NOETHER'S THEOREMS AND SYSTEMS OF CAUCHY-KOVALEVSKAYA TYPE

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ABSTRACT. Noether's Theorem relating symmetries and conservation laws is refined to provide a one-to-one correspondence between nontrivial variational symmetry groups and nontrivial conservation laws for "normal" systems of Euler-Lagrange equations. Besides these, "underdetermined" systems admit nontrivial depencies among the equations, and thus by Noether's Second Theorem admit infinite-dimensional groups of nontrivial variational symmetries with corresponding trivial conservation laws, while "overdetermined" systems have nontrivial integrability conditions, adding further complications.

1. INTRODUCTION AND HISTORY. In 1916, inspired by recent developments in classical mechanics and relativity, E. Noether, [14], formulated and proved two remarkable theorems relating symmetry groups and conservation laws for conservative systems arising from variational principles. The first of these results, justly famous as Noether's Theorem, provides an effective general means of computing conservation laws when used in conjunction with Lie's theory of symmetry groups of differential equations, [19]. With the refinement of Bessel-Hagen, [4], then, all the tools were available to conduct a systematic investigation into the symmetry properties and corresponding conservation laws of the equations of mathematical physics, but, amazingly, such did not occur. On the

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contrary, this important result went unappreciated for 30 years until Hill, [10], popularized a limited, special version of Noether's general theorem among the physics community. As a result, a sizable proportion of subsequent theoretical work in this area has unfortunately been devoted to proving various special cases of Noether's Theorem, followed by rediscoveries of more or less general versions, despite the fact that Noether's original paper provides the most general connection between (generalized) symmetries+ and conservation laws. In contrast, although many authors have looked at the conservation laws associated with geometrical symmetries, only very recently have there been attempts to completely classify conservation laws for some equations of importance in mathematical physics, [3],[15], [16],[23],[25]. Much work remains to be done in this practical direction, and it can be safely said that Noether's Theorem remains the most widely quoted but most under-utilized result in the entire mathematical physics literature.

On the theoretical side, far less attention has been payed to the role played by trivial conservation laws and trivial symmetries in the Noether correspondence, triviality in each case referring to the fact that no new information on the equations or their solutions is provided by the relevant object. Indeed, Noether's Theorem will provide a truly effective means for computing and completely classifying conservation laws only when nontrivial symmetries give rise to nontrivial conservation laws and conversely. For conservation laws, there are, in fact, two distinct kinds of triviality, a fact that causes much of the complication in this aspect of the theory. In the first kind, the conserved density itself vanishes for all solutions of the

+ As far as I can determine, Noether herself was the first to introduce and study generalized symmetries (also mis-named "Lie-Bäcklund transformations", [2]), which have since been rediscovered many times.

system in question; in the second kind the law holds not just for solutions but for all functions. In either case, from the standpoint of the solution set of the system, no new information results. Similarly a trivial (generalized) symmetry is one whose infinitesimal generator vanishes on all solutions, and hence has correspondingly trivial group action. In both cases, symmetries and conservation laws, one is really only interested in equivalence classes of such objects, two of them being equivalent if they differ by a trivial one. The most desirable and effective form of Noether's Theorem, then, would determine a one-to-one correspondence between equivalence classes of conservation laws and equivalence classes of variational symmetries. (As is well known, not every symmetry of the Euler-Lagrange equations gives rise to a conservation law-only those leaving the variational problem itself invariant, called "variational symmetries", are relevant)

For such a result to hold, one must improve certain nondegeneracy conditions on the system in question, including a "local solvability" criterion, which naturally leads one to consider systems for which the Cauchy-Kovalevskaya existence theorem is applicable. The requisite class of differential equations is the normal systems, which are characterized by the existence of at least one noncharacteristic direction at each point, and include practically every system of importance in physical applications. For such systems, one can indeed prove the above refined version of Noether's Theorem. The first person to recognize the importance of such a normality condition is Vinogradov, who in very recent work, [27], [28], [29] uses a closely related condition to prove a similar correspondence using complicated cohomological machinery. The proofs in the present case are entirely elementary and will appear in [18]. (Incidentally, the correct concept of triviality in the case of conservation laws apparently first appeared in [22], but no attempt to incorporate this into the Noether correspondence was made until

Vinogradov's work.)

From this new vantage point, a natural question to appear is the range of validity of this refined version of Noether's theorem, or, to put it another way, how does one characterize normal systems, meaning systems, which under a change of variables are in Kovalevskaya form. Bourlet, [6], was the first to ascertain the existence of "un-normal" systems, but it was not until the under-appreciated work of Finzi, [7] (see also [9]) that the true nature of these systems was revealed. Finzi proved the striking result that a system has the property that every direction is characteristic if and only if it has some kind of "integrability condition". The un-normal systems split naturally into two further distinct classes; the over-determined case, where this integrability condition prescribes further relations amongst lower order derivatives, and the under-determined case, in which there is a nontrivial differential relation among the Euler-Lagrange equations. This latter case is precisely that dealt with by Noether's Second Theorem, which states that such a relation exists if and only if there is a infinite-dimensional group of variational symmetries depending on an arbitrary function. These nontrivial variational symmetries give rise to conservation laws using the original version of Noether's Theorem, but the resulting laws are readily seen to be trivial. Thus under-determined systems of Euler-Lagrange equations are uniquely prescribed by the property that they have nontrivial symmetries giving rise to trivial conservation laws.

The overdetermined case is harder to fathom, and, as yet I know of no counterexample to the refined version of Noether's Theorem relating nontrivial symmetries with nontrivial conservation laws. In particular, does there exist a system of Euler-Lagrange equations which has a nontrivial conservation law coming from a trivial variational symmetry? The answer to this question remains unclear, but as indicated at the end of section 5, if such an example exists it must be quite complicated. (In [8], it is remarked that there is such an example, but the paper referred to, [11], does not actually contain one.)

Lack of space precludes the inclusion of proofs in this paper; these will appear in the forthcoming book, [18]. For the same reason, indications of the vast range of applications of Noether's Theorem for producing new and interesting conservation laws for equations of mathematical, physical and engineering applications must be foregone in this brief summary. Suffice it to say that the methods are completely constructive, to the extent that one could envision symbol-manipulating programs systematically computing conservation laws directly by these techniques. The interested reader can refer to [2], [18], [19] and the other papers referred to in the bibliography for some indication of the range of possibilities.

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2. SOLVABILITY FOR SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS.

We will be concerned with systems of partial differential equations

$$\Delta_\nu(x,u^{(n)}) = 0 \quad , \quad \nu = 1,\dots,q \quad , \tag{1}$$

involving p independent variables $x = (x^1,\dots,x^p) \in X \approx \mathbb{R}^p$ and q dependent variables $u = (u^1,\dots,u^q) \in U \approx \mathbb{R}^q$ defined over an open subset $M \subset X \times U$. Here $u^{(n)}$ denotes all the partial derivatives $u^\alpha_J = \partial^k u^\alpha / \partial x^{j_1} \dots \partial x^{j_k}$ of the u's up to order n, and the functions Δ_ν are smooth or even analytic in their arguments. All our considerations are local, justifying restrictions to Euclidean space, but extensions to vector bundles and smooth manifolds are immediate. By <u>solution</u> $u = f(x)$ we mean a smooth (C^∞) solution for convenience, although with care the differentiability requirements on both the equations and their solutions can be considerably weakened. Since, in our final analysis, the system (1) appears as the Euler-Lagrange equations of some variational problem, we can justifiably restrict our attention to systems having the same number of equations as unknowns, although extensions can, in some instances, be easily envisioned.

In the study of algebraic properties of such a system, including symmetries and conservation laws, a persistent question that arises is the characterization of all differential functions, meaning a function $P(x,u^{(m)})$ depending on x,u and derivatives of u, which vanish for all solutions of the system. The answer to this question rests on certain nondegeneracy hypotheses.

DEFINITION. A system of differential equations (1) is of maximal rank if the Jacobian matrix $(\partial\Delta_\nu/\partial u_J^\alpha)$ with respect to all variables $u^{(n)}$ has rank q whenever $(x,u^{(n)})$ is a solution. The system is locally solvable if for every point $(x_o,u_o^{(n)})$ solving (1) there exists a solution $u=f(x)$ defined in a neighborhood of x_o whose derivatives have the prescribed values $u_o^{(n)}=f^{(n)}(x_o)$.

The maximal rank condition is purely algebraic in nature, which reflects the fact that (1) determines a smooth submanifold of the "jet space" $M^{(n)}$, with coordinates $(x,u^{(n)})$. The local solvability condition addresses the differential properties of system, reflecting the discovery of H. Lewy, [12], of systems of differential equations which have no solutions. For systems of ordinary differential equations, the local solvability problem is the same as the usual initial value problem, whereas for partial differential equations it is of a quite different character from the usual Cauchy or boundary-value problems since the "initial data" $(x_o,u_o^{(n)})$ is prescribed merely at one point x_o rather than on a whole submanifold of the space X. It is closely related to the Riquier existence theory discussed in

Ritt, [21]; also Nirenberg, [13], considers this problem in the context of elliptic systems.

Beside the Lewy-type counter-examples having no solutions, the principal source of systems of differential equations which fail to be locally solvable are those with integrability conditions. For example, the system

$$u_{xx} + v_{xy} + v_x = 0 \quad , \quad u_{xy} + v_{yy} - u_x = 0 \quad , \tag{2}$$

which forms the Euler-Lagrange equations for the variational problem

$$\iint [\tfrac{1}{2}(u_x + v_y)^2 - uv_x]\,dxdy$$

is not locally solvable. Indeed, differentiating the first equation with respect to x, the second with respect to y and subtracting, we find $u_{xx} + v_{xy} = 0$, hence $v_x = 0$, etc. Thus any assignation of initial data $(x^o, y^o, u^o, v^o, u_x^o, u_y^o, v_x^o, v_y^o, u_{xx}^o, u_{xy}^o, u_{yy}^o, v_{xx}^o, v_{xy}^o, v_{yy}^o)$ satisfying (2), but with $v_x^o \neq 0$, will fail the local solvability test.

The appearance of integrability conditions suggests that we should not only look at the system (1) itself, but also all prolongations of it obtained by differentiation. To discuss these we introduce the <u>total derivative</u> operators $D_1, \ldots, D_p$, which differentiate differential functions $P(x, u^{(m)})$ with respect to $x^1, \ldots, x^p$, treating u as a function of x; for instance $D_x(u_{xy} + u^2) = u_{xxy} + 2uu_x$. The m-th prolonged system $\Delta^{(m)}$ corresponding to (1) is the system of differential equations

$$D_J \Delta_\nu = 0 \quad , \quad \nu = 1, \ldots, n \ , \ \#J \leq m$$

obtained by differentiating (1) in all possible ways up to order m, so $D_J = D_{j_1} \cdots D_{j_k}$, where $1 \leq j_\kappa \leq p$, $k = \#J \leq m$.

DEFINITION. A system of differential equations is <u>non-degenerate</u> if it and all its prolongations $\Delta^{(m)}$ are both of maximal rank and locally solvable.

THEOREM 1. Let Δ be a nondegenerate system of differential equations. Let $P(x,u^{(m)})$ be a differential function. Then $P(x,u^{(m)})=0$ for all solutions $u=f(x)$ to Δ if and only if there exist differential operators $\mathcal{D}_\nu = \Sigma Q_J^\nu D_J$ with

$$P = \mathcal{D}_1\Delta_1 + \dots + \mathcal{D}_q\Delta_q \ .$$

The principal tool available to prove the nondegeneracy of a given system is the Cauchy-Kovalevskaya theorem, which is concerned with analytic systems in <u>Kovalevskaya form</u>, [20],

$$\frac{\partial^{n_\nu} u^\nu}{\partial t^{n_\nu}} = \Gamma_\nu(y,t,\widetilde{u^{(n)}}) \ , \ \nu = 1,\dots,q \tag{3}$$

in which $(y,t) = (y^1,\dots,y^{p-1},t)$ are the independent variables, each dependent variable u^α appears up to some order n_α in each of the equations, with the particular derivatives $\partial^{n_\alpha} u^\alpha/\partial t^{n_\alpha}$ appearing only on the left-hand side of the α-th equation. (This is the meaning of the symbol $\widetilde{u^{(n)}}$ in (3).) More generally, an arbitrary system of differential equations (1) can be transformed into one in Kovalevskaya form by a suitable change of variable provided we can find a noncharacteristic direction.

DEFINITION. Let the point $(x_0,u_0^{(n)}) \in M^{(n)}$ be a solution to (1). The system of differential equations Δ is <u>normal</u> at $(x_0,u_0^{(n)})$ if it has at least one non-characteristic direction there. The system is normal if it is normal at every such point.

Recall that a p-tuple $\omega = (\omega^1,\dots,\omega^p)$ determines a <u>characteristic direction</u> (and gives the normal direction to a characteristic surface) if the $q\times q$ matrix $M(\omega) = M(\omega;x_0,u_0^{(n)})$ with polynomial entries

$$M_\alpha^\nu(\omega) = \sum_{\#J = n_\alpha} \omega^J \frac{\partial \Delta_\nu}{\partial u_J^\alpha}(x_o, u_o^{(n)}) \qquad (4)$$

is singular, i.e. $\det M(\omega) = 0$. In (4), the sum is over all multi-indices $J = (j_1, \ldots, j_k)$, $1 \le j_\kappa \le p$, of order $k = \#J$ equal to the maximal order of derivatives of u^α which appear in (1), and $\omega^J = \omega^{j_1}\omega^{j_2}\ldots\omega^{j_k}$. Otherwise, if $\det M(\omega) \neq 0$, ω determines a noncharacteristic direction, and we can apply the Cauchy-Kovalevskaya existence theorem to any noncharacteristic surface through x_o with ω as its normal direction there. Thus the only way for a system to fail to be normal at a point $(x_o, u_o^{(n)})$, whereby there is no way to apply the Cauchy-Kovalevskaya theorem there in any direction, is for the matrix $M(\omega)$ to be singular for all directions ω. Such systems exist; for instance the matrix for (2) is

$$M(\omega) = M(\xi, \eta) = \begin{pmatrix} \xi^2 & \xi\eta \\ \xi\eta & \eta^2 \end{pmatrix}$$

which is singular for all values of $\omega = (\xi, \eta)$, so every direction for (2) is characteristic.

In the case of analytic systems, the Cauchy-Kovalevskaya theorem immediately implies that an analytic, normal system is nondegenerate in the sense of the above definition. Surprisingly, the converse of this statement is also true - an analytic system which is not normal either fails the maximal rank condition or the local solvability condition. (The C^∞ case is more delicate owing to the appearance of Lewy-type examples there, and little is known in general.)

THEOREM 2. Let Δ be an analytic system of differential equations. Then Δ is nondegenerate if and only if it is normal.

The proof rests on a remarkable result due to Finzi, [7], culminating the historical investigations into the algebraic nature of characteristics. In essence, Finzi's theorem says that a system of differential equations is not normal if and only if it has some kind of integrability condition such as (2).

THEOREM 3. Let Δ be a system of differential equations. Then Δ is not normal if and only if there exist homogeneous differential operators. $\mathcal{D}_1,\dots,\mathcal{D}_q$ of some order k such that the linear combination

$$\mathcal{D}_1\Delta_1 + \dots + \mathcal{D}_q\Delta_q = R \tag{5}$$

depends on derivatives of u^α up to order $n_\alpha + k - 1$ only, for $\alpha = 1,\dots,q$.

Since u^α appears in $\Delta_1,\dots,\Delta_q$ to order n_α, if $\mathcal{D}_1,\dots,\mathcal{D}_q$ were any old k-th order differential operators, one would expect u^α to appear in (5) up to order $n_\alpha + k$. Finzi's theorem says that for unnormal system, one can find special operators such that (5) depends on derivatives of order at least one less than might otherwise be expected. (Such indeed was the case with (2) where $k = 1$.)

The operators $\mathcal{D}_\nu$ in (5) are k-th order, so the combination R would appear as a consequence of the equations in the k-th (and higher) order prolongations of Δ. On the other hand, the derivatives of u^α up to order $n_\alpha + k - 1$ already appear in the previous prolongation $\Delta^{(k-1)}$. At this stage there are two possibilities: a) either R vanishes as a consequence of the equations in the previous prolongation, in which case the integrability condition (5) is illusory, or b) $R = 0$ introduces new relations among the $(n_\alpha+k-1)$st order derivatives not appearing in $\Delta^{(k-1)}$, thereby introducing new integrability conditions into the system. These are called respectively under-determined and over-determined systems, and have the formal definition:

DEFINITION. A system of differential equations (1), in which u^α appears up to order n_α, is <u>under-determined</u> if there exist homogeneous k-th order differential operators $\mathcal{D}_1,\ldots,\mathcal{D}_q$, not all zero, such that the combination (5) vanishes as a consequence of the (k-1)st prolongation $\Delta^{(k-1)}$. In the contrary case that the combination (5) does not vanish as a result of $\Delta^{(k-1)}$ the system is called over-determined. (Note that a system can be both under- and over-determined if it has several relations of the form (5) holding.)

For example, the system (2) is over-determined, whereas the closely related system

$$\Delta_1 = u_{xx} + v_{xy} = 0 \quad , \quad \Delta_2 = u_{xy} + v_{yy} = 0 \qquad (6)$$

is under-determined since $D_y\Delta_1 - D_x\Delta_2 = 0$ for all solutions. Finzi's result thus gives a complete trichotomy for analytic systems of differential equations; either the system is normal, in which case it satisfies both nondegeneracy criteria and is in a well-defined sense precisely determined; or it is under-determined and some prolongation violates the maximal rank condition; or it is over-determined and some prolongation fails to be locally solvable. For a purely under-determined system, one can go further and prove that there is at least one arbitrary function in the general solution to the system. Conversely, for an over-determined system, one can prove that it is not possible to prescribe Cauchy data or boundary data arbitrarily and expect to have a solution. Only for normal systems are the natural Cauchy and boundary-value problems well-posed.

3. CONSERVATION LAWS. Consider a system of partial differential equations (1). By a conservation law we mean a divergence expression

$$\text{Div}\ P = D_1P_1 + \cdots + D_pP_p = 0 \qquad (7)$$

with the p-tuple $P = (P_1,\ldots,P_p)$ depending on x,u and derivatives of u , which vanishes for all solutions $u = f(x)$ of the given system. (If one of the independent variables is time t , so that (7) takes the form $D_tT + \mathrm{Div}\, X = 0$, the corresponding entry of P is called the conserved density and has the property that $\int T\, dx$ is constant for all solutions $u = f(x,t)$ which decay sufficiently rapidly as $|x| \to \infty$.) There are two types of trivial conservation laws which hold for any system.

1) If $P = 0$ for all solutions to Δ, then its divergence (7) also vanishes on solutions.

2) If $\mathrm{Div}\, P \equiv 0$ for all functions $u = f(x)$ then (7) automatically holds for all solutions. For example,

$$D_x(u_y) + D_y(-u_x) = 0$$

is a conservation law for any system involving $u = f(x,y)$. Such trivial conservation laws, known as null divergences, [17], have been characterized as "total curls" using the variational complex that arises in the global theory of the calculus of variations on manifolds, [27], [25], [1].

THEOREM 4. A p-tuple $(P_1,\ldots,P_p)$ is a null divergence if and only if there exist differential functions Q_{ij}, $i,j = 1,\ldots,p$ so that

$$\text{i)}\quad Q_{ij} = -Q_{ji}\,, \qquad \text{ii)}\quad P_i = \sum_{j=1}^{p} D_j Q_{ij}\,. \tag{8}$$

A conservation law is trivial if it is the sum of trivial laws of the above two types, i.e. (8) holds for all solutions of the system of differential equations in question. Two conservation laws are equivalent if they differ by a trivial conversation law, $P - \tilde{P} = P_0$, where P_0 is trivial.

Now suppose the system of differential equations is nondegenerate, so that by theorem 1 (7) vanishes for all solutions if and only if there exist function $Q_\nu^J(x,u^{(n)})$ so that

$$\operatorname{Div} P = \Sigma\, Q_\nu^J D_J \Delta_\nu \ . \tag{9}$$

A simple integration by parts shows that there is an equivalent conservation law $\tilde{P}$ in <u>characteristic form</u>

$$\operatorname{Div} P = Q\cdot\Delta = Q_1\Delta_1 + \cdots + Q_q\Delta_q \ , \tag{10}$$

where the characteristic $Q=(Q_1,\ldots,Q_q)$ is given by $Q_\nu = \boldsymbol{\Sigma}(-D)_J Q_\nu^J$. For example,

$$QD_x\Delta = (-D_xQ)\cdot\Delta + D_x(Q\cdot\Delta)$$

and the second term is a trivial conservation law of the first kind. The characteristic Q is uniquely determined only up to the addition of a <u>trivial</u> <u>characteristic</u>, meaning one which vanishes for all solutions $u=f(x)$, owing to an elementary algebraic lemma.

LEMMA 5. If Δ is nondegenerate, and $Q\cdot\Delta = \tilde{Q}\cdot\Delta$, then $Q_\nu - \tilde{Q}_\nu = 0$ for all solutions $u=f(x)$ to Δ.

Two characteristics are equivalent if they differ by a trivial characteristic. For normal systems of differential equations, each conservation laws is, up to equivalence, uniquely determined by its characteristic and vice versa. This result is fundamental to the systematic study of conservation law and their ultimate connection with symmetries in the case of variational problems.

THEOREM 6. Let Δ be a normal, nondegenerate system of partial differential equations. The conservation laws $\operatorname{Div} P=0$, $\operatorname{Div}\tilde{P}=0$ are equivalent if and only if their corresponding characteristics Q and $\tilde{Q}$ are equivalent.

In other words, there is a one-to-one correspondence between (equivalence classes of) conservation laws and (equivalence classes of) characteristics provided the underlying system is normal. The case of "unnormal" systems will be taken up in section 5.

The direct proof of theorem 6 is quite tricky owing to the two types of triviality for conservation laws. Details will appear in [18]; see also [28].

4. SYMMETRIES AND NOETHERS THEOREM. By a geometrical symmetry group of a system of differential equations we mean a local group of transformations acting on the space $M \subset X \times U$ of independent and dependent variables which transforms solutions of the system to other solution. The group transformations $g: (x,u) \to (\tilde{x},\tilde{u})$ act on functions $u = f(x)$ by a point-wise transformation of their graphs. There is thus an induced action on the derivatives $u^{(n)}$ of such functions, called the prolonged group action and denoted $pr^{(n)}g: (x,u^{(n)}) \to (\tilde{x},\tilde{u}^{(n)})$ determined so that if g transforms $u = f(x)$ to $\tilde{u} = \tilde{f}(\tilde{x})$, then it takes the derivatives $u^{(n)} = f^{(n)}(x)$ of u at the point x to the corresponding derivatives $\tilde{u}^{(n)} = \tilde{f}^{(n)}(\tilde{x})$ at the image point $\tilde{x}$.

For a connected, local Lie group of transformations, we can explicitly determine symmetries by looking at their infinitesimal generators, which are vector fields

$$\underline{v} = \sum_{i=1}^{p} \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi_\alpha(x,u) \frac{\partial}{\partial u^\alpha}$$

on M, the corresponding one-parameter group being found by integrating the system of ordinary differential equations

$$\frac{dx^i}{d\epsilon} = \xi^i(x,u) \, , \quad \frac{du^\alpha}{d\epsilon} = \varphi_\alpha(x,u) \tag{11}$$

determining the flow for $\underline{v}$. There is a corresponding prolonged vector field

$$pr^{(n)}\underline{v} = \underline{v} + \sum_{\alpha,J} \varphi_\alpha^J(x,u^{(n)}) \frac{\partial}{\partial u_J^\alpha} \tag{12}$$

on the jet space $M^{(n)}$ generating the one parameter group of prolonged transformations. The coefficient functions of $pr^{(n)}\underline{v}$ have the explicit form

$$\varphi_\alpha^J = D_J Q_\alpha + \sum_{i=1}^{p} \xi^i u^\alpha_{J,i} ,$$

in which $u^\alpha_{J,i} = \partial u^\alpha_J / \partial x^i$, and $Q = (Q_1, \ldots, Q_q)$,

$$Q_\alpha = \varphi_\alpha - \sum_{i=1}^{p} \xi^i \cdot u^\alpha_i \quad , \quad u^\alpha_i = \partial u^\alpha / \partial x^i \quad , \tag{13}$$

is the characteristic of the given symmetry. The basic Lie-Ovsiannikov technique for computing symmetry groups of differential equations hinges on the basic result:

THEOREM 7. A connected Lie group of transformations G forms a symmetry group of the nondegenerate system of differential equations (1) if and only if

$$\mathrm{pr}\,\underline{v}(\Delta_\nu) = 0 \quad , \quad \nu = 1, \ldots, q \tag{14}$$

for all solutions $u = f(x)$ and all infinitesimal generators $\underline{v}$ of the group G .

In practice, (14) forms a large system of elementary differential equations for the coefficients ξ^i , φ_α of $\underline{v}$ whose general solution gives the most general symmetry group of the given system of differential equations. See [5], [19], [18] for examples and applications thereof.

Noether, [14], generalized the notion of symmetry by allowing the coefficients ξ^i , φ_α of the infinitesimal generator to depend on derivatives of u also. The resulting generalized symmetries have the same formula (12) for their prolongations and criterion (14) to be symmetries of some system of differential equations; however the group transformations themselves no longer have an elementary geometric interpretation.

An easy computation shows that any (generalized) vector field $\underline{v}$ can always be replaced by one in evolutionary form

$$\underline{v}_Q = \sum_{\alpha=1}^{q} Q_\alpha \cdot \partial / \partial u^\alpha ,$$

Q being the characteristic (13), which is a symmetry if and only if $\underline{v}$ itself is. In this case, the group transformations are recovered by solving a system of evolution equations

$$\frac{\partial u^\alpha}{\partial \epsilon} = Q_\alpha(x,u^{(n)}) \ , \ \alpha = 1,\dots,q \ , \tag{15}$$

which replaces the flow equations (11) in the geometrical case; here the symmetries are "non-local". Again generalized symmetries can be systematically determined through an analysis of the symmetry criterion (14), [2].

If the characteristic Q of the vector field $\underline{v}$ vanishes on all solutions of the system Δ, then (14) trivially holds and we obtain a trivial symmetry. The corresponding group transformations do not change solutions at all, and hence shed no new light on the system. Two symmetry groups are equivalent if their infinitesimal generators $\underline{v}$ and $\underline{\tilde{v}}$ differ by a trivial symmetry $\underline{v}_0 = \underline{v} - \underline{\tilde{v}}$, and, as with conservation laws, we really need only be interested in nontrivial inequivalent groups.

The connection between symmetry groups and conservation laws holds only for systems with some form of variational structure. In the present discussion, we presume the existence of a variational principle

$$\mathscr{L}[u] = \int_\Omega L(x,u^{(n)})\,dx$$

for which our system of differential equations are the Euler-Lagrange equations

$$\Delta_\nu = E_\nu(L) \equiv \sum_J (-D)_J(\partial L/\partial u_J^\nu) = 0 \ , \ \nu = 1,\dots,q \ . \tag{16}$$

The variational problem is normal or nondegenerate insofar as its Euler-Lagrange equations are. As for symmetries, those of the variational problem are of principal importance, where we define a connected local Lie group of symmetries or generalized symmetries to be a variational symmetry group if for every infinitesimal generator $\underline{v}$,

$$\mathrm{pr}\,\underline{v}(L) + L\cdot\mathrm{Div}\,\xi = \mathrm{Div}\,B \tag{17}$$

for some p-tuple $B = (B_1, \ldots, B_p)$ of differential functions, and where $\mathrm{Div}\, \xi = \Sigma D_i \xi^i$. In essence, the criterion (17), which is due to Bessel-Hagen, [4], says that the variational integral is unchanged under the group action, except for the addition of boundary terms due to the integral of B over $\partial\Omega$. Again, we can replace a generalized vector field $\underline{v}$ by its evolutionary representative without loss of generality, for which (17) simplifies to

$$\mathrm{pr}\, \underline{v}_Q(L) = \sum_{\alpha, J} D_J Q_\alpha \frac{\delta L}{\delta u_J^\alpha} = \mathrm{Div}\, \tilde{B} \tag{18}$$

for some p-tuple $\tilde{B}$. The connection between variational symmetries and symmetries of the Euler-Lagrange equations is as follows:

THEOREM 8. If $\underline{v}$ generates a one-parameter group of variational symmetries for $\mathscr{L} = \int L dx$, then it generates a symmetry group of the Euler-Lagrange equations $E(L) = 0$.

The converse is not true, the principle source of counter-examples being groups of scaling transformations. The easiest means of computing variational symmetries is usually to first determine symmetries of the Euler-Lagrange equations and then determine which groups satisfy the additional variational criterion (17). In particular, a variational symmetry group is trivial if it generates a trivial symmetry group of the Euler-Lagrange equations, and two variational symmetries $\underline{v}$ and $\tilde{\underline{v}}$ are equivalent if their characteristics agree on all solutions of the Euler-Lagrange equations.

Noether's theorem relating symmetry groups and conservation laws arises through an simple integration by parts on (18), which shows that

$$\mathrm{pr}\, \underline{v}_Q(L) = Q \cdot E(L) + \mathrm{Div}\, A \tag{19}$$

for some well-defined p-tuple A depending on Q and L. Combining (18), (19) and (10) we see that the characteristic Q of a variational symmetry is the characteristic of a conservation law and vice-versa.

THEOREM 9. Let $\Delta = E(L) = 0$ be the system of Euler-Lagrange equations for a variational problem. A q-tuple Q is the characteristic of a conservation law for this system if and only if Q is the characteristic of a variational symmetry. In particular, if the Euler-Lagrange equation are normal, non-degenerate, there is a one-to-one correspondence between (equivalence classes of) conservation laws and (equivalence classes of) one-parameter groups of generalized variational symmetries.

In other words, to each nontrivial variational symmetry there corresponds a nontrivial conservation law and conversely. Thus an effective and systematic means of computing conservation laws for a system of Euler-Lagrange equations is to first determine all variational symmetry groups by checking which symmetries of the system satisfy (17) (actually, this can be done directly, [18]) and then computing the resulting conservation laws using (19). Explicit formulae are available to this latter task, but are not particularly enlightening. This is then, for normal systems, our refined version of Noether's Theorem.

5. NOETHER'S SECOND THEOREM. The connection between variational symmetries and conservation laws for unnormal systems is less transparent. Although theorem 9 still yields a variational symmetry for each conservation law and vice versa, there is no guarantee that nontrivial symmetries will result in only trivial conservation laws, or trivial symmetries might give rise to non-trivial laws. For over-determined systems, the complete answer to this problem has yet to be found. Underdetermined systems, however, fall within the scope of Noether's Second Theorem,

which deals with infinite-dimensional groups of variational symmetries depending on arbitrary functions.

THEOREM 10. Let $\mathcal{L}=\int L dx$ be a variational problem with Euler-Lagrange equations $E(L)=0$. This problem admits an infinite-dimensional group of variational symmetries depending on an arbitrary function $h(x)$ if and only if there is a nontrivial dependency between the Euler-Lagrange equation of the form

$$\mathcal{D}_1 E_1(L) + \dots + \mathcal{D}_q E_q(L) \equiv 0 \tag{20}$$

holding identically, the $\mathcal{D}_\nu$'s being differential operators, not all zero.

Note that if the differential operators $\mathcal{D}_\nu$ in (20) are homogeneous, we recover (5) with $R=0$, meaning that the system of Euler-Lagrange equations is under-determined. In the general under-determined case, presuming the (k-1)st prolongation is of maximal rank, we necessarily have $R = \Sigma \tilde{\mathcal{D}}_\nu E_\nu(L)$ for certain (k-1)st order differential operators $\tilde{\mathcal{D}}_\nu$, so (5) changes into (20) by replacing $\mathcal{D}_\nu$ by $\mathcal{D}_\nu - \tilde{\mathcal{D}}_\nu$. In other words, Noether's Second Theorem says that a system of Euler-Lagrange equations is under-determined if and only if the associated variational problem admits an infinite dimensional group of symmetries depending on an arbitrary function.

The proof of the theorem proceeds in outline as follows. One multiplies (20) by an arbitrary function $h(x)$ and integrates by parts, yielding

$$\mathcal{D}_1^*(h)\cdot E_1(L) + \dots + \mathcal{D}_q^*(h) E_q(L) = \text{Div } P \,, \tag{21}$$

where $\mathcal{D}_\nu^*$ is the (formal) L^2-adjoint of the differential operator $\mathcal{D}_\nu$, and P is some well-determined p-tuple of differential functions depending linearly on h and $E(L)$, whose precise form is unimportant. If we set $Q_\nu = \mathcal{D}_\nu^*(f)$, and use (19) we find that $Q=(Q_1,\dots,Q_q)$ is the characteristic of a variational symmetry of $\mathcal{L}$ depending linearly on an arbitrary function $h(x)$. The calculation clearly works in reverse provided the characteristic Q of the group of variational symmetries depends linearly on the arbitrary function; otherwise we can replace it by its Frechet derivative

$$Q_\nu'[u;h_o,h] = \sum_J \frac{\delta Q_\nu}{\delta h_J}[u;h_o]D_J h \qquad (h_J = D_J h)$$

(evaluated at any convenient $h_o(x)$) without losing the symmetry property. The proof is thereby completed - see Noether, [14], for the details.

Now return to the key intermediary relation (21). Defining Q as above, we see that (21) is precisely of the form of a conservation law with characteristic Q, (10). However, P depends linearly on $E(L)$, and hence vanishes whenever $u = f(x)$ is a solution to the Euler-Lagrange equations. In other words $\operatorname{Div} P = 0$ is a <u>trivial</u> conservation law (of the first kind), but whose characteristic Q is, in general, nontrivial and hence corresponds to a nontrivial symmetry group. Since the arbitrary function $h(x)$ apears in both P and Q, we have, in fact, an entire infinite family of trivial conservation laws which arise from nontrivial symmetries whenever the system of Euler-Lagrange equations is underdetermined. In fact, more than this is ture.

THEOREM 11. Let $\mathcal{L}$ be a variational problem. Suppose there exists a nontrivial variational symmetry of $\mathcal{L}$ such that the corresponding conservation law obtained via Noether's Theorem is trivial. Then the Euler-Lagrange equations for $\mathcal{L}$ are underdetermined and there in fact exists an infinite dimensional family of such conservation laws depending on an arbitrary function.

As an example, the system (6) arises from the variational problem $\mathcal{L} = \iint \frac{1}{2}(u_x + v_y)^2 dxdy$, which admits the infinite dimensional group of variational symmetries generated by

$$\underline{v} = h_y(x,y)\partial_u - h_x(x,y)\partial_v$$

for h an arbitrary function, i.e. the one-parameter group

$$(u,v) \to (u+\epsilon h_y, v-\epsilon h_x)$$

leaves $\mathcal{L}$ unchanged. The corresponding family of trivial conservation laws are

$$D_x[h(u_{xy}+v_{yy})] - D_y[h(u_{xx}+v_{xy})] = 0 \ ,$$

with characteristics $(h_x, -h_y)$.

As for overdetermined systems, one might conjecture the possibility of there being systems with trivial symmetry groups corresponding to nontrivial conservation laws. If such an example exists, it must be quite complicated, and I have been unable to produce it. One reason for the complication is the following result, proved with the aid of the homotopy operator for the variational complex, [1], [18].

THEOREM 12. Suppose Δ is a <u>homogeneous</u> system of differential equations, meaning

$$\Delta_\nu(x,\lambda u^{(n)}) = \lambda^\alpha \Delta_\nu(x,u^{(n)}) \ , \ \nu = 1,\dots,q$$

for all x,u , and all $\lambda \in \mathbb{R}$, where α is some nonzero constant. Then every trivial characteristic of a conservation law corresponds to a trivial conservation law.

Thus for homogeneous systems of differential equations, in particular linear systems, nontrivial conservation law necessarily have nontrivial characteristics. Then, by Theorem 12, if a homogeneous system is not under-determined, Noether's Theorem 9 holds as stated for normal systems. In particular, any example exemplifying the above phenomenon must be at least a nonhomogeneous polynomial system!

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