NL3344 **Lie algebras and Lie groups**

Lie groups were introduced by the nineteenth century Norwegian mathematician Sophus Lie through his studies in geometry and integration methods for differential equations, (Hawkins, 1999). Further developments by W. Killing, É. Cartan and H. Weyl established Lie's theory as a cornerstone of mathematics and its physical applications. General references include (Duistermaat & Kolk, 1999; Sattinger & Weaver, 1986; Varadarajan, 1984).

An r parameter Lie group is defined as an r dimensional manifold that is also a group with smooth multiplication and inversion maps. A key example is the $r = n^2$ dimensional general linear group $\mathrm{GL}(n)$ of (either real or complex) $n \times n$ nonsingular matrices, $\det A \neq 0$, under matrix multiplication. Most Lie groups can be realized as matrix groups, i.e., subgroups of $\mathrm{GL}(n)$. Important examples include the

- special linear group $\mathrm{SL}(n) \subset \mathrm{GL}(n)$ with $\det A = 1$, and $r = n^2 - 1$
- orthogonal group $\mathrm{O}(n) \subset \mathrm{GL}(n, \mathbb{R})$ with $A^T A = I$, and $r = n(n-1)/2$
- unitary group $\mathrm{U}(n) \subset \mathrm{GL}(n, \mathbb{C})$ with $A^\dagger A = I$, and $r = n^2$
- symplectic group $\mathrm{Sp}(2n) \subset \mathrm{GL}(2n, \mathbb{R})$ with $A^T J A = J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, and $r = n(2n+1)$.

A Lie algebra $\mathfrak{g}$ is a vector space equipped with a skew-symmetric, bilinear bracket $[\,\cdot,\cdot\,] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies the Jacobi identity

$$[\,u, [\,v, w\,]\,] + [\,v, [\,w, u\,]\,] + [\,w, [\,u, v\,]\,] = 0.$$

The Lie algebra $\mathfrak{g}$ of left-invariant vector fields on an r-parameter Lie group G can be identified with the tangent space at the identity, and so $\dim \mathfrak{g} = r$. The Lie algebra $\mathfrak{gl}(n)$ of $\mathrm{GL}(n)$ consists of all $n \times n$ matrices under matrix commutator $[\,A, B\,] = AB - BA$. A finite-dimensional Lie algebra with basis $v_1, \ldots, v_r$ is specified by its structure constants c^i_{jk}, defined by the bracket relations $[\,v_j, v_k\,] = \sum_{i=1}^r c^i_{jk} v_i$. Each finite-dimensional Lie algebra corresponds to a unique connected and simply connected Lie group G^*; any other is obtained by quotienting by a discrete normal subgroup: $G = G^*/N$.

A subspace $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra if it is closed under the Lie bracket: $[\,\mathfrak{h}, \mathfrak{h}\,] \subset \mathfrak{h}$. Lie subalgebras are in one-to-one correspondence with connected Lie subgroups $H \subset G$. The subalgebra is an ideal if $[\,\mathfrak{g}, \mathfrak{h}\,] \subset \mathfrak{h}$. A Lie algebra is simple if it contains no nontrivial ideals, and semi-simple if it contains no nontrivial abelian (commutative) ideals. Semi-simple algebras are direct sums of simple algebras. A Lie algebra is solvable if the sequence of subalgebras $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(k+1)} = [\,\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}\,]$ eventually terminates with $\mathfrak{g}^{(n)} = \{0\}$. The Levi decomposition says that every Lie algebra is the semi-direct sum of a semi-simple subalgebra and its radical — the maximal solvable ideal.

The Killing–Cartan classification of complex simple Lie algebras contains four infinite families, denoted A_n, B_n, C_n, D_n, corresponding to the simple Lie groups $\mathrm{SL}(n+1), \mathrm{O}(2n+1), \mathrm{Sp}(2n+1), \mathrm{O}(2n)$. In addition, there are five exceptional simple Lie groups, G_2, F_4, E_6, E_7, E_8, of respective dimensions 14, 52, 78, 133, 248. The last plays an important role in modern theoretical physics. Extending the classification to infinite-dimensional simple Lie algebras leads to the Kac–Moody Lie algebras, of importance in integrable systems, theoretical physics, differential geometry and topology, (Kac, 1990).

Lie groups typically arise as symmetry groups of geometric objects or differential equations. A (Lie) group G acts on a manifold M, e.g., Euclidean space, provided $m \mapsto g \cdot m$ for $g \in G, m \in M$, defines a sufficiently smooth invertible map that respects the group multiplication. Lie classified all transformation groups acting on one- and two-dimensional real and complex manifolds, (Olver, 1995). According to Klein's Erlanger Programm, geometric structure is prescribed by an underlying transformation group; thus Euclidean, affine, conformal, projective geometries are based on the eponymous Lie groups. If G acts transitively, then $M = G/H$ is a homogeneous space, obtained by quotienting by a closed Lie subgroup. The group orbits — minimal invariant subsets — form a system of submanifolds, and the invariant functions are constant on orbits. The infinitesimal generators of the group action form a Lie algebra $\mathfrak{g}$ of vector fields tangent to the orbits whose flows generate the group action.

A linear action $\rho\colon G \to \mathrm{GL}(V)$ on a vector space V is known as a representation. Representation theory plays a fundamental role in quantum mechanics since linear symmetries of the Schrödinger equation induce actions on the space of solutions, which decomposes into irreducible representations. The structure of atoms, nuclei, and elementary particles is governed by the representations of particular symmetry groups, (Hamermesh, 1989). Important special functions, e.g., Bessel, hypergeometric, etc., arise as matrix entries of representations of particular Lie groups, (Vilenkin & Klimyk, 1991). The representation theory of the orthogonal group SO(2) leads to trigonometric functions, and hence Fourier analysis, as the simplest case of harmonic analysis on semi-simple Lie groups, (Warner, 1972).

A Lie group acts on its Lie algebra $\mathfrak{g}$ by the adjoint representation and on the dual space $\mathfrak{g}^*$ by the coadjoint representation. The coadjoint orbits are symplectic submanifolds with respect to the natural Lie-Poisson structure on $\mathfrak{g}^*$, and are of importance in classifying representations, (Kirillov, 1999) geometric mechanics, and geometric quantization, (Woodhouse, 1992). The Euler equations of rigid body motion and of fluid mechanics are realized as the Lie-Poisson equations on, respectively, the Lie algebra of the Euclidean group and the infinite-dimensional diffeomorphism group, (Marsden, 1992).

A transformation group is called a symmetry group of a system of differential equations if it maps solutions to solutions. Symmetry groups are effectively computed by solving the infinitesimal symmetry conditions, which form an overdetermined linear system of partial differential equations, usually amenable to automatic solution by computer algebra packages, (Olver, 1993, 1995). Applications include integration of ordinary differential equations, determination of explicit group-invariant (similarity) solutions of partial differential equations, Noether's theorems relating symmetries of variational problems and conservation laws, (Noether, 1918), bifurcation theory, (Golubitsky & Schaeffer, 1985), asymptotics and blow-up, (Barenblatt, 1979), and the design of geometric numerical integration schemes, (Hairer, Lubich & Wanner, 2002).

Classification of differential equations and variational problems admitting a given symmetry group relies on its differential invariants. The simplest examples are the curvature and torsion of space curves, and the mean and Gaussian curvatures of surfaces under the Euclidean group acting on $\mathbb{R}^3$. Cartan's method of moving frames, and its more recent extensions to general Lie group and Lie pseudo-group actions, (Olver, 2001), provides a general mechanism for construction and classification of differential invariants, with applications to differential geometry, the calculus of variations, soliton theory, computer vision, classical invariant theory, and numerical methods.

Modern developments in applications of Lie group methods have proceeded in a variety of directions. General theories of infinite-dimensional Lie groups and algebras, (Kac, 1990), and Lie pseudo-groups, arising in relativity, field theory, fluid mechanics, solitons, and geometry, remain elusive. Higher order or generalized symmetries, in which the infinitesimal generators also depend upon derivative coordinates, first proposed by Noether, (Noether, 1918), have been used to classify integrable (soliton) systems. Recursion operators are used to generate such higher order symmetries, and, via Noether's theorem, higher order conservation laws, (Olver, 1993). Most recursion operators are derived from a pair of compatible Hamiltonian structures, and demonstrate the integrability of biHamiltonian systems. The higher order symmetries also appear in series expansions of Bäcklund transformations in the spectral parameter.

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