Classification of Invariant Wave Equations

Rafael Hernández Heredero[1] and **Peter J. Olver**[2]

[1] Depto. Física Teórica II, Fac. Ciencias Físicas, Universidad Complutense de Madrid, Ciudad Universitaria s/n, 28040 Madrid, Spain.
rhh@eucmax.sim.ucm.es

[2] School of Mathematics, University of Minnesota, Minneapolis, MN 55455, U.S.A.
olver@ima.umn.edu

1 Introduction

One of the most basic constructions of modern physics is the formulation of field equations (or variational principles) admitting a known symmetry group. This can be done, in the regular case, by assembling suitable combinations of differential invariants of the group. A more subtle question, the problem of classifying invariant differential equations of a specified form admitting a prescribed symmetry group, cannot be systematized so directly. For example, the classification of geometric diffusion equations admitting symmetry groups of visual significance is a problem of importance in computer vision and image processing, [4].

In this work, we have considered the classification of wave equations in both one and several space variables, and a single time variable admitting a prescribed finite-dimensional symmetry group. We determine a complete set of conditions that a transformation group admit an invariant evolutionary or wave equation. Differential invariants completely characterize all possible invariant equations admitted by a symmetry group of the prescribed type. In the planar case (one independent spatial variable and one dependent variable), we then give directions in how to use Lie's complete classification of groups of point and contact transformations in the plane [1], [2] to find a complete list of invariant wave equations. We further study possible equivalence relations between these equations, completing the information given in [3].

We make extensive use of the theory of differential invariants, as presented in [2], [5]. We assume that the variables are, in general, complex-valued. The present work can be viewed as a start towards the classification of differential invariants for surfaces under transformation groups in three-dimensional space, where the group acts completely trivially on the time coordinate. An important task awaiting completion is the complete classification of the differential invariants of Lie's three-dimensional transformation groups.

2 Notations

All our considerations are local, so we can work in Euclidean space. The total space $\widehat{E} \simeq X \times U$, where $U \simeq \mathbb{R}$ has coordinate u, the scalar dependent variable, whereas $X \simeq \mathbb{R}^{p+1}$, has coordinates $x = (x^0, \dots, x^p)$, representing one temporal (x_0) and p spatial independent variables. E represents $\widehat{E}$ considering only spatial variables $(x^1, \dots, x^p)$. We use two jet spaces, the n^{th} jet space $\mathrm{J}^n \widehat{E}$ with coordinates $(x, u^{(n)})$, where $u^{(n)}$ stands for all partial derivatives

$$u_K = \frac{\partial^k u}{(\partial x^0)^{k_0} \cdots (\partial x^p)^{k_p}}, \qquad \text{where} \qquad \begin{array}{l} K = (k_0, \dots, k_p), \\ k = \#K = k_0 + \cdots + k_p \leq n \end{array}$$

and the space $\mathrm{J}^n E$, with spatial derivatives only. We consider both point and contact transformation groups G acting on E. An infinitesimal generator of the group action

$$\mathbf{v}^{(0)} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \varphi \frac{\partial}{\partial u},$$

corresponds to the Lie algebra element $\mathbf{v} \in \mathfrak{g}$, where $\mathfrak{g}$ denotes the Lie algebra of G. The group consists of *affine bundle maps* if it consists of transformations $(x, u) \mapsto (\Phi(x), A(x)u + B(x))$ which are fiber-preserving and affine in the dependent variable u at each point; the infinitesimal generators have $\xi = \xi(x)$, $\varphi = \alpha(x)u + \beta(x)$. For instance, most linear partial differential equations have affine bundle symmetry groups. The infinitesimal generator

$$\mathbf{v}^{[n]} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\# K \leq n} \varphi^K \frac{\partial}{\partial u_K},$$

defines the n^{th} prolongation of $\mathbf{v}$ to $\mathrm{J}^n \widehat{E}$, whose coefficients are given by the standard prolongation formula

$$\varphi^K = D_K Q + \sum_{i=1}^{p} \xi^i u_{Ki}.$$

Here D_K is the total derivative corresponding to the multi-index K, and we use the notation $u_{Ki} = u_{K+e_i} = D_i u_K$, e_i being the multi-index corresponding to x_i. $Q = \varphi - \sum_{i=1}^{p} \xi^i u_i$ is the characteristic function of $\mathbf{v}$. We will denote by $\mathbf{v}^{(n)}$ the corresponding prolongation of $\mathbf{v}$ in $\mathrm{J}^n E$.

We are going to consider *evolutionary* differential equations of the form

$$u_K = F \qquad k_0 > 1$$

where F is a differential function in $\mathrm{J}^n E$, i.e. without t derivatives, and the left hand side u_K contains at least one time derivation: $k_0 > 1$.

3 Invariance Conditions

We begin with the standard infinitesimal criteria for invariance, [2]:

Theorem 1 *An evolutionary-type equation $u_K = F$ admits a connected transformation group G as a symmetry group if and only if*

$$\mathbf{v}^{[n]}(u_K - F) = 0 \qquad \text{whenever} \qquad u_K = F.$$

A relative invariant is a differential function R such that $\mathbf{v}^{(n)}(R) = \eta R$, for all $\mathbf{v} \in \mathfrak{g}$, where η is a multiplier differential function.

Theorem 2 *A regular partial differential equation $\Delta(x, u^{(n)}) = 0$ admits G as a symmetry group if and only if Δ is a relative differential invariant for some differential multiplier of G.*

Applying these criteria to the families of equations of our interest, we find restrictions to their possible symmetry groups:

Theorem 3 *Let G be a connected spatial transformation group, and suppose that $u_K = F[u]$, $k_0 > 0$, is an evolutionary-type equation admitting G as a symmetry group. Assume that $x^1, \ldots, x^s$ appear in the derivative u_K ($k_i \geq 1$, $i = 1, \ldots, s$), and $x^{s+1}, \ldots, x^p$ do not ($k_i = 0$, $i = s+1, \ldots, p$).*

- (i) *If the equation is an evolution equation, $u_t = F$, then there are no conditions on G.*
- (ii) *If the equation is a potential evolution equation, $u_{xt} = F$, where $x = x^1$, then G can be a contact transformation group whose characteristic has the form $Q(x, u^{(1)}) = \theta(x, u_1) + \zeta(x^2, \ldots, x^p)u - \sum_{i=2}^{p} \xi^i(x^2, \ldots, x^p)u_i$.*

(iii) *In all other cases, the group is necessarily a group of affine bundle maps, whose characteristic has coefficients of the form* $\xi^i = \xi^i(x^i, x^{s+1}, \ldots, x^p)$, $i = 1, \ldots, s$, $\xi^i = \xi^i(x^{s+1}, \ldots, x^p)$, $i = s+1, \ldots, p$, $\varphi = \sum_{i=1}^{s}(k_1/2)\partial\xi^i/\partial x^i + \zeta(x^{s+1}, \ldots, x^p)$. *Moreover, if* $k_i \geq 2$, $i \geq 1$, *then the corresponding coefficient has the form* $\xi^i = \alpha^i(x^i)^2 + \beta^i x^i + \gamma^i$, *where* α^i, β^i, γ^i *are functions of* $x^{s+1}, \ldots, x^p$ *only.*

In all cases, a group G *of the prescribed form does admit a nontrivial invariant evolutionary type equation* $u_K = F_0$ *with* $F_0 \not\equiv 0$ *a relative invariant of weight* $Q_u - \sum_{i=1}^{p} k_i D_i \xi^i$. *Moreover, the most general* G*-invariant equation of this form is* $u_K = IF_0$, *where* I *is an arbitrary absolute differential invariant of* G.

Applying this general result to the planar case we obtain the following.

Theorem 4 *Let* G *be a connected spatial transformation group acting on* $E = X \times U \simeq \mathbf{R}^2$ *which is a symmetry group of an evolutionary-type equation*

$$u_{mn} = \frac{\partial^{m+n} u}{\partial x^m \partial t^n} = F[u]. \tag{1}$$

(i) $m = 0, n = 1$: *If the equation is an evolution equation,* $u_t = F$, *then there are no conditions on* G.

(ii) $m = 0, n \geq 2$: *If the equation is purely evolutionary, i.e., of the form* $\partial^n u/\partial t^n = F$, *then the infinitesimal generators of* G *have the form* $\mathbf{v}^{(0)} = \xi(x)\partial_x + [\eta(x)u + f(x)]\partial_u$, *where* ξ, η, f *are arbitrary functions of* x.

(iii) $m = 1, n = 1$: *If the equation is a potential evolution equation,* $u_{xt} = F$, *then* G *can be a contact transformation group whose infinitesimal generators have the form* $\mathbf{v}^{(0)} = \xi(x, u_x)\partial_x + [ku + \theta(x, u_x)]\partial_u$, *where* k *is a constant.*

(iv) $m = 1, n \geq 2$: *If the equation is the potential form of a higher order purely evolution equation,* $u_{xn} = F$ *with* $n \geq 2$, *then the infinitesimal generators of* G *have the form* $\mathbf{v}^{(0)} = \xi(x)\partial_x + [ku + f(x)]\partial_u$, *where* k *is a constant and* ξ, f *are arbitrary functions of* x.

(v) $m \geq 2$: *In all other cases, the infinitesimal generators have the form* $\mathbf{v}^{(0)} = [a_2 x^2 + a_1 x + a_0]\partial_x + [(m-1)a_2 u + b_0]\partial_u$, *where* a_0, a_1, a_2, b_0 *are constants, and thus the symmetry group is at most four-dimensional.*

The right hand side of the equations is characterized as follows:

Theorem 5 *If an evolutionary-type equation*

$$u_{mn} = \frac{\partial^{m+n} u}{\partial x^m \partial t^n} = F[u].$$

admits a spatial transformation group G, *then its right hand side satisfies*

$$\mathbf{v}^{(N)}(F) = (Q_u - mD_x\xi)F,$$

and hence is a relative differential invariant of the form

$$F = \frac{L^{m+1}}{\mathrm{E}(L)}\, I.$$

Here I *is an arbitrary differential invariant of* G, *and* $\omega = L(x, u^{(n)})\, dx$ *is a* G*-invariant one-form having nontrivial Euler-Lagrange expression* $\mathrm{E}(L) \not\equiv 0$.

In [3] we give tables of all the differential invariants involved in the construction of invariant equations, for all of the finite-dimensional planar transformation groups in Lie's list.

4 On changes of variables

Lie's list of algebras acting on the plane is given by representative elements of classes up to a point transformation (or contact transformation in the case of contact algebras). These equivalence transformations can transform the representative invariant equations we obtain into other equations, invariant under the transformed algebra. The type of equation, though, is not generally respected by arbitrary transformations.

Consider a change of variables $\bar{x} = \chi(x,u)$, $\bar{u} = \psi(x,u)$. We first can observe that according to theorems 3 and 4 all families of equations, except the evolution one, admit only symmetry groups that are *imprimitive*, i.e. the fibration $x = c$ is an invariant foliation. If the transformed group admits an invariant equation, it must be generated by vector fields of the form

$$\mathbf{v} = \bar{\xi}(\bar{x})\partial_{\bar{x}} + [k(\bar{x})\bar{u} + f(\bar{x})]\partial_{\bar{u}} = \bar{\xi}(\chi)\partial_{\bar{x}} + [k(\chi)\psi + f(\chi)]\partial_{\bar{u}},$$

so the new independent variable $\bar{x} = \chi(x,u)$ must be given by an invariant foliation $\chi = c$. Conditions on the new dependent variable are found differentiating the equations $\mathbf{v}(\chi) = \bar{\xi}(\chi)$ and $\mathbf{v}(\psi) = \xi\psi_x + \varphi\psi_u = k(\chi)\psi + f(\chi)$.

Theorem 6 *A change of variables $\bar{x} = \chi(x,u)$, $\bar{u} = \psi(x,u)$ is an equivalence transformation of an invariant evolutionary equation* (1) *if the following is satisfied:*

(i) *The new independent variable must be chosen as the function χ defining an invariant foliation $\chi(x,u) = c$, $\bar{x} = \chi$.*

(ii) *The new dependent variable $\bar{u} = \psi(x,u)$ must be such that (a) if $\bar{x} \neq u$, then $\psi_u - \psi_x\chi_u/\chi_x$ must be a relative invariant of weight $\varphi_u - \varphi_x\chi_u/\chi_x$; (b) if $\bar{x} = u$ then ψ_x must be a relative invariant of weight φ_x.*

We find thus the following classes of changes of variables:

1. Changes of variables that change the principal invariant foliation, $x = c$.
2. Changes of variables that do not change the principal invariant foliation, merely rescaling the independent variable $\bar{x} = \chi(x)$. These class admits two subclasses: (a) *affine* changes of variables, with $\bar{u} = c(x)u + d(x)$; (b) *non*-affine changes of variables in u.
3. Any composition of the above.

We have found that only the algebras point-equivalent to $\{\partial_x, x\partial_x - u\partial_u, x^2\partial_x - 2xu\partial_u\}$ admit a non-affinechange of variables respecting the form of its invariant equations, namely the inversion $\bar{u} = 1/u$.

5 Generalizations

So far we have restricted our attention to evolutionary-type equations in which the right hand side is purely a function of the spatial variables and spatial derivatives of the dependent variable. In this section, we relax this condition by permitting the right hand side to also depend on time derivatives of u.

Proposition 1 *If the general equation $u_K = F[u]$ admits a spatial symmetry group G, then the right hand side satisfies*

$$\mathbf{v}^{[n]}(F) = \left(Q_u - \sum_{i=1}^{p} k_i D_i \xi^i\right) F + \left[D_K Q - \left(Q_u - \sum_{i=1}^{p} k_i D_i \xi^i\right) u_K\right], \qquad (2)$$

for all infinitesimal generators $\mathbf{v} \in \mathfrak{g}$.

Our first result characterizes those equations that impose an affine symmetry condition on its symmetry group.

Proposition 2 *Consider a differential equation*

$$u_K = F(t, x, u, u_t, u_1, \ldots, u_p, \ldots, u_L, \ldots), \tag{3}$$

with right hand side depending on variables u_L with temporal derivatives of lower order than the one in the left hand side. That is to say, if $K = (k_0, k_1, \ldots, k_p)$ and $L = (l_0, l_1, \ldots, l_p)$, then $l_0 < k_0$ for all variables u_L in F. If K is not a purely temporal multi-index, $K \neq k_0 e_0$, then any connected spatial symmetry group of equations (3) is composed of affine bundle maps.

For planar equations we have

Theorem 7 *Let G be a connected spatial symmetry group of an equation of type*

$$\frac{\partial^{m+n} u}{\partial x^m \partial t^n} = F(x, u^{(N)}, u_t) = F(x, u, u_x, u_t, u_{xx}, u_{xt}, \ldots, u_{kl}, \ldots), \qquad l < n.$$

i.e., with right hand sides that can depend on temporal derivatives of order $l < n$.

(i) *If the equation is purely evolutionary, i.e., of the form $\partial^n u/\partial t^n = F$, then there are no restrictions on G.*

(ii) *If the equation is the potential form $\partial^{n+1} u/\partial x \partial t^n$ of a purely evolutionary equation, then G can be a contact transformation group whose infinitesimal generators have the form $\mathbf{v}^{(0)} = \xi(x, u_x)\partial_x + (ku + f(x, u_x))\partial_u$, where k is a constant.*

(iii) *All the remaining equations, with $m \geq 2$, have the same type of symmetry groups as the corresponding evolutionary-type equations.*

Theorem 8 *In one spatial variable, if an equation (3) admits a spatial transformation group G, then its right hand side satisfies*

$$\mathbf{v}^{(N)}(F) - (Q_u - mD_x\xi)F = H,$$

where the form of H follows from (2). Thus F is an inhomogeneous relative differential invariant of the form

$$F = \frac{L^{m+1}}{\mathrm{E}(L)}\, I + F_0,$$

where I is an absolute differential invariant of G depending on temporal derivatives of u of order less than n, $\omega = L(x, u^{(n)})\, dx$ is a G-invariant one-form having nontrivial Euler-Lagrange expression $\mathrm{E}(L) \not\equiv 0$, and F_0 is a particular inhomogeneous differential invariant of the same weight as F.

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