/G_0 of G by its global isotropy subgroup $G_0 = \{g \in G \mid g \cdot x = x \text{ for all } x \in M\}$ acts freely.

Theorem 3.6. *Let $H_{\mathbf{v}}(x, u^{(n)})$ be an n^{th} order differential multiplier. If $G^{(n)}$ acts effectively freely on (an open subset of) $\mathrm{J}^n E$, then there exists a nontrivial relative invariant of weight $H_{\mathbf{v}}$.*

The next theorem is originally due to Ovsiannikov, [**15**]; see also [**10**; Corollary 5.13].

Theorem 3.7. *If G is a local transformation group on E, then, for n sufficiently large, G acts effectively freely on the open subset of $\mathrm{J}^n E$ where its orbits have maximal dimension.*

Combining Theorems 3.6 and 3.7, we conclude that any transformation group admits a nontrivial relative differential invariant, provided we allow it to have a sufficiently high order.

Theorem 3.8. *Any differential multiplier $H_{\mathbf{v}}(x, u^{(n)})$ of a transformation group G admits a non-zero relative invariant $R_0(x, u^{(m)}) \not\equiv 0$. Moreover, every other relative invariant of weight $H_{\mathbf{v}}$ has the form $R = I \cdot R_0$, where I is an arbitrary absolute differential invariant of G.*

Example 3.9. The total divergence multiplier $D_{\mathbf{v}}$ is defined as the total divergence of the independent variable coefficients of the infinitesimal generator (2.2), so that

$$D_{\mathbf{v}} = \operatorname{Div} \boldsymbol{\xi} = \sum_{i=1}^{p} D_i \xi^i. \tag{3.4}$$

The total divergence multiplier arises in the study of invariant variational problems. The standard infinitesimal invariance criterion, [**9**], that G be a strict variational symmetry group (i.e., without divergence terms) of a variational problem $\int L(x, u^{(n)})\, dx$ is

$$\mathbf{v}^{(n)}(L) + L \operatorname{Div} \boldsymbol{\xi} = 0. \tag{3.5}$$

But (3.5) is just the condition that the Lagrangian $L(x, u^{(n)})$ is a relative differential invariant for the negative total divergence multiplier $\widetilde{D}_{\mathbf{v}} = -\operatorname{Div} \boldsymbol{\xi}$. For example, if $X = \mathbb{R}^2$, then the usual surface area integral

$$\mathcal{S}[u] = \int \sqrt{1 + u_x^2 + u_y^2}\, dx \wedge dy, \tag{3.6}$$

clearly admits the Euclidean group $G = \mathrm{E}(3)$, consisting of translations and rotations in the three-dimensional space coordinatized by (x, y, u), as a symmetry group. This means that $S = \sqrt{1 + u_x^2 + u_y^2}$ is a relative differential invariant for $D_{\mathbf{v}} = -D_x \xi - D_y \eta$, corresponding to the infinitesimal generator $\mathbf{v}^{(0)} = \xi \partial_x + \eta \partial_y + \varphi \partial_u$.

Example 3.10. A second important differential multiplier is the *characteristic multiplier*

$$K_{\mathbf{v}} = Q_u, \tag{3.7}$$

where $Q(x, u^{(1)})$ is the characteristic of the vector field (2.2) given in (2.4). The importance of this differential multiplier lies in its connection with the Euler-Lagrange equations for invariant variational problems. See [**10**], [**14**], for the proof of the following result.

Theorem 3.11. *Let G be a transformation group. Suppose that $\int L(x,u^{(n)})\,dx$ is a G-invariant variational problem, so that L is a relative differential invariant for the total divergence multiplier. Then its Euler-Lagrange expression* $\mathrm{E}(L)$ *is a relative differential invariant of weight* $-Q_u - \operatorname{Div}\boldsymbol{\xi}$.

If one chooses the G-invariant volume element $\int L\,dx$ as the invariant variational problem, then the Euler-Lagrange equation forms the G-invariant minimal hypersurface equation in E. For example, if $G = \mathrm{E}(3)$ is the Euclidean group in $\mathbb{R}^3$, then the Euler-Lagrange equation for the surface area integral (3.6) is the standard (Euclidean-invariant) three-dimensional minimal surface equation

$$0 = \mathrm{E}(S) = -D_x(S_{u_x}) - D_y(S_{u_y}) = \frac{-(1+u_y^2)u_{xx} + 2u_xu_yu_{xy} - (1+u_x^2)u_{yy}}{(1+u_x^2+u_y^2)^{3/2}}.$$

The right hand side of this equation is a relative invariant of weight $-Q_u - D_x\xi - D_y\eta$.

Using the multiplicative property of relative invariants, we readily establish the following useful result.

Corollary 3.12. *Every relative invariant for the characteristic multiplier $K_{\mathbf{v}} = Q_u$ has the form*

$$F = \frac{L}{\mathrm{E}(L)}\,I, \tag{3.8}$$

where I is an arbitrary differential invariant of G, and $\int L(x,u^{(n)})\,dx$ is a G-invariant variational problem having nontrivial Euler-Lagrange expression $\mathrm{E}(L) \not\equiv 0$.

4. Invariance Conditions.

Our primary purpose is determining symmetry groups of evolutionary-type equations, and so we shall extend the preceding considerations to include an additional independent variable, the "time" t. Now, the total space is $\widehat{E} \simeq \mathbb{R} \times E \simeq Z \times U$, where $U \simeq \mathbb{R}$ has coordinate u, whereas $Z \simeq \mathbb{R} \times \mathbb{R}^p$, has coordinates $t = x^0$, representing time, and $x = (x^1, \ldots, x^p)$, representing the spatial independent variables . The n^{th} jet space $\mathrm{J}^n\widehat{E}$ thus has coordinates† $(x, u^{[n]})$, where $u^{[n]}$ stands for all partial derivatives

$$u_K = \frac{\partial^l u}{(\partial t)^{k_0}(\partial x^1)^{k_1}\cdots(\partial x^p)^{k_p}}, \qquad \text{where} \qquad \begin{array}{l} K = (k_0, k_1, \ldots, k_p), \\ l = \#K = k_0 + k_1 + \cdots + k_p \leq n. \end{array} \tag{4.1}$$

In this framework, an n^{th} order *evolution equation* is a partial differential equation of the form

$$u_t = F[u], \tag{4.2}$$

where $F\colon \mathrm{J}^nE \to \mathbb{R}$ is a smooth differential function, depending on spatial derivatives of u up to order n. More generally, we consider higher order *evolutionary-type equations*

$$u_K = F[u], \tag{4.3}$$

† We shall use square brackets to indicate all derivatives, both spatial and temporal, and parentheses for exclusively spatial derivatives.

where $K = L + e_0$ is a multi-index that contains at least one time derivative in it, so $k_0 \geq 1$, and F is again a spatial differential function. Particular cases include the *wave equation*

$$u_{tt} = F[u], \tag{4.4}$$

where $K = 2e_0 = (2, 0, \ldots, 0)$, and the *potential evolution equation*

$$\frac{\partial^2 u}{\partial t \partial x^i} = u_{it} = F[u], \tag{4.5}$$

where $K = e_0 + e_i$. Note that differentiating the evolution equation (4.2) with respect to x^i, or replacing u by its i^{th} potential function $u \mapsto u_i$ will convert (4.2) into an equation of the form (4.5). In each of these examples, one could, of course, go further and allow F to also depend on some lower order temporal derivatives of u; for example, one might allow the right hand side of the wave equation (4.4) to depend on u_t. However, for most of our results, we will restrict attention to purely spatial right hand sides; extensions will be briefly discussed in Section 6.

We are interested in classifying the *spatial* symmetry groups of such evolutionary-type equations. The restriction to spatial implies that the time variable t is not affected by the group transformations, and so we consider a connected Lie group G of either point or contact transformations, which acts on the space $E = X \times U$ of spatial coordinates, and hence determines the corresponding spatially prolonged actions $G^{(n)}$ on the jet space $\mathrm{J}^n E$. In addition, one can include the time t as an additional variable not affected by the group transformations, and thus induce a temporal prolongation $G^{[n]}$ acting on the extended jet space $\mathrm{J}^n \widehat{E}$. In the point transformation case, this is found by prolonging the extended action $(t, x, u) \mapsto (t, g \cdot (x, u))$ on $\widehat{E} \simeq \mathbb{R} \times E$; in the case of contact transformations, we extend the action to $\mathrm{J}^1 \widehat{E} \simeq \mathbb{R}^2 \times \mathrm{J}^1 E$, so that the t variable is not affected, while the action on the time derivative coordinate u_t is determined by the chain rule:

$$\bar{u}_{\bar{t}} = \left(\frac{\partial \Phi}{\partial u} - \sum_{i=1}^{p} \bar{u}_i \, \frac{\partial \Xi^i}{\partial u} \right) u_t, \qquad \text{where} \qquad \begin{array}{ll} \bar{t} = t, & \bar{x}^i = \Xi^i(x, u^{(1)}), \\ \bar{u} = \Phi(x, u^{(1)}), & \bar{u}_i = \Psi_i(x, u^{(1)}). \end{array} \tag{4.6}$$

In this manner, a contact transformation on $\mathrm{J}^1 E$ extends in a natural way to a contact transformation on $\mathrm{J}^1 \widehat{E}$.

Given a Lie algebra element $\mathbf{v} \in \mathfrak{g}$, the corresponding infinitesimal generator $\mathbf{v}^{(0)}$ is a spatial vector field, as in (2.2), whose extended action on $\widehat{E} = \mathbb{R} \times E$ has the same form, $\mathbf{v}^{[0]} = \mathbf{v}^{(0)}$, since the time variable is not changed. Let $\mathbf{v}^{[n]}$ denote the n^{th} prolongation of $\mathbf{v}$ to $\mathrm{J}^n \widehat{E}$. Note that, according to the prolongation formula, the coefficients of temporal and mixed derivatives in $\mathbf{v}^{[n]}$ are *not* necessarily trivial even though t is unaffected by the group transformations; for instance, using (2.5), the coefficient of ∂_{u_t} is

$$\varphi^t = D_t Q + \sum_{i=1}^{p} \xi^i u_{it} = Q_u u_t. \tag{4.7}$$

Thus $\mathbf{v}^{[1]}(u_t) = Q_u u_t$, and we discover that u_t is a relative differential invariant for the characteristic multiplier Q_u. Proposition 3.5 thus implies that the evolution equation (4.2) admits G as a symmetry group if and only if the right hand side $F[u]$ is also a relative differential invariant of weight Q_u. Combining this observation with the characterization of such relative differential invariants in Corollary 3.12 produces the main result of [**14**].

Theorem 4.1. *An evolution equation $u_t = F[u]$ admits the connected spatial transformation group G as a symmetry group if and only if F is relative invariant of weight Q_u, and hence of the form* (3.8).

In particular, every spatial transformation group admits an invariant evolution equation! The most effective method for analyzing the symmetry groups of differential equations is by use of infinitesimal generators. Our starting point is the standard infinitesimal invariance criterion for differential equations.

Theorem 4.2. *An evolutionary-type equation $u_K = F$ admits a connected transformation group G as a symmetry group if and only if*

$$\mathbf{v}^{[n]}(u_K - F) = 0 \qquad \text{whenever} \qquad u_K = F. \tag{4.8}$$

We proceed to analyze this criterion for spatial transformation groups. The prolongation formula (2.6), coupled with (2.7), (2.4), implies that we can replace (4.8) by

$$D_K Q + \sum_{i=1}^{p} \xi^i u_{Ki} = \mathbf{v}^{(n)}(F) \qquad \text{whenever} \qquad u_K = F. \tag{4.9}$$

Recall that $k = \#K$ is the order of K, which contains at least one t derivative. Note first that the left hand side of (4.9) does not contain any terms involving derivatives of order $k+1$ since the terms $\xi^i u_{Ki}$ cancel the corresponding terms obtained by differentiating the characteristic. (This still holds for contact transformations due to the contact conditions (2.5).)

We begin by assuming that K has order $k = \#K \geq 3$, postponing the analysis of the second order cases, namely the wave equation (4.4) and potential evolution equations (4.5), until later. However, unless specifically stated otherwise, the intervening theorems also apply to second order equations, albeit with slightly different proofs. For $\#K \geq 3$, according (2.11), the only terms on the left hand side of (4.9) which involve derivatives of order k are

$$Q_u \, u_K - \sum_{i=1}^{p} \sum_{j=0}^{p} k_j (D_j \xi^i) \, u_{K-e_j+e_i}, \tag{4.10}$$

where we have used (2.5) to identify the derivatives of the characteristic with respect to the derivative variables u_i. On solutions to (4.3) we can replace u_K by F, and hence the terms involving u_K in (4.10) only depend on spatial derivatives of u. On the other hand, because the coefficients φ, ξ^i do not depend on t, all other terms in the left hand side of (4.9) will involve at least one temporal derivative of u, which cannot be replaced by a spatial derivative. Therefore, the infinitesimal invariance condition (4.9) splits into three components. The first part contains only spatial derivatives of u, and is obtained by equating the terms involving u_K in (4.10) to the right hand side of (4.9), leading to our first key result.

Theorem 4.3. *If the evolutionary type equation $u_K = F[u]$ admits a spatial symmetry group G, then the right hand side satisfies*

$$\mathbf{v}^{(n)}(F) = \left(Q_u - \sum_{i=1}^{p} k_i D_i \xi^i \right) F, \qquad \text{for all} \qquad \mathbf{v} \in \mathfrak{g}. \tag{4.11}$$

Thus, equation (4.11) says that the right hand side F of an invariant differential equation of evolutionary type forms a relative differential invariant for the differential multiplier

$$H^K_{\mathbf{v}}(x, u^{(2)}) = Q_u - \sum_{i=1}^{p} k_i D_i \xi^i. \tag{4.12}$$

In particular, the order of t differentiation on the left hand side of the equation does not affect the type of relative invariant that the right hand side assumes.

Corollary 4.4. *If a purely temporal evolutionary-type equation,*

$$\frac{\partial^n u}{\partial t^n} = F[u], \tag{4.13}$$

admits G as a symmetry group, then F is relative invariant of weight Q_u, and hence of the form (3.8).

Thus if (4.13) admits G, then so does the evolution equation (4.2) with the same right hand side. The converse, though, is *not* true since there are additional invariance conditions that (4.13) must satisfy that are not required for the invariance of the simple evolution equation (4.2). The additional terms on the left hand side of the infinitesimal condition (4.9) will end up providing fairly severe restrictions on the types of transformation groups which have, say, invariant wave equations.

Corollary 4.5. *If an equation of the form* (4.13) *admits a symmetry group G, then the corresponding evolution equation $u_t = F$ is also G-invariant. The converse holds provided G fulfills the symmetry conditions in Theorem 4.11 below.*

The second set of invariance conditions arise from the other k^{th} order terms in (4.10), which are equated to 0. We find

$$D_j \xi^i = 0, \qquad \text{whenever} \qquad k_j > 0, \quad j = 0, \ldots, p, \quad j \neq i. \tag{4.14}$$

In particular, $k_0 \geq 1$ by assumption, and hence (4.14) implies $D_t \xi^i = 0$, $i = 1, \ldots, p$. This automatically requires that $\xi^i = \xi^i(x)$ depends only on the spatial variables, and hence the symmetry group is fiber-preserving.

Proposition 4.6. *If G is a symmetry group of an evolutionary-type equation* (4.3) *with left hand side of order $\#K \geq 3$, then G is necessarily a fiber-preserving transformation group.*

The contact conditions (2.3) then imply that the characteristic of each infinitesimal generator of G takes the form

$$Q(x, u^{(1)}) = \varphi(x, u) - \sum_{i=1}^{p} \xi^i(x) u_i. \tag{4.15}$$

Given a multi-index $K = (k_0, k_1, \ldots, k_p)$, let us divide the spatial variables into two sets: the *principal spatial variables*, which are those appearing in the derivative u_K, and the *parametric spatial variables*, which are all the rest. Thus x^j, $1 \le j \le p$, is principal if $k_j > 0$, and parametric if $k_j = 0$. In particular, for the purely temporal evolutionary-type equation (4.13), all spatial variables are parametric. For notational convenience, let us number the spatial variables so that the first s, namely $x^1, \ldots, x^s$ are principal, while the remainder $x^{s+1}, \ldots, x^p$ are parametric.

Proposition 4.7. *Suppose G is a symmetry group of an evolutionary type equation $u_K = F[u]$ in which $\#K \ge 3$. Let $\mathbf{v} \in \mathfrak{g}$ determine an infinitesimal generator (2.2). Then the coefficients $\xi^{s+1}, \ldots, \xi^p$ corresponding to the parametric spatial variables $x^{s+1}, \ldots, x^p$ depend only on parametric variables:*

$$\xi^i = \xi^i(x^{s+1}, \ldots, x^p), \qquad i = s+1, \ldots, p, \tag{4.16}$$

while the coefficients corresponding to the principal spatial variables $x^1, \ldots, x^s$ have the form:

$$\xi^i = \xi^i(x^i, x^{s+1}, \ldots, x^p), \qquad i = 1, \ldots, s. \tag{4.17}$$

Proposition 4.7 allows us to properly justify the statement in Theorem 4.3. In fact, if G is an arbitrary spatial transformation group, then the function (4.12) is, in fact, not an infinitesimal multiplier. However, the additional conditions contained in Proposition 4.7 are precisely those needed to make (4.12) satisfy the infinitesimal multiplier conditions (3.1). Indeed, we can readily produce a basic relative invariant that is associated with (4.12).

Proposition 4.8. *Let us split the independent variables into parametric and principal variables in accordance with the multi-index K. Suppose G is a spatial transformation group satisfying the conditions in Proposition 4.7. Suppose that*

$$\omega = \sum_{i=1}^{p} A_i(x, u^{(n)})\, dx^i \tag{4.18}$$

is a G-invariant one-form. Then, for each principal variable x^i, the coefficient A_i satisfies[†]

$$\mathbf{v}^{(n)}(A_i) + (D_i \xi^i) A_i = 0, \tag{4.19}$$

and hence defines a relative invariant of weight $-D_i \xi^i$.

[†] Note that there is *not* a summation over i in (4.19).

Proof: The infinitesimal invariance conditions for a horizontal one-form (4.18) on J^n (up to contact forms — see [**10**]) are

$$0 = \mathbf{v}^{(n)}(\omega) = \sum_{i=1}^{p} \{ \mathbf{v}^{(n)}(A_i)\, dx^i + A_i\, \mathrm{D}\xi^i \}, \tag{4.20}$$

where

$$\mathrm{D}\xi^i = \sum_{j=1}^{p} D_j \xi^i \, dx^j \tag{4.21}$$

is the total (or horizontal) differential of ξ^i. On the other hand, according to the conditions (4.14), for any principal variable x^i, only the term when $j = i$ contributes to the coefficient of dx^i. Therefore, the coefficient of dx^i in (4.20) is precisely the left hand side of (4.19). *Q.E.D.*

Theorem 4.9. *If the evolutionary type equation $u_K = F[u]$ admits a spatial symmetry group G, then its right hand side is necessarily of the form*

$$F = \frac{A^K\, L}{\mathrm{E}(L)}\, I. \tag{4.22}$$

Here I is an arbitrary absolute differential invariant, $\int L(x, u^{(n)})\, dx$ is a G-invariant variational problem with Euler-Lagrange expression $\mathrm{E}(L) \not\equiv 0$, and $\omega = A_1\, dx^1 + \cdots + A_p\, dx^p$ is a G-invariant one form, such that the product

$$A^K = \prod_{i=1}^{s} (A_i)^{k_i} \not\equiv 0.$$

It is worth re-emphasizing at this point that not every spatial transformation group admits an invariant evolutionary-type equation of a prescribed form. For equations in more than two spatial variables, Proposition 4.7 provides some restrictions on the types of symmetry groups allowed. Further restrictions are obtained by analyzing the lower order terms in the infinitesimal invariance conditions (4.9).

We have already analyzed the terms depending on k^{th} order derivatives. All remaining terms in (4.8) must vanish since they involve lower order temporal derivatives of u. We number the spatial variables so that $x^1, \ldots, x^s$ are the principal variables, and $x^{s+1}, \ldots, x^p$ are the parametric variables. We now use Leibniz' rule and (2.9) to find that the terms involving derivatives of order $k - 1$ are

$$\sum_{i=0}^{p} k_i (D_i \varphi_u) u_{K-e_i} - \sum_{j=1}^{p} \binom{k_j}{2} (D_j^2 \xi^j) u_{K-e_j} = 0. \tag{4.23}$$

In particular, setting $i = 0$ in the first summation shows that $D_t \varphi_u = 0$, and hence

$$\varphi(x, u) = \eta(x) u + \sigma(x). \tag{4.24}$$

This implies that the group consists of affine bundle maps.

Proposition 4.10. *A connected symmetry group of a evolutionary-type equation which is not an evolution equation or a potential evolution equation consists of affine bundle maps.*

(The case of a wave equation will be demonstrated later.) The additional terms in (4.23) imply that

$$\frac{\partial \eta}{\partial x^i} = \frac{k_i - 1}{2}\,\frac{\partial^2 \xi^i}{(\partial x^i)^2}\,, \qquad k_i \neq 0, \tag{4.25}$$

In the ordinary case, the group is fiber-preserving, and so hence, in terms of the principal variables,

$$\eta(x) = \sum_{i=1}^{s} \frac{k_i - 1}{2}\,\frac{\partial \xi^i}{\partial x^i} + \zeta(x^{s+1}, \ldots, x^p). \tag{4.26}$$

Furthermore, if $k_i \geq 2$, then the $(k-2)^{\mathrm{nd}}$ order derivative u_{K-2e_i} in (4.8) has coefficient

$$\binom{k_i}{2} D_i^2 \eta = \binom{k_i}{3} D_i^3 \xi^i, \qquad \text{or} \qquad D_i^2 \eta = \frac{l_i - 2}{3} D_i^3 \xi^i, \quad k_i \geq 2. \tag{4.27}$$

But differentiating (4.25) and subtracting, we find

$$D_i^3 \xi^i = 0, \qquad \text{whenever} \qquad k_i \geq 2. \tag{4.28}$$

Therefore

$$\xi^i = \alpha^i (x^i)^2 + \beta^i x^i + \gamma^i, \qquad k_i \geq 2, \tag{4.29}$$

where α^i, β^i, γ^i are functions of the parametric variables $x^{s+1}, \ldots, x^p$ only. This implies all the lower order derivative terms are also zero; indeed the only term left unaccounted for is $D_K \sigma = 0$, but this is automatic since σ only depends on spatial coordinates and K contains at least one time derivative. We have completed our analysis of the infinitesimal symmetry conditions (4.8) for $\#K \geq 3$, and therefore characterized the possible symmetry groups for higher order evolutionary-type equations.

The analysis in the second order cases proceeds similarly. The wave equation case (4.4) is completely analogous, using (2.8), (2.10), instead of the higher order counterparts, and left to the reader. On the other hand, if the equation is a potential evolution equation, (4.5) (with $i = 1$ for consistency in notation), then the conditions arising from second order derivatives in (4.8) require

$$\frac{\partial^2 Q}{\partial u_l \partial u_m} = \frac{\partial^2 Q}{\partial u_l \partial x^1} = 0 \qquad \text{whenever} \qquad l, m \neq 1,$$

whereas

$$\frac{\partial^2 Q}{\partial u \partial u_j} = \frac{\partial^2 Q}{\partial u^2} = \frac{\partial^2 Q}{\partial u \partial x^1} = 0 \qquad \text{for all} \qquad j = 1, \ldots, p.$$

These imply that the characteristic must have the special form

$$Q(x, u^{(1)}) = \theta(x, u_1) + \zeta(x^2, \ldots, x^p) u - \sum_{i=2}^{p} \xi^i(x^2, \ldots, x^p) u_i. \tag{4.30}$$

Therefore, potential evolution equations can admit contact symmetry groups, but only of a very special type, with infinitesimal generators of the form

$$-\frac{\partial\theta}{\partial u_1}\frac{\partial}{\partial x^1}+\sum_{i=2}^{p}\xi^i(x^2,\ldots,x^p)\frac{\partial}{\partial x^i}+\left[\zeta(x^2,\ldots,x^p)u+\theta(x,u_1)+u_1\frac{\partial\theta}{\partial u_1}\right]\frac{\partial}{\partial u}. \qquad (4.31)$$

We have thus completed our analysis of the infinitesimal symmetry conditions (4.8), and have thus proved the following general result governing the possible symmetry groups of evolutionary-type equations.

Theorem 4.11. *Let G be a connected spatial transformation group, and suppose that $u_K = F[u]$, $k_0 > 0$, is an evolutionary-type equation admitting G as a symmetry group. Assume that $x^1,\ldots,x^s$ are the principal variables, and $x^{s+1},\ldots,x^p$ the parametric variables.*

(i) If the equation is an evolution equation, $u_t = F$, then there are no conditions on G.

(ii) If the equation is a potential evolution equation, $u_{xt} = F$, where $x = x^1$, then G can be a contact transformation group whose generators have the form (4.31).

(iii) In all other cases, the group is necessarily a group of affine bundle maps, whose infinitesimal generators have coefficients of the form (4.16), (4.17), (4.24), (4.26). *Moreover, if the left hand side contains a principal derivative having order 2 or more, i.e., $k_i \geq 2$, then the corresponding coefficient has the form* (4.29).

In all cases, a group G of the prescribed form does admit a nontrivial invariant evolutionary type equation $u_K = F_0$ with $F_0 \not\equiv 0$ a relative invariant of weight (4.12). *Moreover, the most general G-invariant equation of this form is $u_K = IF_0$, where I is an arbitrary absolute differential invariant of G.*

Proof: The only part left to demonstrate is the existence of a suitably invariant evolution equation. This follows from the general existence result for relative invariants given in Theorem 3.8. *Q.E.D.*

Remark: The equation $u_K = 0$ is invariant under any transformation group that meets the invariance conditions in Theorem 4.11.

Remark: If the right hand side of an evolutionary-type equation is nontrivial, $F \not\equiv 0$, then u_K/F is an (absolute) differential invariant of the group G acting on $\mathrm{J}^n\widehat{E}$.

A useful observation is that in every case, the symmetry group admits an invariant foliation, namely that provided by the vertical fibration $\{x = c\}$ of either E or, in the contact case, J^1E, and hence, by definition, must form an imprimitive group of transformations on E. Indeed, Proposition 4.7 implies that the group is "multiply imprimitive" since any collection of independent variables that includes all the parametric variables also defines an invariant foliation.

Proposition 4.12. *Any connected symmetry group of an evolutionary type equation $u_K = F[u]$ which is* not *an evolution equation is necessarily an imprimitive transformation group.*

This result of of great value in simplifying the classification procedure, since it allows us to immediately eliminate many geometrically important transformation groups (which tend to act primitively) from consideration.

An alternative mechanism for generating invariant evolutionary-type equations whose left hand sides have higher order spatial derivatives is by differentiating lower order equations of evolutionary type. The preceding remark shows how this may be used to provide alternative absolute temporal differential invariants.

Theorem 4.13. *If the evolutionary-type equation $u_K = F[u]$ admits a spatial symmetry group G, then any spatial derivative $u_{K+L} = D_L F$, where $l_0 = 0$, also admits G as a symmetry group provided G satisfies the restrictions for the differentiated evolutionary-type equation prescribed in Theorem 4.11.*

Proof: This is a direct consequence of the standard commutation formula

$$\mathbf{v}^{(n+1)} \cdot D_i = D_i \cdot \mathbf{v}^{(n)} - \sum_{j=1}^{p} (D_i \xi^j) D_j, \tag{4.32}$$

between prolonged vector fields and total derivatives, cf. [**9**]. Suppose $L = e_i$, so that x^i is now a principal variable for $K + L = K + e_i$, whether or not it was one for K. Assume first that G satisfies the conditions of Proposition 4.7, with K replaced by $K + L = K + e_i$. Thus, if F satisfies (4.11), then (4.8) implies

$$\begin{aligned} \mathbf{v}^{(n+1)} D_i F &= D_i \mathbf{v}^{(n)}(F) - (D_i \xi^i) D_i F \\ &= D_i \left[\left(Q_u - \sum_{j=1}^{p} k_j D_j \xi^j \right) F \right] - (D_i \xi^i) D_i F \\ &= \left(Q_u - \sum_{j=1}^{p} l_j D_j \xi^j \right) D_i F + \left[D_i \varphi - k_i D_i^2 \xi^i \right] F. \end{aligned} \tag{4.33}$$

Now (4.25), which, according to our hypothesis, must hold with k_i replaced by $l_i = k_i + 1$, implies that the coefficient of F in the final term in (4.33) vanishes, and so $D_i F$ is a relative invariant of the correct weight for the multi-index L, cf. (4.12). *Q.E.D.*

An interesting question is how to connect this approach with that in Theorem 4.9.

5. Classification in One Space Dimension.

The previous sections dealt with the general theory of invariant equations of evolutionary type in multi-dimensional space. We now restrict our attention to evolutionary-type equations in one space dimension, so that $p = 1$ and there is a single spatial variable, x. The advantage here is that the spatial transformation groups act on a two-dimensional space $E \simeq \mathbb{R}^2$, or, in the complex case $E \simeq \mathbb{C}^2$, and hence we can use Lie's classification of transformation groups acting on two-dimensional manifolds — see the Tables below, which are based on [**8**], as simplified in [**10**]. Our goal now is to classify *all* the invariant equations in a single spatial variable for each of the finite-dimensional transformation groups in the plane. (We leave aside the classification of equations admitting infinite dimensional pseudo-groups, since, by a linearization theorem of Bluman and Kumei, [**2**], [**10**; Theorem 6.46], most of these differential equations can be linearized.) For simplicity, we restrict to the complex case here, although extensions to the five additional real forms, [**5**], [**10**], are readily done utilizing the same methods.

Thus we consider an evolutionary-type equation

$$u_{mn} = \frac{\partial^{m+n} u}{\partial x^m \partial t^n} = F[u], \tag{5.1}$$

where $n \geq 1$, and F depends on x, u, and the spatial derivatives $u_k = D_x^k u$ of u. The symmetry generators are vector fields of the form

$$\mathbf{v}^{(0)} = \xi(x, u, u_x)\partial_x + \varphi(x, u, u_x)\partial_u, \tag{5.2}$$

where the dependence on the derivative $u_x = u_1$ allows us to also admit contact transformation groups. Let $N = \max\{m + n, \operatorname{ord} F\}$ denote the order of the partial differential equation (5.1). Let us begin by restating our basic Theorem 4.11 in the scalar spatial case.

Theorem 5.1. *Let G be a connected spatial transformation group acting on $E = X \times U \simeq \mathbb{R}^2$ which is a symmetry group of an evolutionary-type equation* (5.1).

(*i*) $m = 0, n = 1$: *If the equation is an evolution equation, $u_t = F$, then there are no conditions on G.*

(*ii*) $m = 0, n \geq 2$: *If the equation is purely evolutionary, i.e., of the form $\partial^n u/\partial t^n = F$, then the infinitesimal generators of G have the form*

$$\mathbf{v}^{(0)} = \xi(x)\partial_x + [\eta(x)u + f(x)]\partial_u, \tag{5.3}$$

where ξ, η, f are arbitrary functions of x.

(*iii*) $m = 1, n = 1$: *If the equation is a potential evolution equation, $u_{xt} = F$, then G can be a contact transformation group whose infinitesimal generators have the form*

$$\mathbf{v}^{(0)} = \xi(x, u_x)\partial_x + [ku + \theta(x, u_x)]\partial_u, \tag{5.4}$$

where k is a constant.

(*iv*) $m = 1, n \geq 2$: *If the equation is the potential form of a higher order purely evolution equation, $u_{xn} = F$ with $n \geq 2$, then the infinitesimal generators of G have the form*

$$\mathbf{v}^{(0)} = \xi(x)\partial_x + [ku + f(x)]\partial_u, \tag{5.5}$$

where k is a constant and ξ, f are arbitrary functions of x.

(*v*) $m \geq 2$: *In all other cases, the infinitesimal generators have the form*

$$\mathbf{v}^{(0)} = [a_2 x^2 + a_1 x + a_0]\partial_x + [(m-1)a_2 u + b_0]\partial_u, \tag{5.6}$$

where a_0, a_1, a_2, b_0 are constants. (And thus the symmetry group is at most four-dimensional.)

We also generalize Theorem 4.9 to the scalar case. Now the G-invariant one-form can be taken to be the same as the G-invariant Lagrangian $\omega = L\,dx$.

Theorem 5.2. *In one spatial variable, if an evolutionary-type equation* (5.1) *admits a spatial transformation group* G, *then its right hand side satisfies*

$$\mathbf{v}^{(N)}(F) = (Q_u - m D_x \xi) F, \tag{5.7}$$

and hence is a relative differential invariant of the form

$$F = \frac{L^{m+1}}{\mathrm{E}(L)}\, I. \tag{5.8}$$

Here I is an arbitrary differential invariant of G, and $\omega = L(x, u^{(n)})\,dx$ is a G-invariant one-form having nontrivial Euler-Lagrange expression $\mathrm{E}(L) \not\equiv 0$.

Lie's local classification of non-singular† transformation groups that act on a two-dimensional complex manifold appears in Tables 1 and 2 at the end of the paper. Table 1 provides a complete list of canonical forms for the infinitesimal generators of all possible finite-dimensional transformation groups in the plane. In this case, two transformation groups are equivalent if they can be mapped to each other by a point transformation. Cases 1.1–11 list the transitive imprimitive groups; Cases 2.1–3 list the primitive transformation groups; and Cases 3.1–3 the intransitive cases. As for contact transformation groups, there are three additional cases not equivalent to point transformation groups, given in Table 2. Any other finite-dimensional contact transformation group is equivalent, now via a contact transformation, to one of these three or one of the previously listed point transformation groups. (Some of the point transformation canonical forms are equivalent under a contact transformation — one example is the two transitive actions of SL(2) given in 1.1 and 1.2.)

We proceed to classify the possible evolutionary-type equations that admit a finite-dimensional symmetry groups. We leave aside evolution equations, since any transformation group admits an invariant evolution equation. Thus, according to Proposition 4.12 we can immediately restrict our attention to the imprimitive groups. The first step of the classification is to check if the representative Lie algebra of a class fulfills the invariance conditions. If this is so, then, according to Theorem 5.2, knowledge of the fundamental absolute invariants, a relative invariant $R = L/E(L)$ and a non-trivial Lagrangian L of the group allows to generate all invariant equations (5.1). In Tables 3 and 4 we give all the fundamental absolute invariants and invariant derivatives of the algebras acting over E (two-dimensional action) and $\widehat{E}$. The Lagrangian L is the reciprocal of the coefficient of D_x in the invariant two-dimensional derivative. In Table 5 we give the simplest invariant evolution equations, its right hand side serving as the needed relative invariant R (not necessarily equal to $L/E(L)$), together with the corresponding invariant wave and potential evolution equations.

Regardless of the existence of invariant equations for the representative algebra, the possibility remains that a change of variables could yield a point or contact-equivalent

† A transformation group is *singular* if it admits a fixed point, i.e., a zero-dimensional orbit.

algebra satisfying the invariance conditions, and hence admitting invariant equations in the new coordinates. Alternatively, we can think that the representative algebra admits an invariant equation which is not of the prescribed form (4.2), but that can be converted into one by an appropriate equivalence transformation, which leave its invariance untouched.

The second step is to find the required changes of variables, and the corresponding additional invariant equations. We begin with the case of point transformations, which, by Theorem 5.1, covers all but the potential evolution equations. Consider a change of variables

$$\bar{x} = \chi(x, u), \qquad \bar{u} = \psi(x, u). \tag{5.9}$$

If the transformed group admits an invariant equation, it must be generated by vector fields of the form

$$\bar{\mathbf{v}} = \bar{\xi}(\bar{x})\partial_{\bar{x}} + [k(\bar{x})\bar{u} + f(\bar{x})]\partial_{\bar{u}} = \bar{\xi}(\chi)\partial_{\bar{x}} + [k(\chi)\psi + f(\chi)]\partial_{\bar{u}},$$

with the appropriate form of k depending on the type of the invariant equation considered. This means that

$$\begin{aligned} \mathbf{v}(\chi) &= \bar{\xi}(\chi), \\ \mathbf{v}(\psi) &= \xi\psi_x + \varphi\psi_u = k(\chi)\psi + f(\chi), \end{aligned} \tag{5.10}$$

and thus the level sets of the function ξ form an invariant foliation. In terms of the intermediate variables

$$(\tilde{x}, \tilde{u}) = \begin{cases} (\chi(x), u), & \text{if } \chi_x \neq 0, \\ (\chi(x), x), & \text{if } \chi_x = 0, \end{cases}$$

we have

$$\widetilde{\mathbf{v}} = \widetilde{\xi}(\tilde{x})\partial_{\tilde{x}} + \widetilde{\varphi}(\tilde{x}, \tilde{u})\partial_{\tilde{u}}.$$

Differentiating (5.10) with respect to $\tilde{u}$ we obtain

$$\widetilde{\xi}\psi_{\tilde{x}\tilde{u}} + \widetilde{\varphi}_{\tilde{u}}\psi_{\tilde{u}} + \widetilde{\varphi}\psi_{\tilde{u}\tilde{u}} = k\psi_{\tilde{u}},$$

that is to say

$$\widetilde{\mathbf{v}}(\psi_{\tilde{u}}) - (k - \widetilde{\varphi}_{\tilde{u}})\psi_{\tilde{u}} = 0.$$

Therefore the function $\psi_{\tilde{u}}$ *must be a relative invariant* of weight $k - \widetilde{\varphi}_{\tilde{u}}$. Explicitly in original variables we have

$$(\widetilde{\varphi}_{\tilde{u}}, \psi_{\tilde{u}}) = \begin{cases} \left(\varphi_u - \dfrac{\chi_u}{\chi_x}\varphi_x, \psi_u - \dfrac{\chi_u}{\chi_x}\psi_x\right) & \text{if } \chi_x \neq 0, \\ (\varphi_u, \psi_u), & \text{if } \chi_x = 0. \end{cases}$$

Contact transformations can be treated analogously. Consider a contact transformation,

$$\bar{x} = \chi(x, u, u_x), \qquad \bar{u} = \psi(x, u, u_x), \qquad \chi D_x\psi = \psi D_x\chi.$$

The transformed group is now generated by vector fields of the form

$$\begin{aligned}\overline{\mathbf{v}} &= \bar\xi(\bar x,\bar u_{\bar x})\partial_{\bar x} + [k(\bar x,\bar u_{\bar x}]\bar u + f(\bar x,\bar u_{\bar x}))\partial_{\bar u_{\bar x}} \\ &= \bar\xi(\chi,\bar u_{\bar x})\partial_{\bar x} + \left[k\left(\chi,\frac{D_x\psi}{D_x\chi}\right)\psi + f\left(\chi,\frac{D_x\psi}{D_x\chi}\right)\right]\partial_{\bar u}.\end{aligned}$$

Beginning with an invariant foliation by lines $\chi(x,u,u_x) = \lambda$, $\eta(x,u,u_x) = \mu$ of the space J^1E, we conclude that the function

$$\Psi = \Psi(x,u,u_x) = -\frac{(\chi_u\eta_{u_x}-\chi_{u_x}\eta_u)\psi_x - (\chi_u\eta_x-\chi_x\eta_u)\psi_{u_x}}{\chi_x\eta_{u_x}-\chi_{u_x}\eta_x} + \psi_u$$

must be a relative invariant:

$$\mathbf{v}(\Psi) - \left(k + \frac{(\chi_u\eta_{u_x}-\chi_{u_x}\eta_u)\varphi_x - (\chi_u\eta_x-\chi_x\eta_u)\varphi_{u_x}}{\chi_x\eta_{u_x}-\chi_{u_x}\eta_x} - \varphi_u\right)\Psi = 0.$$

In Table 6 we give the invariant foliations of the actions of the considered groups, needed for determining changes of variables discussed above. We have found that there exists only one class of algebras having two inequivalent representatives with invariant equations. It is class 1.1, and the additional representative is studied in Table 7.

6. Generalizations.

So far we have restricted our attention to evolutionary-type equations in which the right hand side is purely a function of the spatial variables and spatial derivatives of the dependent variable. In this section, we relax this condition by permitting the right hand side to also depend on time derivatives of u. Here the computations become more complicated because there is not an immediate separation in the infinitesimal symmetry criteria into purely spatial and temporal parts.

According to Theorem 3.4, a general scalar differential equation

$$R(x,u^{(n)}) = 0 \tag{6.1}$$

is invariant under a group of transformations if and only if R forms a relative differential invariant of the group, and so satisfies (3.3) for some differential multiplier $H_{\mathbf{v}}$. However, if we solve the differential equation (6.1) for one of the derivatives,

$$u_K = F[u], \qquad F\colon \mathrm{J}^n\widehat{E} \to \mathbb{R}, \tag{6.2}$$

then the two components u_K and F may or may not form individual relative differential invariants. (If they do, then Proposition 3.5 implies that they must have the same weight.) As we have seen, this splitting of the equation into relative invariants does occur if (6.2) is of evolutionary type, meaning that K has at least one time derivative, and F depends on purely spatial derivatives. However, in more general situations, F is an "inhomogeneous relative differential invariant", and the existence is more problematic, cf. [**4**]. This has an advantage and a disadvantage. The disadvantage of studying equations with an isolated variable is then the inhomogeneity of the associated relative invariant. The advantage is that the weights are precisely determined, allowing a more systematic approach. The following result, which is analogous to Theorem 4.3 for an evolutionary-type equation, characterizes the right hand side of the general equation (6.2) as an inhomogeneous relative differential invariant.

Proposition 6.1. *If the general equation $u_K = F[u]$ admits a spatial symmetry group G, then the right hand side satisfies*

$$\mathbf{v}^{[n]}(F) = \left(Q_u - \sum_{i=1}^{p} k_i D_i \xi^i\right) F + \left[D_K Q - \left(Q_u - \sum_{i=1}^{p} k_i D_i \xi^i\right) u_K\right], \tag{6.3}$$

for all infinitesimal generators $\mathbf{v} \in \mathfrak{g}$.

Our first result characterizes those equations that impose an affine symmetry condition on its symmetry group.

Proposition 6.2. *Consider a differential equation*

$$u_K = F(t, x, u, u_t, u_1, \ldots, u_p, \ldots, u_L, \ldots), \tag{6.4}$$

with right hand side depending on variables u_L with temporal derivatives of lower order than the one in the left hand side. That is to say, if $K = (k_0, k_1, \ldots, k_p)$ and $L = (l_0, l_1, \ldots, l_p)$, then $l_0 < k_0$ for all variables u_L in F. If K is not a purely temporal multi-index, $K \neq k_0 e_0$, then any connected spatial symmetry group of equations (6.4) is composed of affine bundle maps.

Proof: Apply Theorem 4.2, and expand the infinitesimal symmetry condition (4.8) to obtain

$$D_K Q + \sum_{i=1}^{p} \xi^i u_{Ki} = \mathbf{v}^{[n]}(F), \tag{6.5}$$

in analogy to (4.9). Consider terms in the left hand side of maximal and submaximal orders k and $k-1$, but with maximal temporal order k_0. The right hand side of (6.5) does not contain temporal derivatives of order k_0, and then equating to zero the corresponding coefficients we obtain conditions (4.14) and $D_j Q_u = 0$, where j runs over principal spatial variables. *Q.E.D.*

In the case of an equation (6.4), the right hand side of (6.5) contains terms depending on temporal derivatives (compare with (4.9)). This means that now the expression on the left can contain them also, and that the term on u_K will not be the only one different from zero, i.e. $G \neq 0$ and instead of Theorem 4.3, Proposition 6.1 must be used.

Theorem 6.3. *Let G be a connected spatial symmetry group of an equation of type*

$$\frac{\partial^{m+n} u}{\partial x^m \partial t^n} = F(x, u^{(N)}, u_t) = F(x, u, u_x, u_t, u_{xx}, u_{xt}, \ldots, u_{kl}, \ldots), \qquad l_0 < k_0. \tag{6.6}$$

i.e., with right hand sides that can depend on temporal derivatives of order $l < n$.

(*i*) *If the equation is purely evolutionary, i.e., of the form $\partial^n u/\partial t^n = F$, then there are no restrictions on G.*

(*ii*) *If the equation is the potential form $\partial^{n+1} u/\partial x \partial t^n$ of a purely evolutionary equation, then G can be a contact transformation group whose infinitesimal generators have the form*

$$\mathbf{v}^{(0)} = \xi(x, u_x)\partial_x + (ku + f(x, u_x))\partial_u, \tag{6.7}$$

where k is a constant.

(*iii*) *All the remaining equations, with $m \geq 2$, have the same type of symmetry groups as the corresponding evolutionary-type equations (5.1).*

We can also generalize Theorem 4.9 and Theorem 5.2 to this case. (The existence of suitable inhomogeneous relative invariants is, however, not immediate.)

Theorem 6.4. *In one spatial variable, if an equation* (6.4) *admits a spatial transformation group G, then its right hand side satisfies*

$$\mathbf{v}^{(N)}(F) - (Q_u - mD_x\xi)F = H, \tag{6.8}$$

where the form of H follows from (6.3). *Thus F is an inhomogeneous relative differential invariant of the form*

$$F = \frac{L^{m+1}}{\mathrm{E}(L)}\, I + F_0, \tag{6.9}$$

where I is an absolute differential invariant of G depending on temporal derivatives of u of order less than n, $\omega = L(x, u^{(n)})\, dx$ is a G-invariant one-form having nontrivial Euler-Lagrange expression $\mathrm{E}(L) \not\equiv 0$, and F_0 is a particular inhomogeneous differential invariant of the same weight as F.

As our final examples, let us apply the previous ideas to general wave equations of type (6.4) and to potential-evolution equations with an additional dependence on u_t in the right hand side.

Example 6.5. In this example we discuss invariant wave equations of type (6.4)

$$u_{tt} = F(x, u, u_t, u_x, u_{xt}, \ldots, u_{m-1}, u_{m-1,t}), \tag{6.10}$$

i.e., those whose right hand sides are allowed to depend on spatial derivatives of u and on first order time derivatives. According to Theorem 6.3, the symmetry group G can be any group of transformations — imprimitive, primitive, and contact. Moreover F must be an inhomogeneous relative invariant (6.9) satisfying

$$\mathbf{v}^{(N)}(F) - Q_u F = u_t D_t Q_u - u_{xt} D_t \xi. \tag{6.11}$$

In Table 8 we give the simplest homogeneous part of F, corresponding to some relative invariant $R = L^2\, I/\mathrm{E}(L)$, and the simplest inhomogeneous part F_0.

Example 6.6. In this example we discuss invariant potential evolution equations

$$u_{xt} = F(x, u, u_t, u_x, u_{xx}, \ldots, u_m) \tag{6.12}$$

whose right hand sides are allowed to depend on spatial derivatives of u and the time derivative u_t. Note that potential evolution equations do not belong to the class (6.4), and can be invariant, in principle, under any kind of transformation group. We find that the right hand side F must be an inhomogeneous relative invariant (6.9) satisfying

$$\mathbf{v}^{(N)}(F) - (Q_u - D_x\xi)F = u_t D_x Q_u. \tag{6.13}$$

In Table 9 we give the simplest homogeneous part R and the simplest inhomogeneous part F_0 of the right hand side F.

Table 1

Lie algebras of point transformations in $\mathbb{C}^2$

	Generators	Dim	Structure
1.1.	$\partial_x, x\partial_x - u\partial_u, x^2\partial_x - 2xu\partial_u$	3	$\mathfrak{sl}(2)$
1.2.	$\partial_x, x\partial_x - u\partial_u, x^2\partial_x - (2xu+1)\partial_u$	3	$\mathfrak{sl}(2)$
1.3.	$\partial_x, x\partial_x, u\partial_u, x^2\partial_x - xu\partial_u$	4	$\mathfrak{gl}(2)$
1.4.	$\partial_x, x\partial_x, x^2\partial_x, \partial_u, u\partial_u, u^2\partial_u$	6	$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$
1.5.	$\partial_x, \eta_1(x)\partial_u, \ldots \eta_k(x)\partial_u$	$k+1$	$\mathbb{C} \ltimes \mathbb{C}^k$
1.6.	$\partial_x, u\partial_u, \eta_1(x)\partial_u, \ldots \eta_k(x)\partial_u$	$k+2$	$\mathbb{C}^2 \ltimes \mathbb{C}^k$
1.7.	$\partial_x, \partial_u, x\partial_x + \alpha u\partial_u, x\partial_u, \ldots, x^{k-1}\partial_u$	$k+2$	$\mathfrak{a}(1) \ltimes \mathbb{C}^k$
1.8.	$\partial_x, \partial_u, x\partial_u, \ldots, x^{k-1}\partial_u, x\partial_x + (ku + x^k)\partial_u$	$k+2$	$\mathbb{C} \ltimes (\mathbb{C} \ltimes \mathbb{C}^k)$
1.9.	$\partial_x, \partial_u, x\partial_x, u\partial_u, x\partial_u, x^2\partial_u, \ldots, x^k\partial_u$	$k+3$	$(\mathfrak{a}(1) \oplus \mathbb{C}) \ltimes \mathbb{C}^k$
1.10.	$\partial_x, \partial_u, 2x\partial_x + (k-1)u\partial_u, x^2\partial_x + (k-1)xu\partial_u,$ $x\partial_u, x^2\partial_u, \ldots, x^{k-1}\partial_u$	$k+3$	$\mathfrak{sl}(2) \ltimes \mathbb{C}^k$
1.11.	$\partial_x, \partial_u, x\partial_x, u\partial_u, x^2\partial_x + (k-1)xu\partial_u,$ $x\partial_u, x^2\partial_u, \ldots, x^{k-1}\partial_u$	$k+4$	$\mathfrak{gl}(2) \ltimes \mathbb{C}^k$
2.1.	$\partial_x, \partial_u, x\partial_x - u\partial_u, u\partial_x, x\partial_u$	5	$\mathfrak{sa}(2)$
2.2.	$\partial_x, \partial_u, x\partial_x, u\partial_x, x\partial_u, u\partial_u$	6	$\mathfrak{a}(2)$
2.3.	$\partial_x, \partial_u, x\partial_x, u\partial_x, x\partial_u, u\partial_u, x^2\partial_x + xu\partial_u, xu\partial_x + u^2\partial_u$	8	$\mathfrak{sl}(3)$
3.1.	$\zeta_1(x)\partial_u, \ldots, \zeta_k(x)\partial_u$	k	$\mathbb{C}^k$
3.2.	$\zeta_1(x)\partial_u, \ldots, \zeta_k(x)\partial_u, u\partial_u$	$k+1$	$\mathbb{C} \ltimes \mathbb{C}^k$
3.3.	$\partial_x, x\partial_x, x^2\partial_x$	3	$\mathfrak{sl}(2)$

In Cases 1.5 and 1.6, the functions $\eta_1(x), \ldots, \eta_k(x)$ satisfy a k^{th} order constant coefficient homogeneous linear ordinary differential equation $\mathcal{D}[u] = 0$. In Cases 3.1 and 3.2, the functions $\zeta_1(x), \ldots, \zeta_k(x)$ are arbitrary. In Cases 1.5 – 1.11 we require $k \geq 1$.

Note: We use $\mathfrak{a}(n)$ to denote the Lie algebra of the affine group of $\mathbb{C}^n$, and $\mathfrak{sa}(n)$ for the Lie algebra of the special affine group consisting of volume-preserving affine transformations.

Table 2

Lie algebras of contact transformations in $\mathbb{C}^2$

	Generators
4.1	$\partial_x,\ x\partial_x,\ \partial_u,\ x\partial_u,\ x^2\partial_u,\ 2u_x\partial_x+u_x^2\partial_u$
4.2	$\partial_x,\ x\partial_x,\ u\partial_u,\ \partial_u,\ x\partial_u,\ x^2\partial_u,\ 2u_x\partial_x+u_x^2\partial_u$
4.3	$\partial_x,\ x\partial_x,\ \partial_u,\ x\partial_u,\ x^2\partial_u,\ u\partial_u,\ (2-x)x\partial_x,\ 2u_x\partial_x+u_x^2\partial_u,$ $xu_x^2\partial_u-2(u-xu_x)\partial_x,\ 2x(2u-xu_x)\partial_x+(2u-xu_x)(2u+xu_x)\partial_u$

It is convenient to separate some special cases of the families of algebras given in Tables 1 and 2. Thus 1.7a, 1.7b and 1.7c denote family 1.7 with $\alpha \neq k$, $\alpha = 0$ and $\alpha = k$ respectively. Any starred designation 1.5*–1.11* means the respective family of algebras with the parameter k set to 1. Analogously, 1.11** is 1.11 with $k = 2$.

In the following tables we give the fundamental differential invariants and invariant derivatives of the algebras of tables 1 and 2. Firstly the action on the jet space of variables (x, u) is considered, denoted as "2-D". The last two columns refer to the action on the jet space of (t, x, u). When no invariant derivative with a component in D_t is given in the third column, D_t is meant to be the second invariant derivative.

Table 3

Differential Invariants

	2-D Invariant Derivative	2-D Fundamental Invariant(s)	Invariant Derivatives	Fundamental Invariants
1.1.	$u^{-1}D_x$	$\frac{u_{xx}}{u^3}-\frac{3}{2}\frac{u_x^2}{u^4}$	$u^{-1}D_x$	$\frac{u_t}{u},\ \frac{u_{xx}}{u^3}-\frac{3}{2}\frac{u_x^2}{u^4}$
1.2.	$\frac{D_x}{\sqrt{u_x-u^2}}$	$\frac{u_{xx}-6uu_x+4u^3}{(u_x-u^2)^{3/2}}$	$u_t^{-1}D_x$	$\frac{u_t}{\sqrt{u_x-u^2}}$
1.3.	$uQ_2^{-1/2}D_x$	$Q_2^{-3/2}S_3$	$u^{-1}u_tD_x$	$u^{-1}u_t,\ Q_2^{-3/2}S_3$
1.4.	$\frac{u_xD_x}{\sqrt{Q_3}}$	$\frac{U_5}{Q_3^3}$	$\frac{u_t}{u_x}D_x$	$\frac{u_{tt}}{u_t}-\frac{u_{xt}}{u_x},\ \frac{u_t^2Q_3}{u_x^4}$
1.5.	D_x	$\mathcal{D}[u]$	D_x	$u_t,\ \mathcal{D}[u]$
1.6.	D_x	$D_x\log(\mathcal{D}[u])$	D_x	$\frac{u_{xt}}{u_t},\ \frac{u_{tt}}{u_t},\ \frac{\mathcal{D}[u]}{u_t}$
1.7a.	$u_k^{1/(\alpha-k)}D_x$	$u_k^{1/(\alpha-k)-1}u_{k+1}$	$u_t^{1/\alpha}D_x$	$\frac{u_{xt}}{u_t^{1-1/\alpha}},\ \frac{u_{tt}}{u_t},\ \frac{u_k}{u_t^{1-k/\alpha}}$
1.7b.	$u_k^{-1/k}D_x$	$u_k^{-1/k-1}u_{k+1}$	$u_{xt}^{-1}D_x$	$u_t\frac{u_{xxt}}{u_{xt}^2},\ \frac{u_k}{u_{xt}^k}$
1.7c.	$\frac{D_x}{u_{k+1}}$	$u_k,\ \frac{u_{k+2}}{u_{k+1}^2}$	$u_t^{1/k}D_x$	$\frac{u_{xt}}{u_t^{1-1/k}},\ \frac{u_{tt}}{u_t},\ u_k$
1.8.	$e^{u_k/k!}D_x$	$e^{u_k/k!}u_{k+1}$	$u_t^{1/k}D_x$	$\frac{u_{xt}}{u_t^{1-1/k}},\ \frac{u_{tt}}{u_t},\ \frac{u_t}{e^{u_k/(k-1)!}}$
1.9.	$\frac{u_kD_x}{u_{k+1}}$	$\frac{u_ku_{k+2}}{u_{k+1}^2}$	$\frac{u_t}{u_{xt}}D_x$	$\frac{u_{tt}}{u_t},\ \frac{u_tu_{xxt}}{u_{xt}^2},\ \frac{u_t^{k-1}u_k}{u_{xt}^k}$
1.10.	$u_k^{-2/(k+1)}D_x$	$u_k^{-2-4/(k+1)}Q_{k+2}$	$u_t^{2/(k-1)}D_x$	$u_t^{-1}u_{tt},\ u_t^{k+1}u_k^{k-1}$
1.10*.	$u_x^{-1}D_x$	$u_x^{-4}Q_3$	$u_x^{-1}D_x$	$u_t,\ u_t^{-1}u_{tt},\ u_x^{-4}Q_3$
1.11.	$\frac{u_k}{\sqrt{Q_{k+2}}}D_x$	$Q_{k+2}^{-3/2}S_{k+3}$	$\frac{u_t^2}{Y_3}D_x$	$\frac{u_{tt}}{u_t},\ \frac{\tilde{Z}_{k3}}{Y_3^2},\ \frac{u_t^{2k-1}u_k}{Y_3^k}$
1.11*.	$\frac{u_x}{\sqrt{Q_3}}D_x$	$Q_3^{-3/2}S_4$	$\frac{u_t}{u_x}D_x$	$\frac{u_{xt}}{u_t},\ \frac{u_{tt}}{u_t},\ \frac{u_t^2Q_3}{u_x^4}$
1.11**.	$\frac{u_{xx}}{\sqrt{Q_4}}D_x$	$Q_4^{-3/2}S_5$	$\sqrt{\frac{u_t}{u_{xx}}}D_x$	$\frac{u_{tt}}{u_t},\ \frac{W_3}{\sqrt{u_tu_{xx}^3}},\ \frac{u_{xxt}}{u_{xx}}$

Table 4

Differential Invariants

	2-D Invariant Derivative	2-D Fundamental Invariants	Invariant Derivatives	Fundamental Invariants
2.1	$u_{xx}^{-1/3} D_x$	$\dfrac{R_4}{u_{xx}^{8/3}}$	$\begin{cases} \dfrac{D_x}{u_{xx}^{1/3}}, \\ \dfrac{u_{xt}}{u_{tt}} D_x - D_t \end{cases}$	$\dfrac{u_{xx}}{u_t^3}, \dfrac{U_2}{u_t^4}, \dfrac{V_3}{u_x^5}$
2.2	$\dfrac{u_{xx}}{\sqrt{R_4}} D_x$	$\dfrac{S_5}{R_4^{3/2}}$	$\begin{cases} \dfrac{\sqrt{U_2} D_x}{u_{xx}}, \\ \dfrac{u_{xt}}{u_{tt}} D_x - D_t \end{cases}$	$\dfrac{U_2}{u_t u_{xx}}, \dfrac{V_3}{u_t^{1/2} u_{xx}^{3/2}}, \dfrac{X_3}{u_{xx}^2}$
2.3	$\dfrac{u_{xx}}{S_5^{1/3}} D_x$	$\dfrac{V_7}{S_5^{8/3}}$	$\begin{cases} \sqrt{\dfrac{u_t}{u_x}} D_x, \\ \left(\dfrac{u_{xt}}{2u_{xx}} + \dfrac{u_t u_{xxx}}{6u_{xx}^2} \right) D_x - D_t \end{cases}$	$\dfrac{L_3}{u_t u_{xx}^3}, \dfrac{M_3}{u_t^{3/2} u_{xx}^{9/2}}, \dfrac{N_3}{u_t^2 u_{xx}^6}$
3.1	D_x	$x, \mathcal{D}[u]$	D_x	$x, u_t, \mathcal{D}[u]$
3.2a.	D_x	$x, D_x \log(\mathcal{D}[u])$	D_x	$x, \dfrac{\mathcal{D}[u]}{u_t}$
3.2b.	D_x	$x, D_x \log(\mathcal{D}[u])$	D_x	$x, \dfrac{u_{xt}}{u_t}, \dfrac{u_{tt}}{u_t}, \dfrac{\mathcal{D}[u]}{u_t}$
3.3	D_x	$x, \dfrac{Q_3}{u_x^2}$	D_x	$x, \dfrac{u_t}{u_x}, \dfrac{Q_3}{u_x^2}$
4.1.	$\dfrac{D_x}{u_{xxx}^{1/3}}$	$\dfrac{\tilde{R}_5}{u_{xxx}^{8/3}}$	$\begin{cases} \dfrac{D_x}{u_{xt}}, \\ \dfrac{u_{tt}}{u_{xt}} D_x - D_t \end{cases}$	$u_t, \dfrac{u_{xxx}}{u_{xt}^3}, \dfrac{A_3}{u_{xt}^3}, \dfrac{B_3}{u_{xt}^3}, \dfrac{C_3}{u_{xt}^3}$
4.2.	$\dfrac{u_{xxx}}{\sqrt{\tilde{R}_5}} D_x$	$\tilde{R}_5^{-3/2} \tilde{S}_6$	$\begin{cases} \dfrac{u_t}{u_{xt}} D_x, \\ \dfrac{u_{tt}}{u_{xt}} D_x - D_t \end{cases}$	$\dfrac{u_t^2 u_{xxx}}{u_{xt}^3}, \dfrac{u_t A_3}{u_{xt}^3}, \dfrac{B_3}{u_{xt}^3}, \dfrac{C_3}{u_t u_{xt}^3}$
4.3.	$\dfrac{u_{xxx}}{T_7^{-5/2}} Z_9 \mathcal{D}_x$	$T_7^{-5/2} Z_9$	$\begin{cases} \left(\dfrac{u_t}{u_{xxx}} \right)^{-1/3} D_x, \\ \dfrac{\tilde{Z}_{33} D_x}{2u_t^3 u_{xxx}} - D_t \end{cases}$	$\dfrac{D_3}{u_t^{8/3} u_{xxx}^{4/3}}, \dfrac{E_3}{u_t^4 u_{xxx}^2}, \dfrac{F_4}{u_t^2 u_{xxx}^2}$

In the previous tables, given functions $\eta_1(x), \ldots, \eta_k(x)$, we let $\mathcal{D}$ be a k-th order linear ordinary differential operator whose kernel is spanned by $\eta_1(x), \ldots, \eta_k(x)$, and let $W(x)$ denote their Wronskian determinant. Furthermore

$$
\begin{aligned}
Q_{k+2} &= (k+1)u_k u_{k+2} - (k+2)u_{k+1}^2, \\
R_4 &= 3u_2 u_4 - 5u_3^2, \\
\tilde{R}_5 &= 3u_3 u_5 - 5u_4^2, \\
S_{k+3} &= (k+1)^2 u_k^2 u_{k+3} - 3(k+1)(k+3)u_k u_{k+1} u_{k+2} + 2(k+2)(k+3)u_{k+1}^3, \\
\tilde{S}_6 &= 9u_3^2 u_6 - 45u_3 u_4 u_5 + 40u_4^3, \\
U_5 &= u_1^2\big[Q_3 D_x^2 Q_3 - \frac{5}{4}(D_x Q_3)^2\big] + u_1 u_2 Q_3 D_x Q_3 - (2u_1 u_3 - u_2^2)Q_3^2, \\
V_7 &= u_2^2\big[S_5 D_x^2 S_5 - \frac{7}{6}(D_x S_5)^2\big] + u_2 u_3 S_5 D_x S_5 - \frac{1}{2}(9u_2 u_4 - 7u_3^2)S_5^2, \\
Z_9 &= u_3^2\big[T_7 D_x^2 T_7 - \frac{9}{8}(D_x T_7)^2\big] + u_3 u_4 T_7 D_x T_7 - \frac{4}{5}(7u_3 u_5 - 5u_4^2)T_7^2, \\
T_7 &= 10u_3^3 u_7 - 70u_3^2 u_4 u_6 - 49u_3^2 u_5^2 + 280u_3 u_4^2 u_5 - 175u_4^4, \\
U_2 &= u_{xx}u_{tt} - u_{xt}^2, \\
V_3 &= u_t u_{xxx} - 3u_{xt}u_{xx}, \\
W_3 &= u_t u_{xxx} + 3u_{xt}u_{xx}, \\
X_3 &= u_{xx}u_{xxt} - u_{xt}u_{xxx}, \\
Y_3 &= u_t u_{xtt} - u_{xt}u_{tt}, \\
\tilde{Z}_{k3} &= (k-1)u_t^3 u_{xxt} - (k-2)u_t^2 u_{xt}^2, \\
L_3 &= 12u_t u_{xx}^2 u_{xxt} - 9u_{xx}^2 u_{xt}^2 - 6u_t u_{xx} u_{xt} u_{xxx} - u_t^2 u_{xxx}^2, \\
M_3 &= 54u_t u_{xx}^4 u_{xtt} - 18u_t^2 u_{xx}^2 u_{xxx} u_{xxt} - 54u_t u_{xx}^3 u_{xt} u_{xxt} + 27u_{xx}^3 u_{xt}^3 - 54u_{xx}^4 u_{xt} u_{tt} \\
&\quad + 27u_t u_{xx}^2 u_{xt}^2 u_{xxx} + 9u_t^2 u_{xx} u_{xt} u_{xxx}^2 + u_t^3 u_{xxx}^3, \\
N_3 &= 288u_t u_{xx}^6 u_{ttt} - 144u_t^2 u_{xx}^4 u_{xxx} u_{xtt} - 432u_t u_{xx}^5 u_{xt} u_{xtt} + 24u_t^3 u_{xx}^2 u_{xxx}^2 u_{xxt} \\
&\quad + 144u_t^2 u_{xx}^3 u_{xt} u_{xxx} u_{xxt} + 216u_t u_{xx}^4 u_{xt}^2 u_{xxt} - 81u_{xx}^4 u_{xt}^4 + 432u_{xx}^5 u_{xt}^2 u_{tt} \\
&\quad - 432u_{xx}^6 u_{tt}^2 - 108u_t u_{xx}^3 u_{xt}^3 u_{xxx} + 144u_t u_{xx}^4 u_{xt} u_{tt} u_{xxx} - 54u_t^2 u_{xx}^2 u_{xt}^2 u_{xxx}^2 \\
&\quad - 12u_t^3 u_{xx} u_{xt} u_{xxx}^3 - u_t^4 u_{xxx}^4, \\
A_3 &= u_{xt}u_{xxt} - u_{tt}u_{xxx}, \\
B_3 &= u_{xt}^2 u_{xtt} - 2u_{xt}u_{tt}u_{xxt} + u_{tt}^2 u_{xxx}, \\
C_3 &= u_{xt}^3 u_{ttt} - 3u_{xt}^2 u_{tt} u_{xtt} + 3u_{xt}u_{tt}^2 u_{xxt} - u_{tt}^3 u_{xxx}, \\
D_3 &= 4u_t^2 u_{xxx} u_{xtt} - 4u_t^2 u_{xxt}^2 + 4u_t u_{xt}^2 u_{xxt} - 4u_t u_{xt} u_{tt} u_{xxx} - u_{xt}^4, \\
E_3 &= 4u_t^3 u_{xxx}^2 u_{ttt} - 12u_t^3 u_{xxx} u_{xxt} u_{xtt} + 6u_t^2 u_{xt}^2 u_{xxx} u_{xtt} + 8u_t^3 u_{xxt}^3 - 12u_t^2 u_{xt}^2 u_{xxt}^2 \\
&\quad + 12u_t^2 u_{xt} u_{tt} u_{xxx} u_{xxt} + 6u_t u_{xt}^4 u_{xxt} - 6u_t^2 u_{tt}^2 u_{xxx}^2 - 6u_t u_{xt}^3 u_{tt} u_{xxx} - u_{xt}^6, \\
F_4 &= 2u_t^2 u_{xxx} u_{xxxt} - 2u_t^2 u_{xxt} u_{xxxx} + u_t u_{xt}^2 u_{xxxx} - 4u_t u_{xt} u_{xxx} u_{xxt} + u_t u_{tt} u_{xxx}^2 \\
&\quad + 2u_{xt}^3 u_{xxx}.
\end{aligned}
$$

Table 5

Evolution, u_{xt} and u_{tt}-invariant equations

	$u_t = F$	$u_{tt} = F$	$u_{xt} = F$
1.1.	u	u	$\cdot$
1.2.	$\sqrt{u_x - u^2}$	$\sqrt{u_x - u^2}$	$\cdot$
1.3.	u	u	$\cdot$
1.4.	$\dfrac{u_x^2}{\sqrt{Q_3}}$	$\cdot$	$\cdot$
1.5.	1	1	1
1.6.	$\mathcal{D}[u]$	$\mathcal{D}[u]$	$\mathcal{D}[u]$
1.7ab.	$u_k^{\alpha/(\alpha-k)}$	$u_k^{\alpha/(\alpha-k)}$	$u_k^{(\alpha-1)/(\alpha-k)}$
1.7c.	u_{k+1}^{-k}	u_{k+1}^{-k}	u_{k+1}^{-k+1}
1.8.	$e^{u_k/(k-1)!}$	$e^{u_k/(k-1)!}$	$e^{(k-1)u_k/k!}$
1.9.	$\dfrac{u_k^{k+1}}{u_{k+1}^k}$	$\dfrac{u_k^{k+1}}{u_{k+1}^k}$	$\dfrac{u_k^k}{u_{k+1}^{k-1}}$
1.10.	$u_k^{-(k-1)/(k+1)}$	$\cdot$	$\cdot$
1.10*.	1	1	u_x
1.11.	$\dfrac{u_k^{k+1}}{Q_{k+2}^{k/2}}$	$\cdot$	$\cdot$
1.11*.	$\dfrac{u_x^2}{Q_3^{1/2}}$	$\dfrac{u_x^2}{Q_3^{1/2}}$	u_x
2.1.	$u_{xx}^{1/3}$	$\cdot$	$\cdot$
2.2.	$\dfrac{R_4}{u_{xx}^{7/3}}$	$\cdot$	$\cdot$
2.3.	$\dfrac{u_{xx}^3}{S_5^{2/3}}$	$\cdot$	$\cdot$
3.1.	1	1	1
3.2.	$\mathcal{D}[u]$	$\mathcal{D}[u]$	$\mathcal{D}[u]$
3.3.	u_x	u_x	u_x
4.1.	1	1	$u_{xxx}^{-1/3}$
4.2.	$\dfrac{u_{xxx}^4}{\tilde{R}_5^{3/2}}$	$\dfrac{u_{xxx}^4}{\tilde{R}_5^{3/2}}$	$\dfrac{u_{xxx}^3}{\tilde{R}_5}$
4.3	$\dfrac{u_{xxx}^4}{T_7^{3/4}}$	$\cdot$	$\cdot$

Table 6
Invariant foliations

Algebra	Invariant Foliation
1.1, 1.2, 1.3, 1.5, 1.6, 1.7, 1.8, 1.9, 1.10, 1.11, 3.1, 3.2	$x = \lambda$
1.4, 1.9*, 1.10*, 1.11*,3.2*,3.3	$x = \lambda,\ u = \mu$
1.5*, 1.6*, 1.7bc*	$ax + bu = \lambda$
3.1*	$u - f(x) = \lambda$
4.1, 4.2	$x = \lambda,\ u_x = \mu$

Table 7
Additional algebra with invariant equations

	Change	Algebra	Invariant Derivative	2-D Fundamental Invariant
1.1	$\psi = u^{-1}$	$\partial_x, x\partial_x - \partial_u,$ $x^2\partial_x - 2x\partial_u$	$\frac{D_x}{e^u}$	$\frac{2u_{xx}-u_x^2}{e^{2u}}$
	Fundamental Invariants	$u_t = F$	$u_{xt} = F$	
	$u_t, \frac{2u_{xx} - u_x^2}{e^{2u}}$	$\frac{2u_{xx} - u_x^2}{e^{2u}}$	$\frac{2u_{xx} - u_x^2}{e^u}$	

Table 8

Invariant equations of the form $u_{xt} = F[x, u_t]$

	R	F_0
1.1	u^2	$\frac{u_x u_t}{u}$
1.2	$u^2 - u_x$	$2uu_t$
1.3	$\sqrt{\frac{u_t}{u} Q_2}$	$\frac{u_x u_t}{u}$
1.4	u_x	$\frac{u_t U_4}{u_x Q_3}$
1.5	1	0
1.6	u_t	0
1.7	$u_t^{(\alpha-1)/\alpha}$	0
1.7b	$u_k^{1/k}$	0
1.8	$u_t^{(k-1)/k}$	0
1.9	$(u_t^{k-1} u_k)^{1/k}$	0
1.10	$u_t^{(k-3)/(k-1)}$	$-\frac{k-1}{k+1} \frac{u_{k+1} u_t}{u_k}$
1.10*	u_x	0
1.11	$u_t^{(k-1)/k} u_k^{1/k}$	$-\frac{k-1}{k+1} \frac{u_{k+1} u_t}{u_k}$
2.1	$u_{xx}^{2/3}$	0
2.2	$\sqrt{u_t u_{xx}}$	0
2.3	$\sqrt{u_t u_{xx}}$	$\frac{u_x {S_5}^{1/3}}{3^{2/3} u_{xx}} + \frac{u_x V_6}{u_{xx} S_5}$
3.1	u_t	0
3.2	u_t	0
3.3	u_t	$\frac{u_t u_{xx}}{u_x}$
4.1	$u_{xxx}^{1/3}$	0
4.2	$(u_t^2 u_{xxx})^{1/3}$	0
4.3	$(u_t^2 u_{xxx})^{1/3}$	$\frac{u_x {T_7}^{1/4}}{u_{xxx}} + \frac{u_x Z_8}{2u_{xxx} T_7}$

Table 9

Invariant equations of the form $u_{tt} = F[x, u_t, u_{xt}, \ldots]$

	R	F_0
1.1	u	0
1.2	u_t	0
1.3	u	0
1.4	u_t	$\dfrac{u_t u_{xt}}{u_x}$
1.5	1	0
1.6	u_t	0
1.7	u_t	0
1.8	u_t	0
1.9	u_t	0
1.10	u_t	0
1.10*	1	0
1.11	u_t	0
2.1	u_t	$\dfrac{u_{xt}^2}{u_{xx}}$
2.2	u_t	$\dfrac{u_{xt}^2}{u_{xx}}$
2.3	u_t	$\dfrac{G_4}{3u_{xx}^2}$
3.1	1	0
3.2	u_t	0
3.2	1	0
4.1	1	$\dfrac{u_{xt} u_{xxt}}{u_{xxx}}$
4.2	u_t	$\dfrac{u_{xt} u_{xxt}}{u_{xxx}}$
4.3	u_t	$\dfrac{-F_4}{u_t u_{xxx}^2}$

We have used the notations

$$U_4 = -6{u_2}^3 + 6u_1u_2u_3 - {u_1}^2u_4,$$

$$G_4 = 3u_t u_{xx}^2 - 12u_{xx}u_{xt}^2 - 8u_t u_{xt} u_{xxx} + 18u_t u_{xx} u_{xxt} - u_t^2 u_{xxxx},$$

$$V_6 = 120{u_3}^4 - 185u_2{u_3}^2u_4 + 30{u_2}^2{u_4}^2 + 45{u_2}^2u_3u_5 - 6{u_2}^3u_6,$$

$$Z_8 = -1400{u_4}^5 + 2870u_3{u_4}^3u_5 - 1085{u_3}^2u_4{u_5}^2 - 770{u_3}^2{u_4}^2u_6 + 252{u_3}^3u_5u_6 + 140{u_3}^3u_4u_7 - 15{u_3}^4u_8.$$

References

[**1**] Bäcklund, A.V., Ueber Flachentransformationen, *Math. Ann.* **9** (1876), 297–320.

[**2**] Bluman, G.W., and Kumei, S., Symmetry-based algorithms to relate partial differential equations: I. Local symmetries, *Euro. J. Appl. Math.* **1** (1990), 189–216.

[**3**] Cariñena, J.F., del Olmo, M.A., and Winternitz, P., On the relation between weak and strong invariance of differential equations, *Lett. Math. Phys.* **29** (1993), 151–163 .

[**4**] Fels, M., and Olver, P.J., On relative invariants, preprint, University of Minnesota, 1995.

[**5**] González–López, A., Kamran, N., and Olver, P.J., Lie algebras of vector fields in the real plane, *Proc. London Math. Soc.* **64** (1992), 339–368.

[**6**] Kichenassamy, S., The Perona–Malik paradox, preprint, University of Minnesota, 1995.

[**7**] Lie, S., *Theorie der Transformationsgruppen*, Vol. 3, B.G. Teubner, Leipzig, 1893.

[**8**] Lie, S., Gruppenregister, *in*: *Gesammelte Abhandlungen*, vol. 5, B.G. Teubner, Leipzig, 1924, pp. 767–773.

[**9**] Olver, P.J., *Applications of Lie Groups to Differential Equations*, Second Edition, Graduate Texts in Mathematics, vol. 107, Springer–Verlag, New York, 1993.

[**10**] Olver, P.J., *Equivalence, Invariants, and Symmetry*, Cambridge University Press, 1995.

[**11**] Olver, P.J., Differential invariants, *Acta Appl. Math.* **41** (1995), 271–284.

[**12**] Olver, P.J., Sapiro, G., and Tannenbaum, A., Classification and uniqueness of invariant geometric flows, *Comptes Rendus Acad. Sci. (Paris), Série I,* **319** (1994), 339–344.

[**13**] Olver, P.J., Sapiro, G., and Tannenbaum, A., Differential invariant signatures and flows in computer vision: a symmetry group approach, *in*: *Geometry–Driven Diffusion in Computer Vision*, B. M. Ter Haar Romeny, ed., Kluwer Acad. Publ., Dordrecht, the Netherlands, 1994.

[**14**] Olver, P.J., Sapiro, G., and Tannenbaum, A., Invariant geometric evolutions of surfaces and volumetric smoothing, *SIAM J. Appl. Math.*, to appear.

[**15**] Ovsiannikov, L.V., *Group Analysis of Differential Equations*, Academic Press, New York, 1982.

[**16**] Osher, S.J., and Rudin, L.I., Feature-oriented image enhancement using shock filters, *SIAM J. Num. Anal.* **27** (1990), 919–940.

[**17**] Sokolov, V.V., On the symmetries of evolution equations, *Russ. Math. Surveys* **43:5** (1988), 165–204.