SYMMETRY GROUPS AND CONSERVATION LAWS IN THE FORMAL VARIATIONAL CALCULUS

Peter J. Olver
Mathematical Institute
Oxford University

Abstract: A formal algebraic machinery is developed for handling questions about conservation laws of partial differential equations. This is applied to find infinite series of conservation laws of linear equations, including the Laplace and wave equations. In addition, the relationship of the conservation laws to symmetry groups for the Korteweg-deVries equation and sine Gordon equation is explicated, and an intruiging interrelationship between these two equations is discovered.

Table of Contents

Introduction

The connection between symmetry groups and conservation laws dates back to the fundamental theorem of E. Noether, [34], which states that for each one-parameter symmetry group of a variational problem there is a corresponding conservation law of the Euler equations associated with that problem. Thus, for instance, conservation of energy is a direct consequence of invariance under time translation and conservation of momentum a result of invariance under spatial translations. Noether's theorem was significantly generalized by E. Bessel-Hagen, [2], by the admission of symmetry groups whose transformations depend on the derivatives of the dependent variables, to be called "generalized symmetries" in this paper, and also allowing so-called divergence transformations. The resulting theorem, which is restated here in Theorem 2.14, gives a one-to-one correspondence between symmetries of the variational problem and conservation laws of the Euler equations. Bessel-Hagen's approach does not seem to have been fully exploited in subsequent research.

Much more recently, it has been found that a number of physically interesting nonlinear problems, such as the Korteweg-deVries (KdV) equation and the sine-Gordon equation, can be interpreted as completely integrable Hamiltonian systems, and, as such, possess an infinite series of conservation laws. Indeed, the existence of infinitely many conservation laws of the KdV equation was the first in an on going series of remarkable discoveries of properties of these equations, which now include solitons, inverse scattering, connections with algebraic geometry and so on. The papers [25], [31] and [44] form a good survey of the current research in this area.

The original motivation for this paper came in an attempt to understand these conservation laws in light of Noether's theorem and symmetry group theory. This problem was raised by Ovsjannikov in a conference on symmetry groups in Calgary, Canada in 1974, [42]. It was soon realized that, in the case of the KdV equation, the generalized symmetries responsible for the conservation laws are nothing but the higher order analogues of the Korteweg-deVries equation, due originally to Lax, [23], and Gardner, [11]. In a previous paper, [35], the author showed how a generalization of a recursion relation noticed by A. Lenard could be applied to find symmetry groups of other equations. In the present paper, the relationship of these generalized symmetries to conservation laws is explored in detail. In general, whereas every symmetry of a variational problem is also a symmetry of the Euler equation, the converse is not necessarily true. Usually, an additional criterion, such as that given in Proposition 3.1 for linear equations or in Theorem 4.2 for evolution equations, is necessary for a symmetry of the Euler equation to be a symmetry of the variational problem and, hence, give rise to a conservation law. The primary reason for considering symmetries of the Euler equation is that they are amenable to straightforward (albeit tedious) computation using the methods of Lie and Ovsjannikov, [40]. Chapter 1 briefly recalls how to find the symmetry group of a p. d. e.; the main tool is the formula of Theorem 1.1 for the prolongation of the infinitesimal generators. The generalization of the symmetry group concept then follows in a straightforward fashion, and is shown to be equivalent to the theory of flows defined by evolution equations. Recursion operators were introduced in [35]; here it is shown that for linear p. d. e.'s, each generalized symmetry gives rise to a recursion operator, and hence an infinite family of generalized symmetries. This confusion has resulted in two schools of symmetry group theory: the followers of Ovsjannikov preferring the geometric approach of vector fields, while the follows of W. Miller, [28], favoring the algebraic approach of derivations (recursion operators). Here we reconcile these two approaches and expound on their relationship.

The techniques employed in this paper are in the spirit of Gel'fand and Dikiĭ, [12], and its further developments by the author, [37]. Using the generalized Euler operators, originally due to Kruskal, et. al., [20], a number of important formulae for integration by parts, an Euler product formula, generalized symmetries of variational problems and Noether's theorem are systematically established in the second chapter. This algebraic machinery forms the basis for the applications in the last two chapters of the paper.

The third chapter is devoted to an exposition of the structure of the conservation laws of self-adjoint linear partial differential equations. The additional condition for a linear symmetry to give rise to a conservation law is that the operator $\Delta \mathcal{D}$ be skew-adjoint, where $\Delta(u) = 0$ is the equation and $\mathcal{D}$ is the recursion operator arising from the symmetry. In this way, every odd order symmetry, by suitable modification, gives rise to a conservation law. This is applied to the Laplace and wave equations. The conservation laws arising from point transformational symmetries in the conformal group have been used by Morawetz, [32], [33], in studies of properties of nonlinear wave equations. The conservation laws arising from generalized symmetries are, to the authors knowledge, new and deserving of further investigation.

In the final chapter, the applications to evolution equations in a single spatial variable are discussed. The first section develops some general results for these purposes. For the Korteweg-de Vries equation, these symmetry group methods yield a new derivation of the conservation laws and the Hamiltonian structure, all based on the recursion operator of Lenard. Some of the formulae are new, such as (4.14) and the expression in Corollary 4.12 for the conserved densities of the conservation laws. Next, by application of Miura's transformation ,[29], corresponding results and formulae are found for the modified KdV equation, and also the potential version of the modified KdV equation, which is

$$u_t = u_{xxx} + \frac{\nu}{2} u_x^2 \quad . \tag{0.1}$$

Finally, we consider the sine-Gordon equation, $u_{xt} = \sin(\lambda u)$. There is found to be a remarkable interrelationship between the sine-Gordon and KdV equations. Namely, when $\nu = \frac{3}{2}\lambda^2$ the potential versions of the higher order analogues of the modified KdV equation (of which (0.1) is the first in a series) are all symmetries and give rise to conservation laws of the sine-Gordon equation. Vice versa, the sine-Gordon (and cosine-Gordon) equations give rise to two new conservation laws of (0.1). These in turn provide two new (nonlocal) conserved quantities for the KdV equation itself. The formulae for these two new quantities cannot be explicitly written down, since they involve the inversion of Miura's transformation. We defer the application of these results to solitons and Backlund transformations to a subsequent publication. (See also [51], [52] for related approaches to these questions.)

I would like to express my extreme gratitude to Prof. Jerry Bona for many stimulating discussions about these results. I would also like to thank Prof. Walter Strauss for suggesting the application of these methods to the conservation laws of linear differential equations.

1. Symmetries and Generalized Symmetries

Given a system of partial differential equations, the symmetry group of the system is the "largest" local group of transformations acting on the independent and dependent variables leaving the solution set invariant. It was Lie's pioneering observation that the continuous (as opposed to discrete) symmetries could readily be computed via infinitesimal methods. In the first section a brief review of the elementary theory of jet spaces and group prolongations is provided. More detailed expositions can be found in [15] and [36]. The infinitesimal prolongation formulae motivate the generalization to groups whose transformations depend on the derivatives of the dependent variables.

The final section deals with one method for generating infinite families of generalized symmetries, based on the idea of a recursion operator.

1.1 Jet Spaces and Prolongations

Throughout this paper we will work with the Euclidean spaces $X \simeq \mathbb{R}^p$ with coordinates $x = (x_1, \ldots, x_p)$ and $U \simeq \mathbb{R}$ with coordinate u. The variables $x_1, \ldots, x_p$ should be thought of as the independent variables and u as the single dependent variable in a given partial differential equation. (Generalizations to systems of p.d.e.'s in many dependent variables are immediate, but, will not be needed in the applications considered here.) Multi-index notation will be used throughout.

If $J = (j_1, \ldots, j_p)$, then $|J| = j_1 + \ldots + j_p$, $J! = j_1!\, j_2! \ldots j_p!$, and, whenever $I \leq J$ (which means $i_\nu \leq j_\nu$)

$$\binom{J}{I} = \frac{J!}{I!(J-I)!} .$$

Given a smooth (C^∞) function $u = f(x)$, the J-th partial derivative will be denoted by $u_J = \partial^J f(x)$, where $\partial^J = \partial_1^{j_1} \ldots \partial_p^{j_p}$, and $\partial_i = \partial/\partial x_i$. Note that there are $p_k = \binom{p+k-1}{k}$ different partial derivatives u_J of order $k = |J|$.

Define the <u>k-jet space</u> $J_k = J_k(X \times U)$ to be the Euclidean space of dimension $p + 1 + p_1 + \ldots + p_k$ with coordinates $(x, u^{(k)}) = (x_1, \ldots, x_p; u; \ldots u_J \ldots)$ for $|J| \leq k$. Given a smooth function $u = f(x)$, there is an induced function $u^{(k)} = j_k f(x)$, called the <u>k-jet</u> or k-th <u>prolongation</u> of f, given by $u_J = \partial^J f(x)$. Thus J_k represents the total space of smooth functions together with all their partial derivatives of order $\leq k$. A smooth function $P: J_k \to \mathbb{R}$ will be called a <u>partial differential function</u> (or p.d.f. for short). This is because the subvariety $\{(x, u^{(k)}): P(x, u^{(k)}) = 0\}$ in J_k is a partial differential equation, the solutions of which are those functions $u = f(x)$ such that $P(x, j_k f(x)) = 0$, or, equivalently, the graph of $j_k f$ lies entirely within the subvariety. We will occasionally find it convenient to use the ∞-jet space J_∞, which is the direct limit of the k-th spaces J_k. Thus for $u = f(x)$ smooth, we can form the ∞-jet $u^{(\infty)} = j_\infty f(x)$, which can be identified with the Taylor series of f at x.

Given a p.d.f. $P: J_k \to \mathbb{R}$, the <u>total</u> derivative, D_i, of P with respect to x_i is the p.d.f. $D_i P: J_{k+1} \to \mathbb{R}$ satisfying

$$D_i P(x, j_{k+1} f(x)) = \partial_i [P(x, j_k f(x))]$$

for any smooth function f. It is easily seen that

$$D_i = \partial_i + \sum_J u_{J_i} \partial_J , \tag{1.1}$$

where $\partial_J = \partial / \partial u_J$, $J_i = (j_1, \ldots, j_{i-1}, j_i + 1, j_{i+1}, \ldots, j_p)$ and the sum runs over all p-multi-indices (although for any fixed P only finitely many summands are needed). Given a p-multi-index K, abbreviate $D^K = D_1^{k_1} D_2^{k_2} \ldots D_p^{k_p}$. Also, given a p-tuple $\underline{\underline{P}} = (P_1, \ldots, P_p)$ of p.d.f.'s, define the <u>total divergence</u>

$$\mathrm{Div}\, \underline{\underline{P}} = D_1 P_1 + D_2 P_2 + \ldots + D_p P_p \,. \tag{1.2}$$

Suppose G is a local Lie group of transformations acting on the space $X \times U$ of independent and dependent variables, as in [43] or [38]. G acts locally on smooth functions $u = f(x)$ via transforming their graphs. There is therefore an induced local group action $pr^{(k)}G$ on J_k, called the k-th <u>prolongation</u> of G and given by

$$pr^{(k)} g \cdot (x, j_k f(x)) = (\tilde{x}, j_k (g \cdot f)(\tilde{x})),$$

where $(\tilde{x}, g \cdot f(\tilde{x})) = g \cdot (x, f(x))$. (The reader should consult [10], [38] or [40] for a detailed discussion of this construction.) Similarly, by taking direct limits, we get the ∞-prolonged action $pr^{(\infty)}G$ on J_∞, whose restriction to any J_k is just $pr^{(k)}G$. The prolonged group action in general is exceedingly complicated to analyze directly. However, the infinitesimal generators of the prolonged action have a relatively simple expression, as is proved in [36] and [38].

Theorem 1.1. *Suppose*

$$\underline{\underline{v}} = \xi^1(x,u)\partial_1 + \dots + \xi^p(x,u)\partial_p + \varphi(x,u)\partial_u$$

is a vector field on $X \times U$ *which generates a local one-parameter group* $\exp(t\underline{\underline{v}})$. *The corresponding infinitesimal generator of the* ∞-*prolongation of this group* $\mathrm{pr}^{(\infty)}[\exp(t\underline{\underline{v}})]$ *is the vector field*

$$\mathrm{pr}^{(\infty)}\underline{\underline{v}} = \sum_{i=1}^{p} \xi^i \partial_i + \sum_J \varphi^J \partial_J, \tag{1.3}$$

whose coefficient functions are given by

$$\varphi^J = D^J(\varphi - \sum \xi^i u_i) + \sum \xi^i u_{J_i} . \tag{1.4}$$

(*Here* $u_i = \partial u/\partial x_i$, *and* J_i *is defined above.*)

The finite prolongations $\mathrm{pr}^{(k)}\underline{\underline{v}}$ are found by truncating the sum in (1.3) for $\sum J \leq k$.

Given a p.d.f $Q(x, u^{(k)})$, define the "*vector field*" with *characteristic* Q on J_∞ as

$$\underline{\underline{v}}_Q = \sum_J D^J Q\, \partial_J .$$

For example,

$$\underline{\underline{v}}_{u_i} = D_i - \partial_i ,$$

where we are re-interpreting the total derivative (1.1) as a vector field. Then formula (1.3) can be rewritten in the concise form

$$\mathrm{pr}^{(\infty)}\underline{\underline{v}} = \underline{\underline{v}}_Q + \sum_{i=1}^{p} \xi^i D_i , \tag{1.6}$$

where $Q = \varphi - \sum_{i=1}^{p} \xi^i u_i$.

Note that

the prolongation preserves the Lie bracket:

$$\mathrm{pr}^{(\infty)}[\underline{\underline{v}}, \underline{\underline{w}}] = [\mathrm{pr}^{(\infty)}\underline{\underline{v}}, \mathrm{pr}^{(\infty)}\underline{\underline{w}}]. \tag{1.7}$$

1.2 Symmetry Groups of Partial Differential Equations

A k-th order p. d. e. in p independent and one dependent variables can be thought of as the subvariety $\{P(x, u^{(k)}) = 0\}$ of J_k given by the vanishing of some p. d. f. $P: J_k \to \mathbb{R}$. A point transformational symmetry group of this p. d. e. is a local transformation group G acting on $X \times U$ which transforms solutions to other solutions. The following theorem gives an infinitesimal criterion for a connected group to be the symmetry group of a given p. d. e.

Theorem 1.2. Let G be a connected local group of transformations acting on $X \times U$. Let $P: J_k \to \mathbb{R}$ be a p. d. f. such that:

a) (Nondegeneracy) The gradient of P with respect to all the variables $(x, u^{(k)})$ never vanishes.

b) (Local Solvability) For every point $(x_0, u_0^{(k)}) \in J_k$ such that $P(x_0, u_0^{(k)}) = 0$, there is a solution $u = f(x)$ defined in a neighborhood of x_0 such that $j_k f(x_0) = u_0^{(k)}$.

Then G is a symmetry group of the p. d. e. $P(x, u^{(k)}) = 0$ if and only if

$$\mathrm{pr}^{(k)}\underline{\underline{v}}[P(x, u^{(k)})] = 0 \tag{1.8}$$

whenever $P(x, u^{(k)}) = 0$, for every infinitesimal generator $\underline{\underline{v}}$ of G. (If assumption b) is dropped, then (1.8) is just a sufficient condition for G to be a symmetry group.)

In practice, to find the symmetry group of a given p.d.e., condition (1.8) yields a large number of elementary p.d.e.'s in the coefficient functions of the infinitesimal generator $\underline{\underline{v}}$, whose general solution is the most general infinitesimal symmetry of the given p.d.e. Examples of the calculation of symmetry groups, and proofs of this theorem may be found in [4], [40] and [38]. Note also that formula (1.7) implies that the space of all infinitesimal symmetries forms a Lie algebra, hence we can exponentiate to find a corresponding Lie group. (Here and elsewhere we will ignore deep theoretical questions in the case that this algebra is infinite dimensional, cf. [22]. Usually the integrability will be obvious from the explicit expressions for the infinitesimal generators.)

The goal of this section is to suitably generalize the notion of symmetry beyond the purely point-transformational symmetries discussed so far. Additional references for these generalized symmetries include [1], [21] and [50]. Essentially we will allow the transformations to depend on the derivatives of the dependent variable in addition to the values of the independent and dependent variables themselves. Thus we formally write down a <u>generalized infinitesimal</u> generator as

$$\underline{\underline{v}} = \sum_{i=1}^{p} \xi^i(x, u^{(k)})\partial_i + \varphi(x, u^{(k)})\partial_u , \tag{1.9}$$

where $\xi_1, \ldots, \xi_p$ and φ are arbitrary p.d.f.'s. The prolongation of $\underline{\underline{v}}$ will be given by Theorem 1.1, or, more concisely, by the exact analogue of formula (1.6):

$$\mathrm{pr}^{(\infty)}\underline{\underline{v}} = \underline{\underline{v}}_Q + \sum \xi^i D_i \,, \tag{1.6'}$$

where $Q = \varphi - \sum \xi^i u_i$. We will call $\underline{\underline{v}}$ a <u>generalized infinitesimal symmetry</u> of a p. d. e. $P(x, u^{(k)}) = 0$ if $\mathrm{pr}^{(\infty)}\underline{\underline{v}}[P(x, u^{(k)})] = 0$ whenever $P = 0$. In practice this means that there exist nonvanishing p. d. f.'s R_J such that

$$\mathrm{pr}^{(\infty)}\underline{\underline{v}}(P) = \sum R_J D^J P \,. \tag{1.10}$$

Notice that by formula (1.6'), $\underline{\underline{v}}$ as given by (1.9) is an infinitesimal symmetry of $P = 0$ if and only if $\tilde{\underline{\underline{v}}} = Q\partial_u = (\varphi - \sum \xi^i u_i)\partial_u$ is an infinitesimal symmetry. We will call Q the <u>characteristic</u> of the vector field $\underline{\underline{v}}$, and two vector fields will be called <u>equivalent</u> if they have the same characteristic. The vector field $Q\partial_u$ will be called the <u>standard form</u> of a vector field with characteristic Q. The above remark shows that we may restrict our attention to vector fields in standard form.

What is the group action corresponding to a generalized infinitesimal symmetry? Now $\exp(t\underline{\underline{v}})$ can no longer act on $X \times U$ since the derivatives of functions must also be involved. In the case $\underline{\underline{v}} = Q\partial_u$ is in standard form, we can let $\exp(t\underline{\underline{v}})$ act on $C^\infty = C^\infty(X)$, the space of smooth functions $f: X \to U$, as follows. Consider the initial value problem

$$\frac{\partial u}{\partial t} = Q(x, u^{(k)}) , \qquad u(x,0) = f(x) . \tag{1.11}$$

Throughout this paper, we will always assume that (1.11) has a unique solution for t sufficiently small. (This may require shrinking the space C^{∞} to require some decay properties of f at $\pm\infty$.) The solution of (1.11) is then

$$u = f(x,t) = \exp(t\underline{\underline{v}})f(x),$$

which defines the group action generated by $\underline{\underline{v}}$. In other words, $\underline{\underline{v}} = Q\partial_u$ is the infinitesimal generator of the flow defined by the evolution equation $u_t = Q$. It is straightforwardly checked that this definition of $\exp(t\underline{\underline{v}})$ gives rise to the correct prolongation formula $pr^{(\infty)}(Q\partial_u) = \underline{\underline{v}}_Q$. Moreover, if $\underline{\underline{v}}$ is an infinitesimal symmetry of a p.d.e., then whenever $u = f(x)$ is a solution, so is $\tilde{u} = \exp(t\underline{\underline{v}})f(x)$. (Interpretations of $\exp(t\underline{\underline{v}})$ for nonstandard $\underline{\underline{v}}$'s can be found, but since we will usually assume our vector fields are in standard form, we will not take the trouble to do this here.)

<u>Example 1.3</u>. Consider the case $p = 2$, and let $G = \mathbb{R}$ be the group of translations in the first coordinate:

$$G: (x,y,u) \to (x+t, y, u) , \qquad t \in \mathbb{R} .$$

The infinitesimal generator of G is $\underline{\underline{v}} = \partial_x$. A standard representative of $\underline{\underline{v}}$ is $\tilde{\underline{\underline{v}}} = -u_x\partial_u$. Indeed, to "exponentiate" $\tilde{\underline{\underline{v}}}$, we must solve the evolution equation

$$u_t = -u_x , \qquad u(x, y, 0) = f(x, y) .$$

The solution is

$$F(x, y, t) = \exp(t\tilde{\underline{\underline{v}}})f(x, y) = f(x - t, y) .$$

On the other hand, using the definition of group action on functions given in section 1.1,

$$\exp(t\underline{\underline{v}})f(x, y) = f(x - t, y) .$$

Therefore $\underline{\underline{v}}$ and $\tilde{\underline{\underline{v}}}$ generate the same group action on C^∞, thereby justifying the term equivalent.

Note that if $Q = \sum R_J D^J P$ for some p.d.f.'s R_J, then $Q\partial_u$ is trivially an infinitesimal symmetry of the p.d.e. $P = 0$. Such vector fields will be called <u>trivial</u> symmetries, and two standard infinitesimal symmetries of $P = 0$ will be termed <u>equivalent modulo</u> P if their difference is a trivial symmetry. Of course, we will mainly be interested in nontrivial symmetries of a given p.d.e., i.e., in equivalence classes of symmetries.

(The reader should not confuse these generalized symmetries with contact transformations, cf. [7]. The two concepts are different.)

1.3 Recursion Operators

Given a partial differential equation, we would like to describe all possible generalized infinitesimal symmetries of it. One method is to restrict our attention to symmetries whose characteristic Q depends only on the derivatives of order $\leq k$. Then equation (1.10) yields a large number of p.d.e.'s in Q which are often solvable by elementary methods, and so we obtain all possible symmetries of order $\leq k$. Of course, this method is limited by having to choose k finite before beginning. In this section, a different tactic is used, and, through a device known as a recursion operator, we show how to find infinite families of symmetries.

Definition 1.4. A recursion operator for the p.d.e. $P(x, u^{(k)}) = 0$ is an operator $\mathcal{D}$ acting on the space of p.d.f.'s such that whenever Q is the characteristic of a symmetry of $P = 0$, then $\mathcal{D}Q$ is also the characteristic of a symmetry.

The basic mechanics behind finding recursion operators, as well as a number of examples, is to be found in [35]. Given a p.d.f. P, define the operator

$$\underset{=}{A}_P = \sum_J \partial_J P \cdot D^J . \tag{1.12}$$

Note that $\underset{=}{A}_P(Q) = \underset{=}{v}_Q(P)$ for any P, Q.

Theorem 1.5. *Suppose $\mathcal{D}$ is an operator on the space of p.d.f.'s such that*

$$\underset{=}{A}_P \mathcal{D} = \mathcal{D}' \underset{=}{A}_P \tag{1.13}$$

for some operator $\mathcal{D}'$ of the form $\mathcal{D}' = \sum R_J D^J$. Then $\mathcal{D}$ is a recursion operator for the p.d.e. $P = 0$.

The proof is immediate using (1.10) and the definition of $\underset{=}{A}_P$. In the special case of linear equations, recursion operators are easy to find. In fact, whenever we have a linear symmetry, we automatically have a recursion operator.

Proposition 1.7. *Suppose $\Delta = \sum a_J(x)D^J$ is a linear partial differential operator. Suppose $\underset{=}{v} = Q\partial_u$ is an infinitesimal symmetry of the linear p.d.e.* $\Delta(u) = 0$ *such that* $Q = \mathcal{D}(u) = \sum b_J(x)u_J$. *Then* $\mathcal{D} = \sum b_J(x)D^J$ *is a recursion operator for* $\Delta(u) = 0$.

Proof. Note first that $\underset{=}{A}_{\Delta(u)} = \Delta$, so the symmetry condition (1.10) implies

$$\Delta(Q) = \Delta\mathcal{D}(u) = 0$$

whenever $\Delta(u) = 0$. Therefore if P is any p.d.f., $\Delta\ \mathcal{D}(P) = 0$ whenever $\Delta(P) = 0$. (Just substitute P for u in the preceding.) But this implies that if P is the characteristic of a symmetry, so is $\mathcal{D}(P)$, which proves the proposition.

Example 1.8. Consider the Laplace equation in $\mathbb{R}^p$, so $\Delta = \sum_{i=1}^{p} \partial_i^2$. It is well known that the point transformational symmetry group of $\Delta(u) = 0$ contains the conformal group in $\mathbb{R}^p$. In case $p \geq 3$, the Lie algebra of the symmetry group is spanned by the $\frac{1}{2}(p+1)(p+2)$ vector fields in the conformal algebra

$$
\begin{aligned}
&\underline{\underline{t}}_i = -\partial_i && i = 1, \ldots, p \\
&\underline{\underline{r}}_{ij} = -x_i\partial_j + x_j\partial_i && i < j = 1, \ldots, p \\
&\underline{\underline{d}} = -\sum_{i=1}^{p} x_i\partial_i && \\
&\underline{\underline{i}}_k = -2x_k \sum_{i=1}^{p} x_i\partial_i + \left(\sum_{i=1}^{p} x_i^2\right)\partial_k + (p-2)x_k u\partial_u && k = 1, \ldots, p.
\end{aligned}
\tag{1.14}
$$

and also the trivial subalgebra spanned by

$$\underline{\underline{v}}^0 = u\partial_u \quad \text{and} \quad \underline{\underline{v}}^\alpha = \alpha(x)\partial_u$$

where α is any solution of the Laplace equation. (The superscripts t, r, d, i stand for translation, rotation, dilation and inversion.) The standard representative of the conformal symmetries are

$$
\begin{aligned}
&\tilde{\underline{\underline{t}}}_i = u_i\partial_u , && i = 1, \ldots, p , \\
&\tilde{\underline{\underline{r}}}_{ij} = (x_iu_j - x_ju_i)\partial_u , && i < j = 1, \ldots, p, \\
&\tilde{\underline{\underline{d}}} = \sum_{i=1}^{p} x_iu_i\partial_u , && \\
&\tilde{\underline{\underline{i}}}_k = [2x_k \sum_{i=1}^{p} x_iu_i - \left(\sum_{i=1}^{p} x_i^2\right)u_k + (2-p)x_k u]\partial_u , && k = 1, \ldots, p.
\end{aligned}
\tag{1.15}
$$

Therefore, according to Proposition 1.7, we have the following $\frac{1}{2}(p+1)(p+2)$ recursion operators for the Laplace equation:

$$\begin{aligned}
\mathcal{T}_i &= D_i & & i = 1, \ldots, p, \\
\mathcal{R}_{ij} &= x_i D_j - x_j D_i & & i < j = 1, \ldots, p, \\
\mathcal{D} &= \sum_{i=1}^{p} x_i D_i & & \\
\mathcal{I}_k &= 2x_k \sum_{i=1}^{p} x_i D_i + \Big(\sum_{i=1}^{p} x_i^2\Big) D_k + (p-2)x_k, & & k = 1, \ldots, p.
\end{aligned} \tag{1.15}$$

Since there are $q = \frac{1}{2}(p+1)(p+2)$ recursion operators, there are $\binom{k+q-1}{k}$ different linear symmetries where the characteristic is of degree k, which may be obtained by applying k-fold products of the recursion operators to u. However, some of these are trivial ; for instance

$$\sum_{i=1}^{p} (\mathcal{T}_i)^2 = \Delta .$$

In the case $p = 3$ it can be shown, [5], that there are 55 different second order symmetries, but there are 20 linear relations between these, so there are actually only 35 nontrivial equivalence classes of symmetries. I do not know how many nontrivial equivalence classes of symmetries of degree k there are in general, nor whether every symmetry is linear and can be obtained by using only these recursion operators, although I strongly suspect this to be the case. The reference [5] should be consulted for further details on this problem.

In my opinion, Proposition 1.7 fully delineates the difference that exists between the two schools of symmetry group theory. On the one hand, Ovsjannikov [40], Bluman and Cole [4] and others have viewed infinitesimal symmetries as vector fields on the space of independent and dependent variables, as in the definition in section 1.2. On the other hand, Miller, Kalnins, Boyer [28] and others have usually restricted their attention to linear equations, and have viewed symmetries as linear partial differential operators. The key point is that the operators of Miller's group are just the <u>recursion operators</u> corresponding to the standard representatives of the symmetry groups of Ovsjannikov, et. al. For linear equations, Proposition 1.7 shows that these two concepts coincide. However, for nonlinear equations, as was shown in [35] and will be discussed in detail in Chapter 4, there is a genuine distinction, and a symmetry does not necessarily give rise to a recursion operator. (See also [37] for an equation with symmetries but no recursion operators.)

2. Formal Variational Calculus

The treatment of problems in the calculus of variations by formal algebraic techniques has received new impetus from the investigations of Kruskal, Miura, Gardner and Zabusky, [20], and Gel'fand and Dikii, [12], [13]. In this section, we continue the systematic development of these methods for the general study of conservation laws which was begun in [37].

The key algebraic step is the introduction of the generalized Euler operators. These give a simplified presentation of Gardner's Poisson Bracket, [11], at the end of the first section. The second section introduces the symmetries, generalized symmetries and divergence transformations of Bessel-Hagen. Noether's theorem then reduces to an exercise in integration by parts once the infinitesimal criterion of invariance is established, giving a complete equivalence between symmetries and conservation laws. The final section considers substitution maps, which are used whenever a change of variables for a p. d. e., such as the Miura transformation, is needed.

2.1 The Euler Operators

The most important operator occurring in the formal variational calculus is the Euler operator or variational derivative. Given a variational problem $I(u) = \int_\Omega L(x, u^{(k)})dx$ where L, the Lagrangian of the problem, is a partial differential function and Ω is an open subset of $\mathbb{R}^p$, the (sufficiently smooth) extremals of I satisfy the well-known Euler equation $E(L) = 0$, where $E = \delta/\delta u$ is the Euler operator. Integration by parts shows that

$$E = \sum_J (-D)^J \partial_J . \tag{2.1}$$

For the present purposes, the most important property of E stems from the following resolution. For technical reasons, we must restrict our attention to the differential algebra A consisting of partial differential polynomials in u and its derivatives with arbitrary smooth functions of x as coefficients. First, define the operators

$$F = E \cdot u - 1 = \sum (-D)^J u\partial_J , \tag{2.2}$$

and

$$N = \sum u_J \partial_J . \tag{2.3}$$

Note that N has the effect of multiplying each monomial by its degree. Further define the "total exterior derivative"

$$D : A \otimes \Lambda^k \mathbb{R}^p \to A \otimes \Lambda^{k+1} \mathbb{R}^p \tag{2.4}$$

by

$$D(P \otimes \omega) = \Sigma\, D_i P \otimes e_i \wedge \omega \ ,$$

where $e_1, \ldots, e_p$ is the standard basis of $\mathbb{R}^p$, and $\omega \in \Lambda^k \mathbb{R}^p$.

Theorem 2.1. The following sequence

$$0 \to A \otimes \Lambda^0 \mathbb{R}^p \xrightarrow{D} A \otimes \Lambda^1 \mathbb{R}^p \xrightarrow{D} \cdots \xrightarrow{D} A \otimes \Lambda^{p-1} \mathbb{R}^p \xrightarrow{D} A \xrightarrow{E}$$

$$\xrightarrow{E} A \xrightarrow{F-N} A \xrightarrow{F+1} A \xrightarrow{F-N} A \xrightarrow{F+1} \cdots$$

is exact.

A proof of this result in the case $p = 1$, along with examples, may be found in [39]. This more general case, and a further generalization to differential algebras involving more than one dependent variable may be found in C. Shakiban's thesis, [45].

Corollary 2.2. Suppose L is a p.d.f. Then $L = \mathrm{Div}\, \underline{P}$ for some p-tuple $\underline{P}$ of p.d.f.'s if and only if $E(L) = 0$.

This corollary is well known, cf. [13] or [20], and holds in the more general case of p.d. functions rather than just p.d. polynomials.

Corollary 2.3. *Suppose* Q *is a p.d.p. Then* $Q = 0$ *is the Euler equation for some variational problem with Lagrangian* L, *meaning* $Q = E(L)$, *if and only if* $F(Q) = N(Q)$.

This second corollary is false for p.d. functions; see [45] for a counterexample.

The remainder of this section contains some rather technical formulae that will be needed in the sequel. The reader is advised to just skim this material at first and then proceed to the more relevant applications.

Given a p-multi-index K, define the K-generalized Euler operator

$$E^{(K)} = \sum_{K \le J} \binom{J}{K} (-D)^{J-K} \partial_J . \tag{2.5}$$

These are the multi-dimensional analogues of the generalized Euler operators introduced in [20] and [37]. Note first that for any other p-multi-index J,

$$E^{(K)} D^J = \begin{cases} E^{(K-J)} & K \ge J \\ 0 & \text{otherwise} \end{cases} \tag{2.6}$$

Thus, using Corollary 2.2, the Euler operators $E^{(K)}$ can be used to investigate where a given p.d.f. is a higher order total derivative of some other p.d.f.; we leave the details of this to the reader. Our immediate goal is to establish some general integration by parts formulae.

Lemma 2.4. Given a multi-index J,

$$QD^J P = \sum_{I \leq J} \binom{J}{I} D^I [P(-D)^{J-I} Q] . \tag{2.7}$$

for any p.d.f.'s P and Q.

The proof of (2.7) is a straightforward exercise in induction. Consequently, we have

Lemma 2.5. For any p.d.f.'s P and Q

$$\underline{\underline{v}}_P(Q) = \sum D^K (PE^{(K)} Q) . \tag{2.8}$$

Moreover,

$$\underline{\underline{v}}_P(Q) = PE(Q) + \mathrm{Div}\, \underline{\underline{R}} \tag{2.9}$$

where $\underline{\underline{R}} = (R_1, \ldots, R_p)$, and

$$R_i = \sum \frac{1}{\#K} D^{K_i^-} (PE^{(K)} Q) ,$$

the sum being over all multi-indices K with $k_i > 0$. Here $\#K$ denotes the number of nonzero entries in K, and $K_i^- = (k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_p)$.

The proof is self-evident from the definition of $\underline{\underline{v}}_P$, (1.5).

Next an important product formula for the Euler operators is established.

Proposition 2.6. Let K be a multi-index and P and Q be p.d.f.'s. Then

$$E^{(K)}(P\cdot Q) = \sum_{K\leq J} \binom{J}{K} [E^{(J)}(P)(-D)^J Q + E^{(J)}(Q)(-D)^J P] \tag{2.10}$$

Proof. Note that by Leibnitz' rule, $E^{(K)}(PQ)$ is a sum of two terms, one of which is

$$\sum_{J\geq K} \binom{J}{K} (-D)^{J-K}[Q\,\partial_J P] = \sum_{K\leq J}\ \sum_{L\leq J-K} \binom{J}{K}\binom{J-K}{L} (-D)^L Q(-D)^{J-K-L}\partial_J P$$

$$= \sum_{L} \binom{L+K}{K} (-D)^L Q \sum_{L+K\leq J} \binom{J}{L+K} (-D)^{J-K-L}\partial_J P.$$

The other term is found by interchanging the roles of P and Q. This proves (2.10).

If we define the operator

$$\underline{\underline{B}}_P = \sum_{I} E^{(I)}(P)(-D)^I , \tag{2.11}$$

then formula (2.10), in the case $K = 0$ can be written concisely as

$$E(PQ) = \underline{\underline{B}}_P(Q) + \underline{\underline{B}}_Q(P) . \tag{2.10'}$$

Lemma 2.7. *For any multi-index* K,

$$E^{(K)}\cdot E = (-1)^{|K|} \partial_K \cdot E . \tag{2.12}$$

The proof of (2.12) is very similar to the one-dimensional case, which may be found in [37, Lemma 2.7], and is omitted here for the sake of brevity.

Corollary 2.8. *Given a multi-index* K, *for any p.d.f.* P *in the image of* E

$$\underline{\underline{B}}_{D^K P}(Q) = (-1)^{|K|} \underline{\underline{v}}_{D^K Q}(P) \tag{2.13}$$

for any p.d.f. Q. *In particular, for such* P, $\underline{\underline{B}}_P = \underline{\underline{A}}_P$.

Proof. Let $P = E(L)$ for some L. Then, by (2.6),

$$
\begin{aligned}
B_{D^K P}(Q) &= \sum_I E^{(I)}(D^K E(L))(-D)^I Q \\
&= \sum_{K \leq I} (-1)^{|K|} \partial_{I-K} E(L)\, D^I Q \\
&= (-1)^{|K|} \sum_J D^{J+K} Q\, \partial_J P ,
\end{aligned}
$$

which proves the lemma.

As a consequence of these considerations we have an elementary proof of a generalization of a result of Gel'fand and Dikii, [13; Theorem 1],

Proposition 2.9. *Suppose* K *is a multi-index and* P *and* Q *are p.d.f.'s in the image of* E. *Then*

$$E(QD^K P) = \underline{\underline{v}}_{D^K P}(Q) + (-1)^{|K|} \underline{\underline{v}}_{D^K Q}(P) . \tag{2.14}$$

In particular, if $|K|$ *is odd,*

$$\{P, Q\}^K = [P, Q]^K ,$$

where $\{\ ,\ \}^K$ *is the Gardner-Poisson bracket,*

$$\{P, Q\}^K = E(QD^K P)$$

and $[\ ,\ ]^K$ *is the generalized Lie bracket,*

$$[P, Q]^K = \underline{\underline{v}}_{D^K Q}(P) - \underline{\underline{v}}_{D^K P}(Q) ,$$

which satisfies

$$[\underline{\underline{v}}_{D^K P}, \underline{\underline{v}}_{D^K Q}] = \underline{\underline{v}}_{D^K R} , \quad R = [P, Q]^K ,$$

(the last bracket being the ordinary Lie bracket of vector fields).

The Poisson bracket of Gardner, [11], which leads to the complete integrability of the KdV equation as a Hamiltonian system corresponds to the special case $p = 1$ and $K = k = 1$. Proposition 2.9 therefore provides a whole family of "Poisson brackets" for p.d.f.'s corresponding to each multi-index K with $|K|$ odd.

2.2 Symmetries of Variational Problems

Syppose $I(u) = \int_\Omega L(x, u^{(k)})\,dx$ is a variational problem.

A local Lie group of transformations acting on $X \times U$ will be called a (point transformational) symmetry group of I if for every subdomain $\Omega' \subset \Omega$, every smooth function $\tilde{u} = f(x)$ defined over Ω' and every group element g such that $u = g \cdot f(x)$ is a smooth function defined over a domain $\tilde{\Omega}' \subset D$, then

$$\int_{\tilde{\Omega}'} L(\tilde{x}, j_k(g \cdot f)(\tilde{x}))\,d\tilde{x} = \int_{\Omega'} L(x, j_k f(x))dx \ .$$

In other words the value of integral I over arbitrary subsets of Ω is unchanged by the group action of G. The following result is the standard infinitesimal criterion of invariance for a variational problem; see [14] or [18] for proofs.

Proposition 2.10. Suppose G is a connected local group of transformations acting on $X \times U$. A variational problem with Lagrangian L is invariant under G if and only if

$$\mathrm{pr}^{(\infty)}\underline{\underline{v}}(L) + \left(\sum_{i=1}^{p} D_i \xi^i\right)L = \underline{\underline{v}}_Q(L) + \mathrm{Div}(L\underline{\underline{\xi}}) = 0 \quad (2.15)$$

for every infinitesimal generator $\underline{\underline{v}} = \sum_{i=1}^{p} \xi^i \partial_i + \varphi\,\partial_u$ of G, where

$Q = \phi - \Sigma\, \xi^i u_i$ is the characteristic of $\underline{\underline{v}}$, and

$\underline{\underline{\xi}} = (\xi_1, \ldots, \xi_p)$.

For the case of generalized symmetries, the infinitesimal criterion (2.15) obviously generalizes to when the coefficient functions of the infinitesimal generator $\underline{\underline{v}}$ are p.d.f.'s. Moreover, in the case $\underline{\underline{v}} = Q\partial_u$ is in standard form, then it is easy to check that $\underline{\underline{v}}_Q(L) = 0$ if and only if $I(u)$ is invariant under the one parameter group $\exp(t\underline{\underline{v}})$ defined by the appropriate evolution equation, (1.11). However, owing to the presence of the divergence term in (2.15'), it is not necessarily true that a vector field is a symmetry if and only if its standard form is. To rectify this situation, we make the following generalization of the notion of symmetry, due to E. Bessel-Hagen, [2].

Definition 2.11. A (generalized) vector field $\underline{\underline{v}} = \sum \xi^i \partial_i + \varphi\, \partial_u$ is an infinitesimal symmetry modulo divergence of $I = \int L\, dx$ if there is a p-tuple $\underline{\underline{R}} = (R_1, \ldots, R_p)$ of p.d.f.'s such that

$$\underline{\underline{v}}_Q(L) = \operatorname{Div} \underline{\underline{R}} \quad , \tag{2.16}$$

where $Q = \varphi - \sum \xi^i u_i$.

If we make use of the property of the Euler operator as given in Corollary 2.2, condition (2.16) is equivalent to

$$E(\underset{=}{v}_Q(L)) = 0 . \tag{2.17}$$

Moreover, the integration by parts formula of Lemma 2.5 shows that this in turn is equivalent to

$$E(Q \cdot E(L)) = 0. \tag{2.18}$$

Thus finding the characters of infinitesimal symmetries (mod divergence) of a variational problem with Lagrangian L is equivalent to finding the kernel of the operator $Q \mapsto E(P.Q)$, where $P = E(L)$.

<u>Proposition 2.12.</u> <u>If a vector field</u> $\underset{=}{v}$ <u>is an infinitesimal symmetry modulo divergence of a variational problem with Lagrangian</u> L, <u>then</u> $\underset{=}{v}$ <u>is an infinitesimal symmetry of the associated Euler equation</u> $P = E(L) = 0$.

Proof. Let Q be the character of $\underline{\underline{v}}$. We make use of the Euler product formula (2.10') and also formula (2.14)

$$
\begin{aligned}
0 &= E(QP) \\
&= \underline{\underline{B}}_P(Q) + \underline{\underline{B}}_Q(P) \\
&= \underline{\underline{v}}_Q(P) + \sum_K E^{(K)}(Q)(-D)^K P .
\end{aligned}
$$

This immediately implies that $v_Q(P) = 0$ whenever $P = 0$, hence Q is the character of a symmetry of $P = 0$.

Note that the converse to this proposition is not true. In general, a symmetry of the Euler equation must satisfy some additional restriction in order to be a symmetry of the associated variational problem.

Example 2.13. Consider the Lagrangian $L = \sum_{i=1}^{p} u_i^2$. The Euler equation associated with L is the Laplace equation $\Delta(u) = 0$ in $\mathbb{R}^p$. Here we investigate which of the conformal symmetries (1.14) are also symmetries of the variational problem $I = \int L\,dx$. First the translations and rotations are (ordinary) symmetries of I. For the dilatation subgroup,

$$Q = \sum x_i u_i \text{ , and}$$

$$\underline{\underline{v}}_Q(L) + \mathrm{Div}(L\underline{\underline{\xi}}) = \frac{1}{2}(2-p)L \text{ .}$$

Thus, except for the case $p = 2$, the dilatations are not symmetries of I. However, if we consider

$$\underline{\underline{m}} = \underline{\underline{d}} + (\frac{1}{2}p - 1)u\partial_u \text{ ,}$$

then the new characteristic is $Q' = \sum x_i u_i + (\frac{1}{2}p - 1)u$, and

$$\underline{\underline{v}}_{Q'}(L) = \underline{\underline{v}}_Q(L) + \frac{1}{2}(p-2)L \text{ ,}$$

hence $\underline{\underline{m}}$ is a symmetry of I. Finally, in the case of inversions $\underline{\underline{i}}_k$,

$$\underline{\underline{v}}_Q(L) + \mathrm{Div}(L\underline{\underline{\xi}}) = (2-p)uu_k = D_k(\frac{1}{2}(2-p)u^2) \text{ ,}$$

hence these are symmetries mod divergence of I, but not ordinary symmetries.

2.3 Conservation Laws and Noether's Theorem

Given a partial differential equation $P(x, u^{(k)}) = 0$, a conservation law consists of a p-tuple $\underline{\underline{R}} = (R_1, \ldots, R_p)$ of p.d.f.'s such that the equation

$$\mathrm{Div}\, \underline{\underline{R}} = 0 \tag{2.19}$$

is satisfied for all solutions $u = f(x)$ of the original equation. Under mild nondegeneracy assumptions on P, this is equivalent to the existence of p.d.f.'s S_J such that $\mathrm{Div}\, \underline{\underline{R}} = \sum S_J D^J P$. In the special case one of the independent variables is time t, so that (2.19) is of the form $D_t T + \sum D_i R_i = 0$, then T is called the conserved density. A straightforward application of Stokes' theorem shows that the quantity $\int_{-\infty}^{\infty} T(x, t, u^{(k)})dx$ is independent of t whenever $u = f(x)$ is a solution of $P = 0$, provided the solution $u(x, t)$ decays sufficiently fast for large $|x|$.

A conservation law such that $\mathrm{Div}\, \underline{\underline{R}} = 0$ holds identically is called trivial. For instance, in the case $p = 2$, $\underline{\underline{R}} = (u_y, -u_x)$ gives rise to a trivial conservation law since $D_x(u_y) + D_y(-u_x) = 0$. In general, according to Theorem 2.1, the trivial (polynomial) conservation laws are those in the image of $\mathcal{D} : A \otimes \Lambda^{p-2} \mathbb{R}^p \to A \otimes \Lambda^{p-1} \mathbb{R}^p$.

A second type of triviality occurs when the entries of $\underline{\underline{R}}$ are in the "ideal" generated by P, i.e., $R_i = \sum S_i^J D^J P$ for some nonsingular p.d.f.'s S_i^J. In general, we will only be interested in nontrivial conservation laws.

From now on assume that the p.d.e. under consideration arises from a variational problem, $P = E(L) = 0$. The main thrust of this section is to find for such p.d.e.'s a relationship between symmetry groups and conservation laws. The main theorem in this respect is due to E. Noether, [34]. The generalization stated here is due to E. Bessel-Hagen, [2].

<u>Theorem 2.14.</u> <u>Suppose</u> $I = \int L(x, u^{(k)})\, dx$ <u>is a variational problem with Euler equation</u> $P = E(L) = 0$. <u>If</u> $\underline{\underline{v}} = Q\partial_u$ <u>is a standard infinitesimal symmetry mod divergence of</u> I, <u>so that</u> $\underline{\underline{v}}_Q(L) = \operatorname{Div} \underline{\underline{R}}^0$ <u>for some p-tuple</u> $\underline{\underline{R}}^0$, <u>then the conservation law</u> $\operatorname{Div} \underline{\underline{R}} = 0$ <u>holds for the Euler equation where</u>

$$R_i = R_i^0 + \sum_K \frac{1}{\#K} D^{\bar{K}_i}(QE^{(K)}(L)) \ . \tag{2.20}$$

<u>Conversely, if we have a conservation law</u> $\underline{\underline{R}}$ <u>of the Euler equation, so</u>

$$\operatorname{Div} \underline{\underline{R}} = \sum S_J D^J P$$

<u>for some p.d.f.'s</u> S_J, <u>then</u> $\underline{\underline{v}} = Q\partial_u$ <u>is an infinitesimal symmetry mod divergence of</u> I, <u>where</u>

$$Q = \sum (-D)^J S_J \ . \tag{2.21}$$

The proof of this theorem is nothing more than an exercise in integration by parts using the formula of Lemma 2.5. Note further that as P is <u>nondegenerate</u> in the terminology of Gel'fand-Dikii [12], so that $\sum S_J D^J P = 0$ if and only if each $S_J = 0$, Theorem 2.14 gives a one-to-one correspondence between nontrivial conservation laws and nontrivial infinitesimal symmetries modulo divergence. The characteristic Q of the symmetry corresponding to a given conservation law will also be called the <u>characteristic</u> of that conservation law. Thus we have reduced the problem of finding the conservation laws of a given Euler equation to the problem of finding the symmetries of the corresponding variational problem. Moreover, according to Proposition 2.12, these symmetries occur among the symmetries of the Euler equation itself. It remains to find sufficient conditions for a symmetry of the Euler equation to be one of the variational problem. This will be done in two special cases -- linear equations and evolution equations -- in the following two chapters.

2.4. Substitution Maps

A useful device in the study of partial differential equations is the finding of (nonlinear) substitutions or change of variables that simplify the equation under consideration. Here we formalize this concept under the name of a substitution map.

Definition 2.15. _Suppose_ P _is a p.d.f., then the_ **_substitution map_** $\mathcal{S}$ _associated with_ P _is the map that replaces the variable_ u _in a p.d.f. by the expression_ P. _In other words_

$$\mathcal{S}[Q(x,u,\dots,u_J,\dots)] = Q(x,P,\dots,D^J P,\dots) .$$

Note first that $\mathcal{S}$ is a differential algebra morphism, meaning

$$\begin{aligned} \mathcal{S}(Q+Q') &= \mathcal{S}(Q) + \mathcal{S}(Q'), \\ \mathcal{S}(Q\cdot Q') &= \mathcal{S}(Q)\cdot \mathcal{S}(Q'), \\ \mathcal{S}(D_i Q) &= D_i \mathcal{S}(Q) . \end{aligned} \tag{2.21}$$

As an example, consider the Hopf-Cole transformation relating non-vanishing solutions of the heat equation to solutions of Burgers' equation, cf. [9], [16].

For our purposes, the key observation is that if $E(L) = 0$ for some p.d.f. L, then also $E(\mathcal{S}L) = 0$ for any substitution map $\mathcal{S}$. This is a trivial consequence of Corollary 2.3 and the last identity in (2.21). We will, however, need to find $E(\mathcal{S}L)$ explicitly in terms of $E(L)$ itself.

<u>Proposition 2.16.</u> <u>Suppose</u> P <u>and</u> L <u>are p.d.f.'s, and</u> $\mathcal{S}$ <u>is the substitution map associated with</u> P. <u>Then</u>

$$E(\mathcal{S}L) = \underline{\underline{B}}_P \mathcal{S} E(L), \tag{2.23}$$

<u>where</u> $\underline{\underline{B}}_P$ <u>is the operator defined in (2.11).</u>

<u>Proof.</u> Note first that

$$\partial_J \mathcal{S} L = \sum_I \partial_J D^I P \cdot \mathcal{S} \partial_I L .$$

Therefore

$$\begin{aligned}
E(\mathcal{S}L) &= \sum_J (-D)^J \sum_I \partial_J D^I P \cdot \mathcal{S}\partial_I L \\
&= \sum_J \sum_I \sum_{L \le J} \binom{J}{L} (-D)^L \partial_J D^I P \cdot \mathcal{S}(-D)^{J-L} \partial_I L \\
&= \sum_I \sum_K \sum_L \binom{L+K}{L} (-D)^L \partial_{L+K} D^I P \, \mathcal{S}(-D)^K \partial_I L \\
&= \sum_I \sum_K E^{(K)} D^I P \cdot \mathcal{S}(-D)^K \partial_I L \\
&= \sum_I \sum_{I \le K} E^{(K-I)} P (-D)^{K-I} \mathcal{S}(-D)^I \partial_I L \\
&= \underline{\underline{B}}_P \mathcal{S} E(L) \ .
\end{aligned}$$

We have made use of (2.6).

As an illustration, suppose $P = u^2 + \mu u_x$. Then $\underline{\underline{B}}_P = 2u - \mu D$, so (2.23) says

$$E(\mathcal{S} L) = (2u - \mu D) \ \mathcal{S}E(L) \ .$$

This means that if we replace u by P in a variational problem, then the new Euler equation is obtained by replacing u by P in the original Euler equation and then applying the operator $2u - \mu D$.

3. Applications to Linear Equations

In the first chapter it was shown how every linear symmetry of a linear p. d. e. gives rise to a recursion operator, and hence an infinite family of symmetries. Here we restrict our attention to self-adjoint linear p. d. e.'s, and find necessary and sufficient conditions for a linear symmetry to give rise to a conservation law via Noether's theorem. This in turn implies that the number of independent nontrivial quadratic conservation laws is the same as the number of independent nontrivial odd-order symmetries. In addition for each recursion operator, there is also a recursion formula for both linear and nonlinear conservation laws, given in Theorem 3.3. The second section of this chapter relates the theory to the Laplace and wave equations, and displays some of the higher order conservation laws obtainable.

3.1. Conservation Laws of Self-Adjoint Equations

In this chapter we will study linear p. d. e.'s in p independent and one dependent variable, which will be written as

$$\Delta(u) = (\sum a_J(x)D^J)u = \sum a_J(x)u_J = 0 . \qquad (3.1)$$

It will always be assumed that (3.1) arises as the Euler equation of a variational problem with quadratic Lagrangian; equivalently, Δ is

a self-adjoint differential operator.

Suppose that Q is the characteristic of a conservation law of (3.1). Recall that this is equivalent to Q being a solution of

$$E(Q\Delta(u)) = 0. \qquad (3.2)$$

For the moment, we restrict our attention to quadratic conservation laws, so $Q = \mathcal{D}(u)$ where $\mathcal{D}$ is a linear partial differential operator. By Proposition 2.12, Q is the characteristic of a symmetry of (3.1). Moreover, Proposition 1.7 shows that this is true only when $\mathcal{D}$ is a recursion operator for (3.1). Therefore $\mathcal{D}$ must satisfy $\Delta\mathcal{D} = \hat{\mathcal{D}}\Delta$ for some other p. d. o. $\hat{\mathcal{D}}$. The next result gives the necessary and sufficient condition for a recursion operator of (3.1) to give rise to a conservation law.

Proposition 3.1. A linear p. d. f. $Q = \mathcal{D}(u)$ is the characterisitc of a conservation law of the self-adjoint linear p. d. e. $\Delta(u) = 0$ if and only if the operator $\Delta\mathcal{D}$ is skew-adjoint, which means

$$\Delta\mathcal{D} = -\mathcal{D}^{*}\Delta \ . \tag{3.3}$$

Proof. Using the Euler product formula (2.10') and formula (2.13), we have

$$E(Q\Delta(u)) = \underline{\underline{v}}_Q[\Delta(u)] + \underline{\underline{B}}_Q[\Delta(u)] \ .$$

Now

$$\underline{\underline{v}}_Q[\Delta(u)] = \Delta\underline{\underline{v}}_Q(u) = \Delta(Q) = \Delta\mathcal{D}(u).$$

On the other hand, if $\mathcal{D} = \sum_J b_J D^J$, then

$$\begin{aligned}
\underline{\underline{B}}_Q[\Delta(u)] &= \sum_K (-D)^K \Delta(u) E^{(K)}\Big(\sum_J b_J u_J\Big) \\
&= \sum_K \sum_{K\leq J} \binom{J}{K} (-D)^K \Delta(u)(-D)^J b_J \\
&= \sum_J (-D)^J [b_J \Delta(u)] \\
&= \mathcal{D}^{*}\Delta(u) \ .
\end{aligned}$$

This completes the proof.

Corollary 3.2. *If the operators $\mathcal{D}$ and Δ commute, and Δ is self-adjoint, then $Q = \mathcal{D}(u)$ is the characteristic of a conservation law of $\Delta(u) = 0$ if and only if $\mathcal{D}$ is skew-adjoint.*

This gives a complete characterizationof the generalized linear symmetries which give rise to conservation laws. The next theorem shows that every recursion operator for the symmetries of (3.1) yields also a recursion operator for conservation laws.

Theorem 3.3. *Suppose the linear partial differential operator $\mathcal{D}$ is a recursion operator for the self-adjoint linear p.d.e. $\Delta(u) = 0$. This means that $\Delta \mathcal{D} = \hat{\mathcal{D}} \Delta$ for some linear partial differential operator $\hat{\mathcal{D}}$. Let $\mathcal{S}$ be the substitution map associated with $\mathcal{D}(u)$. Then if Q is any (not necessarily linear) characteristic of a conservation law of $\Delta(u) = 0$, then so is $\tilde{Q} = \hat{\mathcal{D}}^* \mathcal{S} Q$. In other words, $\mathcal{R} = \hat{\mathcal{D}}^* \mathcal{S}$ is a recursion operator for characteristics of conservation laws. In particular, if $\mathcal{D}$ satisfies (3.3), then $\mathcal{R} = -\mathcal{D}\mathcal{S}$.*

Proof. First note that

$$\mathcal{S}\,\Delta(u) = \Delta(\mathcal{S}u) = \Delta\mathcal{D}(u) = \hat{\mathcal{D}}\Delta(u).$$

Therefore, using the remarks preceding Proposition 2.16 ,

$$\begin{aligned} E[\tilde{Q}\Delta(u)] &= E[\hat{\mathcal{D}}^* \mathcal{S} Q \cdot \Delta(u)] \\ &= E[\mathcal{S} Q \cdot \hat{\mathcal{D}} \Delta(u)] \\ &= E[\mathcal{S}(Q \cdot \Delta(u))] \\ &= 0 . \end{aligned}$$

Theorem 3.3 implies that given a single conservation law, and given a single linear symmetry $\mathcal{D}(u)$ of $\Delta(u) = 0$, we can construct an infinite family of conservation laws by repeated application of the recursion operator $\mathcal{R}$ associated with $\mathcal{D}$ to the characteristic of the given law. For the case of quadratic conservation laws, much more can be said. Let $\mathcal{U}$ denote the algebra of all linear recursion operators of $\Delta(u) = 0$, so

$$\mathcal{U} = \{ \mathcal{D} : \Delta\mathcal{D} = \hat{\mathcal{D}}\Delta \}.$$

Note that the operator $\hat{\mathcal{D}}$ is uniquely determined. Furthermore, if $\mathcal{D} \in \mathcal{U}$, then also $\hat{\mathcal{D}}^* \in \mathcal{U}$ because

$$\Delta\hat{\mathcal{D}}^* = (\hat{\mathcal{D}}\Delta)^* = (\Delta\mathcal{D})^* = \mathcal{D}^*\Delta .$$

Moreover, the operator $\frac{1}{2}(\mathcal{D} - \hat{\mathcal{D}}^*)$ satisfies the condition of Theorem 3.1 since

$$\Delta(\mathcal{D} - \hat{\mathcal{D}}^*) = (\hat{\mathcal{D}} - \mathcal{D}^*)\Delta = -(\mathcal{D} - \hat{\mathcal{D}}^*)^*\Delta .$$

Therefore to each operator $\mathcal{D} \in \mathcal{U}$ there is a well-defined operator $\mathcal{D}^c = \frac{1}{2}(\mathcal{D} - \hat{\mathcal{D}}^*)$ which gives rise to a conservation law of

$\Delta(u) = 0$. Examination of the highest order terms of these operators shows that if $\mathcal{D} = \mathcal{D}_n$ + lower order terms, and $\Delta = \Delta_m$ + lower order terms, where n and m indicate the respective orders, then

$$\Delta\mathcal{D} = \Delta_m \mathcal{D}_n + \ell.o.t. = \mathcal{D}_n \Delta_m + \ell.o.t.,$$

hence $\hat{\mathcal{D}} = \mathcal{D}_n + \ell.o.t.$. Moreover, $\hat{\mathcal{D}}^* = (-1)^n \mathcal{D}_n + \ell.o.t.$, hence $\mathcal{D}^c$ is an n-th order operator if and only if n is odd. This proves the following.

Theorem 3.4. Let $\mathcal{U}$ denote the symmetry algebra of the self-adjoint linear p.d.e. $\Delta(u) = 0$, and let $\mathcal{U}^c$ denote the subspace of operators giving rise to conservation laws, i.e., satisfying (3.3). There is a well-defined map $\mathcal{D} \to \mathcal{D}^c = \frac{1}{2}(\mathcal{D} - \hat{\mathcal{D}}^*)$ from $\mathcal{U} \to \mathcal{U}^c$. Let L denote the collection of all linearly independent leading terms of operators in $\mathcal{U}$, and let L^0 denote the leading terms of odd order. If $\beta: L \to \mathcal{U}$ is a map that assigns to each leading term a basis element $\beta(\mathcal{D}_n) = \mathcal{D}_n + \ell.o.t.$ of $\mathcal{U}$, then the map $\beta^c: L^0 \to \mathcal{U}^c$ defined by $\beta^c(\mathcal{D}_n) = \beta(\mathcal{D}_n)^c$ gives a basis of $\mathcal{U}_c$ and thus a complete set of independent quadratic conservation laws of $\Delta(u) = 0$.

In a less precise fashion, the "number" of different odd order leading terms of linear recursion operators of $\Delta(u) = 0$ is the same as the number of independent quadratic conservation laws. To further rule out trivial conservation laws in which the characteristic is a function of $\Delta(u)$, $\mathcal{D} = \mathcal{D}'\Delta$ for some $\mathcal{D}'$, we let $\{\Delta\}$ denote the left ideal of all differential operators of the form $\mathcal{D}'\Delta$, and consider the coset space $\mathcal{U}/\{\Delta\}$. These correspond to nontrivial recursion operators and each different odd order leading terms in $\mathcal{U}/\{\Delta\}$ will give rise to an independent, nontrivial conservation law.

It often turns out that $\mathcal{U}$ can be realized as the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of point transformational symmetries, cf. [19]. In this case, if we let $\mathcal{D}_1, \ldots, \mathcal{D}_n$ be a basis of the first order recursion operators associated with a basis of $\mathfrak{g}$, then by the Poincaré-Birkhoff-Witt theorem a basis of $\mathcal{U}$ is given by all the products $\mathcal{D}_{i_1} \mathcal{D}_{i_2} \cdots \mathcal{D}_{i_k}$ with $i_1 \le i_2 \le \ldots \le i_k$. We can assume that each $\mathcal{D}_i$ satisfies $\mathcal{D}_i = -\hat{\mathcal{D}}_i^*$. (Otherwise replace $\mathcal{D}_i$ by $\mathcal{D}_i^c$.) Then according to Theorem 3.4, a basis of $\mathcal{U}^c$ is given by all the expressions

$$\frac{1}{2}\left[\mathcal{D}_{i_1} \mathcal{D}_{i_2} \cdots \mathcal{D}_{i_k} + \mathcal{D}_{i_k} \mathcal{D}_{i_{k-1}} \cdots \mathcal{D}_{i_1} \right] \tag{3.4}$$

corresponding to $i_1 \le i_2 \le \cdots \le i_k$ and k odd.

3.2. Applications to the Laplace and Wave Equations

To indicate the possible applications of the theory of the preceding section, we show how to construct families of conservation laws of the Laplace and wave equations. These conservation laws have been used by Morawetz, [32], [33] and Strauss, [46], to give existence theorems and decay properties for solutions of certain nonlinear wave equations. The author suspects that the conservation laws found here will shed additional light into these questions.

For the Laplace equation, $\Delta = \sum_{i=1}^{p} D_i^2$, which is, of course, self-adjoint. In Example 1.8, the symmetry group of the Laplace equation was found. The standard forms of the infinitesimal generators are given in (1.15) and the corresponding recursion operators in (1.16). Now

$$\begin{aligned} \Delta \mathcal{T}_i &= \mathcal{T}_i \Delta , \\ \Delta \mathcal{R}_{ij} &= \mathcal{R}_{ij} \Delta , \\ \Delta \mathcal{D} &= (\mathcal{D} + 2)\Delta , \\ \Delta \mathcal{I}_k &= (\mathcal{I}_k + 4x_k)\Delta . \end{aligned} \tag{3.5}$$

The operators $\mathcal{T}_i$ and $\mathcal{R}_{ij}$ are skew-adjoint and so, by Corollary 3.2, give rise to conservation laws. Next,

$$(\mathcal{D} + 2)^* = -\mathcal{D} + 2 - p$$

so by the discussion preceding Theorem 3.4, the operator

$$\mathcal{M} = \mathcal{D} + \frac{1}{2}p - 1 = \sum_{i=1}^{p} x_i D_i + \frac{1}{2}p - 1$$

gives rise to a conservation law. Finally, $(\mathcal{I}_k + 4x_k)^* = -\mathcal{I}_k$, hence $\mathcal{I}_k$ satisfies condition (3.3) and yields another conservation law. The characteristics of these conservation laws are

$$\begin{aligned} T_i &= u_i \,, \\ R_{ij} &= x_i u_j - x_j u_i \,, \\ M &= \sum_{i=1}^{p} x_i u_i + (\tfrac{1}{2}p - 1)u \,, \\ I_k &= 2x_k \sum_{i=1}^{p} x_i u_i - (\sum_{i=1}^{p} x_i^2)u_k + (2-p)x_k u \end{aligned} \tag{3.6}$$

The actual forms of the conservation laws $\sum D_i R_i = 0$ are rather complicated; we refer the reader to [46] for details.

Since the symmetry algebra $\mathcal{U}$ of the Laplace equation is the universal enveloping algebra of the conformal group, for each k-fold product, with k an odd integer, of the $q = \frac{1}{2}(p+1)(p+2)$ operators

$$\begin{aligned} \mathcal{T}_i &= D_i && i = 1,\ldots,p \\ \mathcal{R}_{ij} &= x_i D_j - x_j D_i \,, && i < j = 1,\ldots,p \\ \mathcal{M} &= \sum_{i=1}^{p} x_i D_i + (\tfrac{1}{2}p - 1) \,, && \\ \mathcal{I}_k &= 2x_k \sum_{i=1}^{p} x_i D_i - (\sum_{i=1}^{p} x_i^2)D_i + (2-p)x_k && k = 1,\ldots,p \end{aligned} \tag{3.7}$$

there is a conservation law of the Laplace equation, which are given by expression (3.4).

In the case $p = 3$, with independent variables x, y, z, we have the following table of "representative" third-order operators giving rise to characteristics of conservation laws. (To obtain all possible characteristics, just permute the variables x, y, z in all possible ways.) The first class of operators comes from the recursion Theorem 3.3, and are of the form $\mathcal{D}\mathcal{D}'\mathcal{D}$ for each pair of operators $\mathcal{D}$, $\mathcal{D}'$ chosen from the list (3.7). The second class arises from formula (3.4) and are not obtainable by recursion.

TABLE 1

Class 1 Third-Order Recursion Operators for Conservation Laws of the Laplace Equation in Three Dimensions

Type	Operator
I_x	T_x^3
I_{xy}	$T_x T_y T_x$
II_{xy}	$T_x R_{xy} T_x$
II_{xyz}	$T_x R_{yz} T_x$
III_x	$T_x M T_x$
IV_x	$T_x I_x T_x$
IV_{xy}	$T_x I_y T_x$
V_{xy}	$R_{xy} T_x R_{xy}$
V_{xyz}	$R_{xy} T_z R_{xy}$
VI_{xy}	R_{xy}^3
VI_{xyz}	$R_{xy} R_{yz} R_{xy}$
VII_{xy}	$R_{xy} M R_{xy}$
$VIII_{xy}$	$R_{xy} I_x R_{xy}$
$VIII_{xyz}$	$R_{xy} I_z R_{xy}$
IX_x	$M T_x M$

X_{xy}	$M R_{xy} M$
XI	M^3
XII_x	$M I_x M$
$XIII_x$	$I_x T_x I_x$
$XIII_{xy}$	$I_x T_y I_x$
XIV_{xy}	$I_x R_{xy} I_x$
XIV_{xyz}	$I_x R_{yz} I_x$
XV_x	$I_x M I_x$
XVI_x	I_x^3
XVI_{xy}	$I_x I_y I_z$

TABLE 2

Class 2 Third-Order Recursion Operators for Conservation Laws of the Laplace Equation in Three Dimensions

Type	Operator
I'_{xyz}	$T_x T_y T_z$
II'_{xy}	$T_x R_{xy} T_y - \frac{1}{2}(T_x^2 + T_y^2)$
II'_{xyz}	$T_x R_{xy} T_z - \frac{1}{2} T_y T_z$
III'_{xy}	$T_x M T_y$
IV'_{xy}	$T_x I_x T_y + R_{xy} T_x - M T_y$
IV'_{xyz}	$T_x I_y T_z + T_x R_{yz} + T_z R_{xy}$
V'_{xyz}	$R_{xy} T_x R_{yz} + \frac{1}{2}(T_y R_{yz} - T_x R_{xz})$
V''_{xyz}	$R_{xy} T_y R_{yz} - \frac{1}{2}(T_x R_{yz} + T_y R_{xz} + T_z R_{xy})$
VI'_{xyz}	$R_{xy} R_{yz} R_{xz} - \frac{1}{2}(R_{xy}^2 - R_{yz}^2 + R_{xz}^2)$
VII'_{xyz}	$R_{xy} M R_{yz} - \frac{1}{2} R_{xz} M$
$VIII'_{xyz}$	$R_{xy} I_x R_{yz} + \frac{1}{2}(R_{yz} I_y - R_{xz} I_x)$
$VIII''_{xyz}$	$R_{xy} I_y R_{yz} - \frac{1}{2}(R_{xy} I_z + R_{xz} I_y + R_{xy} I_z)$
$XIII'_{xy}$	$I_x T_x I_y + M I_y + R_{xy} I_x$
$XIII'_{xyz}$	$I_x T_y I_z + R_{xy} I_z + R_{yz} I_x$
XIV'_{xy}	$I_x R_{xy} I_y - \frac{1}{2}(I_x^2 + I_y^2)$
XIV'_{xyz}	$I_x R_{xy} I_z - \frac{1}{2} I_y I_z$
XV'_{xy}	$I_x M I_y$
XVI'_{xyz}	$I_x I_y I_z$

Type	Operator
XVI''_{xy}	$MR_{xy}T_x + \frac{1}{2}(R_{xy}T_x + T_yM)$
$XVII'_{xyz}$	$MR_{xy}T_z + \frac{1}{2}R_{xy}T_z$
$XVIII'_{xy}$	$I_xR_{xy}T_x + R_{xy}M + \frac{1}{2}(T_yI_x - T_xI_y)$
$XVIII''_{xy}$	$I_xR_{xy}T_y + R_{xy}^2 - \frac{1}{2}(T_xI_x + I_yT_y)$
$XVIII'_{xyz}$	$I_xR_{xy}T_z + R_{xy}R_{xz} - \frac{1}{2}I_yT_z$
$XVIII''_{xyz}$	$I_xR_{yz}T_x + R_{yz}M$
$XVIII'''_{xyz}$	$I_xR_{yz}T_z + R_{yz}R_{xz} - \frac{1}{2}T_yI_z$
XIX'_{x}	$I_xMT_x + I_xT_x + M^2 + M$
XIX'_{xy}	$I_xMT_y + MR_{xy} + \frac{1}{2}(I_xT_y + T_yI_x)$
XX'_{xy}	$I_xMR_{xy} + \frac{1}{2}(I_xR_{xy} + MI_y)$
XX'_{xyz}	$I_xMR_{yz} + \frac{1}{2}I_xR_{yz}$

Next, consider the wave equation

$$\square(u) = (D_t^2 - \Delta)\, u = u_{tt} - \sum_{i=1}^{p} u_{ii} = 0 , \tag{3.8}$$

in p spatial variables. Note that formally replacing t by $\sqrt{-1}\, x_{p+1}$ changes the wave equation into the Laplace equation in $p+1$ dimensions. Therefore the symmetry algebra of (3.8) will be given by replacing x_{p+1} by $-\sqrt{-1}\, t$ in the symmetry algebra of the Laplace equation. The recursion operators are

$$\begin{aligned}
\mathcal{T}_i &= D_i & & i = 1, \dots, p \\
\mathcal{T}_t &= D_t & & \\
\mathcal{R}_{ij} &= x_i D_j - x_j D_i & & i < j = 1, \dots, p \\
\mathcal{R}_{it} &= x_i D_t + t D_i & & i = 1, \dots, p \\
\mathcal{M} &= \sum x_i D_i + t D_t + \tfrac{1}{2}(p-1) & & \\
\mathcal{I}_k &= 2x_k\left(\sum x_i D_i + t D_t\right) + \left(t^2 - \sum x_i^2\right) D_k + (1-p) x_k, & & k = 1, \dots, p \\
\mathcal{I}_t &= 2t\left(\sum x_i D_i\right) + \left(t^2 + \sum x_i^2\right) D_t - (1-p) t & &
\end{aligned} \tag{3.9}$$

Again, recursion operators corresponding to conservation laws are found by taking k-fold products of the operators in (3. 9) and using formula (3.4). As examples we explicitly display some of the more elementary conserved densities for the ase $p = 3$.

TABLE 3

Some Conserved Densities of Conservation Laws with Third-Order Characteristics of the Wave Equation in Three Spatial Dimensions

Type	Characteristic	- Density[+]
I_x	u_{xxx}	$u_{xx}u_{xt}$
I_{xy}	u_{xxy}	$u_{xy}u_{xt}$
I'_{xyz}	u_{xyz}	$u_{xy}u_{zt}$
I_{xt}	u_{xxt}	$\frac{1}{2}(u_{xx}u_{tt} + u_{xt}^2)$
I_{tx}	u_{xtt}	$u_{xt}u_{tt}$
I_t	u_{ttt}	$\frac{1}{2}(u_{xt}^2 + u_{yt}^2 + u_{zt}^2 + u_{tt}^2)$
II_{xy}	$xu_{xxy} - yu_{xxx} + u_{xy}$	$u_{xt}(xu_{xy} - yu_{xx})$
II'_{xy}	$xu_{xyy} - yu_{xxy} + \frac{1}{2}u_{yy} - \frac{1}{2}u_{xx}$	$u_{xx}(yu_{yt} + \frac{1}{2}u_t) - u_{yy}(xu_{xt} + \frac{1}{2}u_t)$
II_{xyz}	$yu_{xxz} - zu_{xxy}$	$u_{xx}(yu_{zt} - zu_{yt})$
II'_{xyz}	$xu_{xyz} - yu_{xxz} + \frac{1}{2}u_{yz}$	$u_{zt}(xu_{xy} - yu_{xx} + \frac{1}{2}u_y)$
II_{xt}	$xu_{xxt} + tu_{xxx} + u_{xt}$	$\frac{1}{2}x(u_{xx}^2 + u_{xy}^2 + u_{xz}^2 + u_{xt}^2) + tu_{xx}u_{xt}$
II_{xyt}	$yu_{xxt} + tu_{xxy}$	$\frac{1}{2}y(u_{xt}^2 + u_{xx}^2 + u_{xy}^2 + u_{xz}^2) + tu_{xy}u_{xt}$

[+] This is the negative of the density.

II'_{xyt}	$xu_{xyt} - yu_{xxt} + \frac{1}{2}u_{yt}$	$u_{xt}(xu_{yt} - yu_{xt})$
II'_{xty}	$xu_{xyt} + tu_{xxy} + \frac{1}{2}u_{yt}$	$u_{xt}(xu_{yt} + tu_{xy})$
II'_{xyzt}	$yu_{xzt} - zu_{xyt}$	$u_{xt}(yu_{zt} - zu_{yt})$
II'_{xytz}	$yu_{xzt} + tu_{xyz}$	$u_{xt}(yu_{zt} + tu_{yz})$
II'_{xt}	$xu_{xtt} + tu_{xxt} + \frac{1}{2}u_{xx} + \frac{1}{2}u_{tt}$	$u_{xt}(xu_{tt} + tu_{xt} + \frac{1}{2}u_{x}) - \frac{1}{2}u_{t}u_{tt}$
II'_{xyt}	$yu_{xtt} + tu_{xyt} + \frac{1}{2}u_{xy}$	$u_{xt}(yu_{tt} + tu_{yt})$
II_{txy}	$xu_{ytt} - yu_{xtt}$	$u_{tt}(xu_{yt} - yu_{xt})$
II_{tx}	$xu_{ttt} + tu_{xtt} + u_{xt}$	$\frac{1}{2}x(u_{tt}^2 + u_{xt}^2 + u_{yt}^2 + u_{zt}^2) + tu_{xt}u_{tt}$

4. Applications to Nonlinear Equations

For the most part, this section will be devoted to the study of evolution equations in a single spatial variable.

The necessary and sufficient condition for a symmetry of the equation to give rise to a conservation law is found. The remaining three sections show how to apply these methods to the Korteweg-de Vries, modified KdV and sine-Gordon equations. This finally results in the remarkable connection among all three of these alluded to in the Introduction. The exposition here is self-contained and constitutes an independent way of arriving at many previously known formulae as well as some new ones.

4.1. Conservation Laws of Evolution Equations

Consider the evolution equation

$$u_t = K(x, u, u_x, \ldots), \tag{4.1}$$

where K is a p.d.f. in u and its derivatives with respect to a single indpendent variable x. In this section, the theory relating conservation laws and symmetries of Chapter 2 will be applied to equations of this special type. Note that by successive substitutions using (4.1) and its derivatives, any p.d.f. in the derivatives of u with respect to both x and t is equivalent to a p.d.f. in just the x derivatives of u, which we call an x-p.d.f. for short. Therefore, every conservation law is equivalent to an identity

$$D_t T + DX = \sum T_j D^j (u_t - K), \tag{4.2}$$

where $D = D_x$, $T_j = \partial T/\partial u_j$ and $u_j = \partial^j u/\partial x^j$. Integrating (4.2) by parts shows that

$$D_t T + DX' = Q(u_t - K), \tag{4.3}$$

for some X', and $Q = E(T)$ will be called the characteristic of the conservation law.

As it stands, (4.1) can never be the Euler equation of a variational problem since the term u_t cannot occur in such an equation. (When K is a polynomial, Corollary 2.3 can be invoked to show this.) There are two basic tricks available for transforming (4.1) into the Euler equation of some variational problem:

a) If (4.1) is differentiated with respect to x, we obtain

$$u_{xt} = DK. \tag{4.4}$$

Now if $DK = E(L)$ for some x-p.d.f. L, then (4.4) is the Euler equation corresponding to the Lagrangian $\tilde{L} = \frac{1}{2}u_x u_t + L$.

b) If the substitution map S_0 associated with u_x is applied to (4.1), we obtain

$$u_{xt} = K_0 = S_0 K = K(x, u_x, u_{xx}, \ldots). \tag{4.5}$$

Again, if $K_0 = E(L)$, then (4.5) is the Euler equation corresponding to $\tilde{L} = \frac{1}{2}u_x u_t + L$.

For example, the Korteweg-deVries equation is $u_t = u_{xxx} + \nu u u_x$, where ν is a constant. In this case, trick b) yields $u_{xt} = u_{xxxx} + \nu u_x u_{xx}$, which is the Euler equation for $\tilde{L} = \frac{1}{2} u_x u_t + \frac{1}{2} u_{xx}^2 - \frac{\nu}{6} u_x^3$, cf. [48]. Here trick a) would not work. On the other hand, trick a) can be used for the "potential KdV" equation $u_t = u_{xxx} + \frac{1}{2}\nu u_x^2$, yielding the same Lagrangian. Note that the x-derivative of a solution of the potential KdV equation is a solution of the KdV equation. The next proposition shows that this is no accident; whenever an evolution equation is amenable to trick b), there is a "potential version" of that equation, the x-derivatives of whose solutions are solutions of the original equation.

Proposition 4.1. *Suppose* $P = E(L)$ *is a p.d.f. in one independent variable. Then* $P = DK$ *for some* K *if and only if* $P = \mathcal{S}_0\tilde{K}$ *for some* $\tilde{K}$. *In this case* $u_t = \tilde{K}$ *is called the potential version of* $u_t = K$.

The proof may be found in [37]. Therefore, we will be considering equations of the form

$$u_{xt} = P = E(L) \tag{4.6}$$

for some L, since these include evolution equations amenable to one of the two tricks, (which correspond to those P which do not depend on u). Consider what happens to the conservation laws of (4.1) under the two tricks. If trick a) is used, then (4.2) implies that $D_tT + DX = 0$ is a conservation law of (4.4) if and only if $T_0 = 0$, i.e., T does not depend on u. Conversely, any conservation law of (4.4) is also one for (4.1). As for trick b), if (4.2) gives a conservation law of (4.1), then $D_tT_0 + D_xX_0 = 0$, where $T_0 = \mathcal{S}_0 T$, $X_0 = \mathcal{S}_0 X$, is a conservation law of (4.5). Vice versa, any conservation law of (4.5) in which the density does not depend on u comes from a conservation law of (4.1) in this fashion. Thus we must consider conservation laws of (4.6) in which the density and flux are x-p.d.f.'s and the density is independent of u.

Theorem 4.2. An x-p.d.f. Q is the characteristic of a conservation law of (4.6) of the above type if and only if

i) The vector field $\underline{\underline{v}} = Q\partial_u$ is an infinitesimal symmetry of (4.6),

and ii)
$$-DQ = E(T). \tag{4.7}$$

Proof. The necessity of condition i) follows from Proposition 2.12. Now the conservation law must take the form

$$D_t T + DX = \sum_j T_j D^{j-1}(u_{xt} - P) .$$

Integrating by parts with respect to x shows that the characteristic Q satisfies $Q = \sum (-D)^{j-1} T_j$, hence (4.7) is also necessary.

Conversely, suppose Q satisfies conditions i) and ii) for some x-p.d.f. T. (It can be shown, [37], that T can be chosen so as not to depend on u.) Then $D_t T$ is the x-derivative of some flux X provided

$$0 = E(\sum T_{j+1} D^j P)$$

$$= E(P\ Q)$$

$$= \underline{\underline{B}}_P(Q) + \underline{\underline{B}}_Q(P)$$

$$= \underline{\underline{v}}_Q(P) - \sum D^j P \partial_{j+1} DQ ,$$

where we have used (2.10'), (2.12) and (2.13). However, since by Proposition 4.1, DQ does not depend on u, the infinitesimal criterion of invariance of condition i) is

$$\sum D^j P \partial_{j+1} D Q = D_t D Q = \underline{\underline{v}}_Q(P),$$

which proves the result.

Corollary 4.3. *Suppose $u_t = K$ is an evolution equation with $K_0 = \mathcal{S}_0 K = E(L)$ for some p.d.f. L. Then an x-p.d.f. Q is the characteristic of a conservation law with density T if and only if*

i) *The flows $u_t = K$ and $u_t = DQ$ commute.*

ii) $Q = E(T)$.

Proof. Applying $\mathcal{S}_0$ to (4.3) yields

$$D_t T_0 + D_x X_0' = Q_0(u_{xt} - K_0),$$

hence $Q_0 = \mathcal{S}_0 Q$ is the characteristic of the conservation law of (4.5) with density T_0. Proposition 2.16 implies that

$$E(T_0) = D \mathcal{S}_0 E(T) = DQ_0,$$

so condition ii) of the theorem is fulfilled. Also, the infinitesimal criterion for i) is

$$\underline{\underline{v}}_{DQ}(K) = \underline{\underline{v}}_K(DQ).$$

Applying $\mathcal{S}_0$ to this equation shows

$$\underline{\underline{v}}_{Q_0}(K_0) = \sum D^j K_0 \frac{\partial}{\partial u_{j+1}} (DQ_0) .$$

This is precisely the infinitesimal criterion for condition i) of the theorem.

Corollary 4.4. *Suppose $u_t = K$ is an evolution equation so that $DK = E(L)$ for some L. Then an x-p.d.f. Q is the characteristic of a conservation law with density T depending only on $x, t, u_x, u_{xx}, \ldots$ if and only if*

i) *The flows $u_t = K$ and $u_t = D^{-1}Q$ commute,*

and ii) $Q = E(T)$.

Note that Proposition 4.1 implies the existence of $D^{-1}Q$. The proof of Corollary 4.4 is similar to that of Corollary 4.3.

Actually, the restriction that T doesn't depend on u is not vital. Note first that if $DK = E(L)$, then by Proposition 4.1, K does not depend on u. Given Q, not necessarily in the image of D, we can interpret the "flow" $u_t = D^{-1}Q$ as the solution of the integro-differential equation $u_t = \int_a^x Q\,dx$, where we must fix $u_t(a,t) = 0$ for some a (e.g., $a = -\infty$). (Equivalently, it is the solution of $u_{xt} = Q$ satisfying an appropriate boundary condition.) Now for K as above, the infinitesimal criterion that the "flows" $u_t = K$ and $u_t = D^{-1}Q$ commute is now

$$\sum_{j=1}^{\infty} D^{j-1} Q \frac{\partial}{\partial u_j} DK = \sum_{j=0}^{\infty} D^j K \frac{\partial Q}{\partial u_j} .$$

We could also write this as

$$\underset{=}{v}_{D^{-1} Q}(K) = \underset{=}{v}_{K}(D^{-1} Q) .$$

Under these generalizations, Corollary 4.4 holds without the restriction on T.

4.2. The Korteweg-deVries Equation

The Korteweg-deVries equation is

$$u_t = K_1 = u_{xxx} + \nu u u_x , \tag{4.8}$$

and is well-known to possess an infinite family of conservation laws, [30]. In this section we will reinterpret these conservation laws in light of the group-theoretic methods developed in the preceding section.

Lax [23] and Gardner [11] discovered an infinite family of "higher order KdV equations," which proved to be a set of mutually commuting flows described by evolution equations. Subsequently, A. Lenard noticed that the operator

$$\mathcal{D} = D^2 + \frac{2}{3} \nu u + \frac{1}{3} \nu u_x D^{-1} \tag{4.9}$$

is a recursion operator for this family of evolution equations, so that the j-th KdV equation is given by $u_t = K_j = \mathcal{D}^j(u_x)$. In [35], the author gave another proof that $\mathcal{D}$ is a recursion operator for (4.8), by showing that it formally satisfied the condition of Theorem 1.5. To be completely rigorous, the operator D^{-1} must be explained. Note that $D^{-1}P$ is not well-defined for every p.d.f. P; it makes sense only when $P \in \operatorname{im} D = \ker E$. Also $D^{-1}P$ is defined only up to an additive constant. For the present purposes, since the p.d.f.'s under consideration are always polynomial, we can uniquely define $D^{-1}P$ for $P \in \operatorname{im} D$ without constant term to be the polynomial Q without constant term such that $DQ = P$. It is now necessary to show that $K_j = \mathcal{D}^j(u_x)$ is actually defined for each positive integer j. Thus we must show that $K_j \in \operatorname{im} D$ for each j.

Lemma 4.5. *If* $K_j \in \operatorname{im} D$, *then* $uK_j \in \operatorname{im} D$.

Proof. Note first that the formal adjoint of the operator $\mathcal{D}$ satisfies

$$\mathcal{D}^* = D^2 + \frac{2}{3}\nu u - \frac{1}{3}\nu D^{-1} u_x = D^{-1}\mathcal{D} D . \tag{4.10}$$

(The operator $D^{-1}u_x$ takes a p.d.f., multiplies it by u_x, and then applies D^{-1}.) Now by assumption, $K_i = \mathcal{D}^i(u_x)$ for $i = 1, \ldots, k$, hence $J_i = D^{-1}K_i = \mathcal{D}^{*i}(u)$ also exist for $i = 1, \ldots, k$. Therefore,

$$\begin{aligned} E(uK_j) &= E(u\ \mathcal{D}^j(u_x)) \\ &= E(u_x\ \mathcal{D}^{*j}(u)) \\ &= E(u_x D^{-1}K_j) \\ &= -E(uK_j) , \end{aligned}$$

the last step being a simple integration by parts. Thus $E(uK_j) = 0$, hence by Corollary 2.2, $uK_j \in \operatorname{im} D$.

Lemma 4.6. *If K_j and uK_j are both in $\operatorname{im} D$, then $K_{j+1} \in \operatorname{im} D$.*

Proof. Note first that formally

$$\mathcal{D} = D(D + \frac{1}{3}\nu u D^{-1} + \frac{1}{3}\nu D^{-1} u).$$

Therefore,

$$K_{j+1} = \mathcal{D} K_j = D[(D + \frac{1}{3}\nu u D^{-1} + \frac{1}{3}\nu D^{-1} u)K_j] \qquad (4.11)$$

which exists by hypothesis.

Theorem 4.7. *For each positive integer j, $K_j = \mathcal{D}^j(u_x)$ is well-defined. Moreover,*

$$E(K_i D^{-1} K_j) = 0 \qquad (4.12)$$

for any i and j.

Proof. The first statement is proved by induction using the preceding two lemmata. To show (4.12), by (4.10)

$$\begin{aligned} E(K_i D^{-1} K_j) &= E(\mathcal{D}^i(u_x)\, \mathcal{D}^{*j}(u)) \\ &= E(u\, \mathcal{D}^{i+j}(u_x)) \\ &= E(uK_{i+j}) = 0 . \end{aligned}$$

Note that if we let $J_j = D^{-1}K_j$, then (4.12) becomes

$$E(J_j DJ_i) = 0 \ , \tag{4.13}$$

hence the J_i's are in involution with respect to Gardner's Poisson bracket, (2.14). They key result for the application of Corollary 4.3 to prove that the K_j's give rise to generalized symmetries and to conservation laws is the following.

Lemma 4.8. *For each positive integer* j, $K_j \in \operatorname{im} DE$. *(Equivalently,* $J_j \in \operatorname{im} E$.)

The lemma can be inferred from the results of Lax [24], or McKean & van Moerbeke, [27]. However, owing to the key position of this result, the author feels obligated to present an independent and self-contained proof based on the machinery of this paper. Unfortunately, the proof is rather technical, and borders on an exercise in computational dexterity using the full power of our formal calculus of variations. Therefore, we will first state and prove the main consequences of Lemma 4.8, and relegate its proof to the end of this section.

Corollary 4.9. *The flows* $u_t = K_i$ *and* $u_t = K_j$ *for any* i *and* j *commute.*

Proof. Applying formula (2.14) to (4.13) shows

$$\underline{\underline{v}}_i(J_j) - \underline{\underline{v}}_j(J_i) = 0 ,$$

where $\underline{\underline{v}}_i = \underline{\underline{v}}_{K_i}$. Moreover, since the $\underline{\underline{v}}$'s commute with D,

$$\underline{\underline{v}}_i(K_j) = \underline{\underline{v}}_j(K_i) ,$$

which is the infinitesimal criterion for commutation of flows.

Corollary 4.10. For each integer j,

$$E(J_{j+1}) = \frac{1}{3}\nu(2j+3)J_j \tag{4.14}$$

Proof. Since $\mathcal{D}^*$ is the recursion operator for the J's

$$\begin{aligned} E(J_{j+1}) &= \frac{1}{3}\nu E[(2u - D^{-1}u_x)J_j] \\ &= \frac{2}{3}\nu E(uJ_j) - \frac{1}{3}E^{(1)}(u_x J_j). \end{aligned}$$

To proceed, we need the following lemma.

Lemma 4.11. If $P \in \operatorname{im} E$, then

$$E^{(1)}(u_x P) = (1-M)P , \tag{4.15}$$

where $M = \sum_{i=1}^{\infty} iu_i \partial/\partial u_i$ is the operator that multiplies each monomial by the number of x-derivatives in it.

Proof. By Proposition 2.6 and (2.12),

$$E^{(1)}(u_x P) = \sum_{i=1}^{\infty} (-1)^{i-1}\, i\, u_i E^{(i)}(P) + P$$
$$= -MP + P .$$

Returning to the proof of (4.14), by (2.2), and (4.15),

$$E(J_{j+1}) = \frac{1}{3}\nu[\, 2N + M + 1]J_j .$$

It must therefore be shown that

$$(2N + M)J_j = (2j + 2)J_j . \tag{4.16}$$

(Note that $N + \frac{1}{2}M$ is the operator that multiplies each monomial by its rank, as defined in [20].) This follows by induction from the easily established commutation formula

$$(2N + M)\ \mathcal{D}^* = \mathcal{D}^*(2N + M + 2) . \tag{4.17}$$

This proves (4.14).

Combining Corollaries 4.9 and 4.10 with Corollary 4.3 yields the result.

Corollary 4.12. For each pair of integers i and j, $\frac{1}{3}\nu(2i + 3)J_i$ is the characteristic of a conservation law of $u_t = K_j$ with density J_{i+1}.

We now turn to a proof of Lemma 4.8. This will be proved by induction on j, so assume $J_j \in \text{im } E$. By Corollary 2.3, we must show that

$$E(uJ_{j+1}) = (N+1)J_{j+1} \quad . \tag{4.18}$$

Since $J_{j+1} = \mathcal{D}^* J_j$, integration by parts yields

$$E(uJ_{j+1}) = E[(u_{xx} + \frac{2}{3}\nu u^2 - \frac{1}{3}\nu u D^{-1} u_x)J_j] \quad .$$

<u>Lemma 4.13.</u> <u>If</u> $P \in \text{im } E$, <u>then</u> :

$$\text{i)} \quad E(u_{xx}P) = (\underline{\underline{v}}_2 + D^2)P \tag{4.19}$$

<u>where</u> $\underline{\underline{v}}_2 = \underline{\underline{v}}_{u_{xx}}$;

$$\text{ii)} \quad E(u^2P) = (\underline{\underline{v}}_1 + 2u)P \quad , \tag{4.20}$$

<u>where</u> $\underline{\underline{v}}_1 = \underline{\underline{v}}_{u^2}$.

iii) $D^{-1}u_xP$ <u>exists and</u>

$$E(uD^{-1}u_xP) = (u + D^{-1}u_x - \alpha)P \quad , \tag{4.21}$$

<u>where</u> α <u>is the operator</u>

$$\alpha = \sum_{\ell=1}^{\infty} \left(\sum_{k=0}^{\ell=1} \binom{\ell}{k+1} u_k u_{\ell-k} \right) \frac{\partial}{\partial u_\ell} \tag{4.22}$$

<u>Proof.</u> These three statements are direct consequences of the Euler product formula (2.10). Here we will prove (4.21), leaving the other two to the reader. Now

$$E(uD^{-1}(u_xP)) = D^{-1}(u_xP) + \sum_{k=0}^{\infty} (-1)^k u_k E^{(k+1)}(u_xP).$$

Moreover, again by (2.10), and also (2.12),

$$E^{(k+1)}(u_xP) = \delta_{k0}P + \sum_{\ell=k+1}^{\infty} (-1)^{k+1}\binom{\ell}{k+1} u_{\ell-k}\frac{\partial}{\partial u_\ell} .$$

(Here δ_{k0} is the Kronecker symbol.) Combining these proves (4.21).

Thus by Lemma 4.13,

$$E(uJ_{j+1}) = (\mathcal{D}^* + \gamma)J_j ,$$

where

$$\gamma = \underline{\underline{v}}_2 + \frac{1}{3}\nu u + \frac{2}{3}\nu\beta , \qquad (4.23)$$

$$\beta = \underline{\underline{v}}_1 + \frac{1}{2}\alpha .$$

We are thus left with the task of proving

$$\gamma J_j = N\mathcal{D}^* J_j .$$

This we again do by induction, which amounts to proving that

$$[\gamma, \mathcal{D}^*] = [N\mathcal{D}^*, \mathcal{D}^*].$$

On the one hand,

$$\begin{aligned}[N\mathcal{D}^*, \mathcal{D}^*] &= \frac{1}{3}\nu(2u - D^{-1}u_x)\mathcal{D}^* \\ &= \frac{2}{3}\nu uD^2 - \frac{1}{3}\nu u_x D + \frac{1}{3}\nu u_{xx} + \frac{4}{9}\nu^2u^2 \\ &\quad - \frac{1}{3}\nu D^{-1}(u_{xxx} + \frac{2}{3}\nu uu_x) - \frac{2}{9}\nu^2uD^{-1}u_x + \frac{1}{9}\nu^2(D^{-1}u_x)^2.\end{aligned}$$

To compute the other commutator, we first show

<u>Lemma 4.14.</u> <u>The following identities formally hold:</u>

$$[\underline{\underline{v}}_2, u] = u_{xx}$$

$$[\underline{\underline{v}}_2, D^{-1}u_x] = D^{-1}u_{xxx}$$

$$[u, D^2] = -2u_x D - u_{xx}$$

$$[u, D^{-1}u_x] = uD^{-1}u_x - D^{-1}uu_x$$

$$[\beta, D^2] = uD^2 + \frac{1}{2}u_x D$$

$$[\beta, u] = u^2$$

$$[\beta, D^{-1}u_x] = 2D^{-1}uu_x \ .$$

<u>Proof.</u> The first four are left to the reader. Note that $\beta = \sum \beta_j \frac{\partial}{\partial u_j}$, where the β_j's are quadratic differential polynomials which satisfy the recursion relation $D\beta_j + \frac{1}{2}uu_{j+1} = \beta_{j+1}$.

Therefore,

$$
\begin{aligned}
D\cdot\beta &= \sum_j (D\beta_j + \beta_j D)\cdot\frac{\partial}{\partial u_j} \\
&= \sum_j (D\beta_j - \beta_{j+1})\frac{\partial}{\partial u_j} + \beta D \qquad (4.24)
\end{aligned}
$$

$$D\beta = -\frac{1}{2}uD + \beta D \ .$$

The fifth identity follows from a second application of (4. 24). The seventh identity holds since $\beta_0 = u^2$. Finally, from (4.24), formally

$$D^{-1}\beta = \beta D^{-1} + \frac{1}{2} D^{-1} u .$$

Therefore

$$\begin{aligned} \beta D^{-1} u_x &= D^{-1} \beta u_x - \frac{1}{2} D^{-1} u u_x \\ &= D^{-1} u_x \beta + 2D^{-1} u u_x , \end{aligned}$$

because $\beta_1 = \frac{5}{2} u u_x$.

Now we use the lemma to compute

$$\begin{aligned} [\gamma, \vartheta^*] &= [\underline{\underline{v}}_2 + \frac{1}{3}\nu u + \frac{2}{3}\nu \beta, D^2 + \frac{2}{3}\nu u - \frac{1}{3}\nu D^{-1} u_x] \\ &= \frac{2}{3}\nu u D^2 - \frac{1}{3}\nu u_x D + \frac{1}{3}\nu u_{xx} + \frac{4}{9}\nu^2 u^2 \\ &\quad - \frac{1}{3}\nu D^{-1}(u_{xxx} + \nu u u_x) - \frac{1}{9}\nu^2 u D^{-1} u_x . \end{aligned}$$

To complete the proof, upon comparison of the two commutators, we need

$$D^{-1} u_x D^{-1} u_x = u D^{-1} u_x - D^{-1} u u_x .$$

However, this is readily shown by applying D to both sides. This finishes the proof of Lemma 4.8.

4.3. The Modified Korteweg-de Vries Equation

Next we apply the theory to study the symmetries and conservation laws of the Modified KdV (MKdV) equation

$$u_t = \widetilde{K}_1 = u_{xxx} + \nu u^2 u_x . \tag{4.25}$$

Miura, [29], noticed a remarkable nonlinear transformation between the solutions of the MKdV equation and those of the KdV equation. Namely, if $u = f(x)$ is a solution of (4.25), then

$$v = u^2 + \mu u_x , \quad \text{where} \quad \mu = \sqrt{-6/\nu} , \tag{4.26}$$

is a solution of (4.8). In fact, if $\mathcal{S}$ is the substitution map associated with $u^2 + \mu u_x$, then

$$(\mu D + 2u)(u_t - \widetilde{K}_1) = \mathcal{S}(u_t - K_1) .$$

In [35] it was formally verified that

$$\widetilde{\mathcal{D}} = D^2 + \frac{2}{3}\nu u^2 + \frac{2}{3}\nu u_x D^{-1} u \tag{4.27}$$

is a recursion operator for (4.25), hence $\widetilde{K}_j = \widetilde{\mathcal{D}}^j(u_x)$ for each j is the characteristic of a symmetry of (4.25). Moreover, each K_j is formally derivable from the corresponding $\mathbf{K}_j$ via Miura's transformation

$$(\mu D + 2u)\widetilde{K}_j = \mathcal{S} K_j . \tag{4.28}$$

In order to rigorously justify these observations, and in preparation for subsequent applications to the sine-Gordon equation, we proceed in a less direct fashion. Recall that $J_j = D^{-1}K_j = \mathcal{D}^{*j}(u)$, where $\mathcal{D}^*$ is given in (4.11). Define

$$\tilde{J}_{j+1} = -\frac{\nu}{6}(\mu D - 2u)\,\mathcal{S}J_j \quad . \tag{4.29}$$

The claim is that

$$D\tilde{J}_{j+1} = \tilde{K}_{j+1} \quad . \tag{4.30}$$

To see this, let

$$\tilde{\mathcal{D}}^* = D^2 + \frac{2}{3}\nu u^2 - \frac{2}{3}\nu u D^{-1}u_x \tag{4.31}$$

be the adjoint of $\tilde{\mathcal{D}}$. A short computation shows that

$$\tilde{\mathcal{D}}^*(\mu D - 2u)\,\mathcal{S} = (\mu D - 2u)\,\mathcal{S}\mathcal{D}^* \tag{4.32}$$

by virtue of the definition of μ in (4.26). Therefore $\tilde{J}_{j+1} = (\tilde{\mathcal{D}}^*)^j(u_{xx} + \frac{1}{3}\nu u^3) = (\tilde{\mathcal{D}}^*)^{j+1}(u)$. From this we can easily check (4.30), since $D\tilde{\mathcal{D}}^* = \tilde{\mathcal{D}}D$. Note that by this subterfuge we avoided having to invert the operator $(\mu D + 2u)$ to find $\tilde{K}_j$. Next we discuss how the symmetries given by $\tilde{K}_j$ are related to conservation laws.

<u>Lemma 4.15.</u> <u>For each positive</u> j,

$$E[(\mu D - 2u)^{-1}\tilde{J}_{j+1}] = \frac{\nu}{3}(2j - 1)\tilde{J}_j \quad . \tag{4.33}$$

Proof. Using (4.29), (4.14) and Proposition 2.16

$$\begin{aligned} E[(\mu D - 2u)^{-1}\tilde{J}_{j+1}] &= -\frac{\nu}{6}\, E(\mathcal{S}\, J_j) \\ &= -\frac{\nu}{6}(2u - \mu D)\mathcal{S}\, E(J_j) \\ &= -\frac{\nu^2}{18}(2j - 1)(2u - \mu D)\mathcal{S}\, J_{j-1} \\ &= \frac{\nu}{3}(2j - 1)\tilde{J}_j \ . \end{aligned}$$

Theorem 4.16. For each pair of positive integers i and j, the flows $u_t = \tilde{K}_i$ and $u_t = \tilde{K}_j$ commute. Moreover $\frac{\nu}{3}(2i - 1)\tilde{J}_i$ is the characteristic of a nontrivial conservation law of $u_t = \tilde{K}_j$ with conserved density $(\mu D - 2u)^{-1}\tilde{J}_{j+1} = -\frac{\nu}{6}\, \mathcal{S}\, J_j$.

Proof. The second statement follows directly from Corollary 4.3. To prove the first, it suffices to show that the Gardner-Poisson bracket of $\tilde{J}_i$ and $\tilde{J}_j$ is zero. First, from (4.13),

$$E(\mathcal{S}\, J_i \cdot \mathcal{S} D J_j) = 0 \ .$$

Next note that

$$\mathcal{S} \cdot D \cdot \mathcal{D}^{*} = -\frac{\nu}{6}(\mu D + 2u)D(\mu D - 2u)\mathcal{S} \ . \tag{4.34}$$

Therefore

$$E(\mathcal{S} J_j \cdot (\mu D + 2u)D(\mu D - 2u)\, \mathcal{S} J_{j-1}) = 0 \ .$$

Integrating by parts, using (4.29), and replacing $i+1$ by i, shows that

$$E(\tilde{J}_i D\tilde{J}_j) = 0 . \tag{4.35}$$

This in turn completes the proof of the theorem.

Finally, consider the potential versions of the modified KdV equations. These will be used in the next section. According to the definitions at the beginning of section 4.1, these are the evolution equations

$$u_t = \tilde{P}_j , \tag{4.36}$$

where $\quad \tilde{P}_j = \mathcal{S}_0 \tilde{J}_j = \mathcal{S}_0 D^{-1}\tilde{K}_j$.

Thus if $u = f(x,t)$ is a solution of (4.36), then $u' = \partial f/\partial x$ is a solution of the corresponding MKdV equation $u_t = K_j$. The P_j's have a recursion operator

$$\mathcal{D}_0 = D^2 + \frac{2}{3}\nu u_x^2 - \frac{2}{3}\nu u_x D^{-1} u_{xx} . \tag{4.37}$$

Theorem 4.17. <u>For each pair of positive integers i and j the flows $u_t = \tilde{P}_i$ and $u_t = \tilde{P}_j$ commute. Moreover $\frac{\nu}{3}(2i-1)\tilde{P}_i$ is the characteristic of a nontrivial conservation law of $u_t = \tilde{P}_j$ with conserved density</u> $(\mu D - 2u_x)^{-1}\tilde{P}_{i+1}$.

<u>Proof</u> By Proposition 2.16.

$$\begin{aligned} E((\mu D - 2u_x)^{-1}\tilde{P}_{i+1}) &= E(\mathcal{S}_0(\mu D - 2u)^{-1}\tilde{J}_{i+1}) \\ &= D\,\mathcal{S}_0 E((\mu D - 2u)^{-1}\tilde{J}_{i+1}) \\ &= \frac{\nu}{3}(2i - 1)D\,\mathcal{S}_0\,\tilde{J}_i \\ &= \frac{\nu}{3}(2i - 1)D\,\tilde{P}_i\ . \end{aligned}$$

The commutation of flows then follows directly from (4.5). The statement concerning conservation laws is a direct consequence of Corollary 4.3.

4.4. The Sine-Gordon Equation

As a final illustration of the possible applications of the theory, consider the sine-Gordon equation, which will be written in characteristic coordinates,

$$u_{xt} = \sin \lambda u \quad . \tag{4.38}$$

Note that although the sine-Gordon is not an evolution equation, it still fits into our general framework, and arises as the Euler equation for the variational problem with Lagrangian

$$\tilde{L} = \frac{1}{2} u_x u_t - \frac{1}{\lambda} \cos \lambda u \; .$$

The sine-Gordon equation is known to possess infinitely many conservation laws, and infinitely many symmetries, which were derived in [21] from the well-known Backlund transformation associated with it. In [35] it was formally shown that these symmetries have the recursion operator

$$\mathcal{D}_0 = D^2 + \lambda^2 u_x^2 - \lambda^2 u_x D^{-1} u_{xx} \; . \tag{4.39}$$

The remarkable thing to notice is that $\mathcal{D}_0$ is just the recursion operator for the potential modified KdV equations, where $\nu = \frac{3}{2}\lambda^2$. As a consequence of Theorem 4.17, we find that the PMKdV equations provide conservation laws for the sine-Gordon equation.

Theorem 4.18. *For each integer* j, *the potential modified KdV equation defined by (4.36) is a symmetry group of the sine-Gordon equation (4.38) provided* $\nu = \frac{3}{2}\lambda^2$. *Moreover,* $\frac{1}{2}\lambda^2(2j-1)\tilde{P}_j$ *is the characteristic of a nontrivial conservation law of the sine-Gordon equation, with conserved density* $(\mu D - 2u_x)^{-1}\tilde{P}_{j+1}$, *where* $\mu = \sqrt{-6/\nu} = \lambda^{-1}\sqrt{-1}$.

The proof follows directly from Theorem 4.2. Notice that since $\tilde{P}_j$ is the characteristic of a conservation law of (4.38),

$$E(\tilde{P}_j \sin \lambda u) = 0.$$

This reciprocally implies that $-\lambda \sin \lambda u$ is the characteristic of a conservation law of each PMKdV equation $u_t = \tilde{P}_j$, with conserved density $\cos \lambda u$. By the connection between the solutions of the PMKdV equations and the corresponding MKdV equations, that $\left(\sqrt{\frac{2}{3}\nu}\int_{-\infty}^{x} u\,dx\right)$ is a conserved density of each of the Modified KdV equations $u_t = K_j$ (where we assume the solutions satisfy appropriate decay properties at $-\infty$.) This further implies that if $v(x,t)$ is a solution of one of the KdV equations, and u is the solution of $v = \mu u_x + u^2$, with $u(-\infty, t) = 0$, then $\int_{-\infty}^{\infty} \cos\left(\sqrt{\frac{2}{3}\nu}\int_{-\infty}^{x} u\,dx\right)dx$ is an additional conserved quantity of each of the KdV equations! A second conserved quantity is found by reversing the roles of sin and cos.

Furthermore, since the flows determined by the sine-Gordon equation and the PMKdV equations commute, we also have the flow determined by

$$u_t = \sin\left(\sqrt{\frac{2}{3}\nu}\int_{-\infty}^{x} u\,dx\right)$$

commuting with the MKdV equations. The existence of the Miura transformation indicates a corresponding symmetry of the KdV equations. This symmetry, however, does not seem to arise from a partial differential equation. For instance, if we apply the operator $2u + \mu D$ to the above equation, we obtain, setting $\lambda = \sqrt{\frac{2}{3}\nu}$,

$$2uu_t + \mu u_{xt} = 2u\sin\left(\lambda\int u\,dx\right) + \lambda\mu u\cos\left(\lambda\int u\,dx\right)$$

or, equivalently,

$$v_t = \sqrt{-4}\, u \exp\left[-\sqrt{-\frac{2}{3}\nu}\int_{-\infty}^{x} u\,dx\right] \tag{4.38}$$

where $v = u^2 + \mu u_x$ will be the solution of the KdV equation. The right-hand side of (4.38) does not seem to have a simple expression in terms of v. If we let

$$w = \exp[-\frac{1}{\mu}\int_{-\infty}^{x} u\,dx] \;, \quad \mu = \sqrt{\frac{-6}{\nu}} \;,$$

then w satisfies

$$w_{xx} - \frac{6}{\nu} v w = 0 \;,$$

and 4.38 is

$$v_t = \sqrt{\frac{\nu}{3}}\,(w^2)_x = \sqrt{\frac{4\nu}{3}}\; w w_x \;.$$

Note the mysterious appearance of Hill's equation. These matters certainly warrant further research. We will report on the connections with the Bäcklund transformation and soliton solutions in a subsequent publication.

Bibliography

[1] R. L. Anderson, S. Kumei and C. E. Wulfman, "Invariants of the Equations of Wave Mechanics: Rigid Rotator and Symmetric Top," J. Math. Phys. 14 (1973), pp. 1527-1531.

[2] E. Bessel-Hagen, "Über die Erhaltungssätze der Elektrodynamik," Math. Ann., 84 (1921), pp. 258-276.

[3] G. W. Bluman and J. D. Cole, "The General Similarity Solution of the Heat Equation," J. Math. Mech., (11) 18 (1969), pp. 1025-1042.

[4] G. W. Bluman and J. D. Cole, Similarity Methods for Differential Equations, Springer-Verlag, App. Math. Sci. No. 13, New York, 1974.

[5] C. P. Boyer, E. G. Kalnins and W. Miller, Jr., "Symmetry and Separation of Variables for the Helmholtz and Laplace Equations," Nagoya Math. J., 60 (1976), pp. 35-80

[6] E. Candotti, C. Palmieri and B. Vitale, "On the Inversion of Noether's Theorem in the Lagrangian Formalism," Nuovo Cimento, 70A (1970), pp. 233-246.

[7] C. Carathéodory, Calculus of Variations and Partial Differential Equations of the First Order, Holden-Day, Inc., San Francisco, 1965.

[8] A. Cohen, Introduction to the Lie Theory of One Parameter Groups, with Applications to the Solution of Differential Equations, Stechert, New York, 1911.

[9] J. D. Cole, "On a Quasilinear Parabolic Equation Used in Aerodynamics," Quart. Appl. Math., 9 (1951), pp. 225-236.

[10] L. P. Eisenhart, Continuous Groups of Transformations, Princeton University Press, Princeton, N. J., 1933.

[11] C. S. Gardner, "Korteweg-de Vries Equation and Generalizations. IV. The Korteweg-de Vries Equation as a Hamiltonian System," J. Math. Phys., 12 (1971), pp. 1548-1551.

[12] I. M. Gel'fand and L. A. Dikiĭ, "Asymptotic Behaviour of the Resolvent of Sturm-Liouville Equations and the Algebra of the Korteweg-de Vries Equation," Russian Math. Surveys 30 (1975), pp. 77-113.

[13] I. M. Gel'fand and L. A. Dikiĭ, "A Lie Algebra Structure in a Formal Variational Calculation," Funk. Anal., 10 (1975), pp. 18-25.

[14] I. M. Gel'fand and S. V. Fomin, Calculus of Variations, Prentice-Hall, Englewood Cliffs, N. J., 1963.

[15] M. Golubitsky and V. Guillemin, Stable Mappings and their Singularities, Springer-Verlag, Grad. Texts in Math. No. 14, New York, 1973.

[16] E. Hopf, "The Partial Differential Equation $u_t + uu_x = \mu u_{xx}$," Comm. Pure Appl. Math., 3 (1950) pp. 201-230.

[17] N. H. Ibragimov, "Invariant Variational Problems and Conservation Laws," Theor. and Math. Phys., 1 (1969), pp. 350-359. (Russian)

[18] N. H. Ibragimov, "Invariance and Conservation Laws of Continuum Mechanics," in Symmetry, Similarity and Group Theoretic Methods in Mechanics, P. G. Glockner and M. C. Singh, eds., Univ. of Calgary Press, Calgary, Canada, 1974.

[19] N. Jacobson, Lie Algebras, Interscience Publishers, New York, 1962.

[20] M. D. Kruskal, R. M. Miura, C. S. Gardner and N. J. Zabusky, "Korteweg-de Vries Equation and Generalizations. V. Uniqueness and Nonexistence of Polynomial Conservation Laws," J. Math. Phys. 11 (1970), pp. 952-960.

[21] S. Kumei, "Invariance Transformations, Invariance Group Transformations, and Invariance Groups of the Sine-Gordon Equations," J. Math. Phys. 16 (1975), pp. 2461-2468.

[22] M. Kuranishi, "On the Local Theory of Continuous Infinite Pseudo-Groups, I, II," Nagoya Math. J., 15 (1959), pp. 225-260 and 19 (1961), pp. 55-91.

[23] P. D. Lax, "Integrals of Nonlinear Equations of Evolution and Solitary Waves," Comm. Pure Appl. Math., 21 (1968), pp. 467-490.

[24] P. D. Lax, "Perodic Solutions of the KdV Equation," Comm. Pure Appl. Math., 28 (1975), pp. 141-188.

[25] P. D. Lax, "Almost Periodic Solutions of the KdV Equation," SIAM Review, 18 (1976), pp. 351-375.

[26] P. D. Lax, "A Hamiltonian Approach to the KdV and Other Equations," in Group Theoretical Methods in Physics, Fifth International Colloquium, Academic Press, New York, 1977.

[27] H. P. McKean and P. van Moerbeke, "The Spectrum of Hill's Equation," Invent. Math., 30 (1975), pp. 217-274.

[28] W. Miller, Jr., "Symmetry and Separation of Variables," in Encyclopedia of Applicable Mathematics, vol. 4., Addison-Wesley, Reading, Mass., 1977.

[29] R. M. Miura, "Korteweg-de Vries Equations and Generalizations. I. A Remarkable Explicit Nonlinear Transformation," J. Math. Phys., 9 (1968), pp. 1202-1204.

[30] R. M. Miura, C. S. Gardner and M. D. Kruskal, "Korteweg-de Vries Equation and Generalizations. II. Existence of Conservation Laws and Constants of Motion," J. Math. Phys., 9 (1968), pp. 1204-1209.

[31] R. M. Miura, "The Korteweg-de Vries Equation: A Survey of Results," SIAM Review, 18 (1976), pp. 412-459.

[32] C. S. Morawetz, "Time Decay for the Nonlinear Klein-Gordon Equation," Proc. Roy. Soc. A306 (1968), pp. 291-296.

[33] C. S. Morawetz, "Decay for Solutions of the Exterior Problem for the Wave Equation," Comm. Pure Appl. Math., 28 (1975), pp. 229-264.

[34] E. Noether, "Invariante Variations Probleme," Kgl. Ger. Wiss. Nachr. Gottingen, Math-Phys. Kl., (1918), pp. 235-257.

[35] P. Olver, "Evolution Equations Possessing Infinitely Many Symmetries," J. Math. Phys., 18 (1977), pp. 1212-1215.

[36] P. Olver, "Symmetry Groups and Group Invariant Solutions of Partial Differential Equations," J. Diff. Geom., to appear.

[37] P. Olver, "Euler Operators and Conservation Laws of the BBM Equation," preprint.

[38] P. Olver, "How to Find the Symmetry Group of a Differential Equation," preprint, to appear in Lectures on Bifurcation Theory, D. Sattinger, in preparation.

[39] P. Olver and C. Shakiban, "A Resolution of the Euler Operator. I," Proc. Amer. Math. Soc., to appear.

[40] L. V. Ovsjannikov, Group Properties of Differential Equations, Novosibirsk, 1962. (transl. by G. W. Bluman, unpublished).

[41] L. V. Ovsjannikov, "On the Infinite Transformation Groups Determined by Differential Equations," Dokl. Akad. Nauk SSSR., 148 (1963), pp. 36-39.

[42] L. V. Ovsjannikov, "Some Problems Arising in Group Analysis of Differential Equations," in Symmetry, Similarity, and Group Theoretic Methods in Mechanics, Univ. of Calgary Press, Calgary, Canada, 1974.

[43] R. S. Palais, A Global Formulation of the Lie Theory of Transformation Groups, Memoirs of the Amer. Math. Soc. No. 22, Providence, R.I., 1957.

[44] A. C. Scott, F. Y. F. Chu and D. W. McLaughlin, "The Soliton: A New Concept in Applied Science," Proc. IEEE, 61 (1973), pp. 1443-1483.

[45] C. Shakiban, Ph.D. Thesis, Brown University, 1978. (unpublished)

[46] W. A. Strauss, "Nonlinear Invariant Wave Equations," Erice Conference, 1977, Lecture Notes in Physics, Springer-Verlag, to appear.

[47] H. D. Wahlquist and F. D. Estabrook, "Backlund Transformations for Solutions of the Korteweg-de Vries Equation," Phys. Rev. Lett., 31 (1973), pp. 1386-1390.

[48] G. B. Whitham, Linear and Nonlinear Waves, John Wiley, New York, 1974.

[49] V. E. Zakharov and L. D. Faddeev, "Korteweg-de Vries Equation: A Completely Integrable Hamiltonian System," Funk. Anal. Appl., 5 (1972), pp. 280-287.

[50] N. H. Ibragimov and R. L. Anderson, "Lie-Bäcklund Tangent Transformations," J. Math. Anal. 59 (1977), pp. 146-162.

[51] H. Steudel, "An Infinite Set of Conservation Laws Derived by Noether's Theorem for Several Nonlinear Evolution Equations," VII Internationale Konferenz über Nichtlineare Schwingungen, Akademie-Verlag, Berlin, 1977.

[52] M. Wadati, "Infinitesimal Transformations and Conservation Laws; Field Theoretic Approach to Theory of Soliton," Nonlinear Evolution Equations Solvable by the Spectral Transform, F. Calogero, ed., Pitman, London, 1978.