Moving Frames

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History of Moving Frames

Classical contributions:

M. Bartels ($\sim$1800), J. Serret, J. Frénet, G. Darboux, É. Cotton,

Élie Cartan

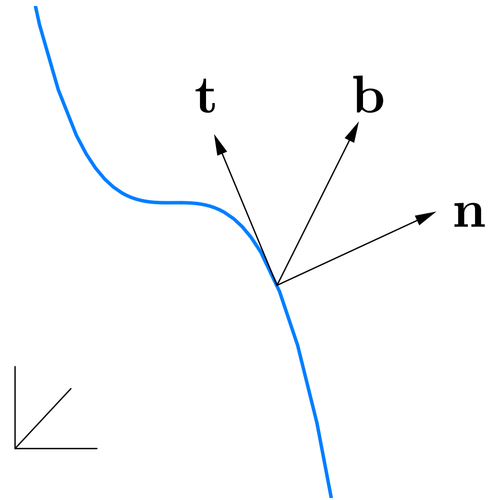
Modern developments: (1970's)

S.S. Chern, M. Green, P. Griffiths, G. Jensen, T. Ivey, J. Landsberg, ...

The equivariant approach: (1997 –)

PJO, M. Fels, G. Marí–Beffa, I. Kogan, J. Cheh, J. Pohjanpelto, P. Kim, M. Boutin, D. Lewis, E. Mansfield, E. Hubert, E. Shemyakova, O. Morozov, R. McLenaghan, R. Smirnov, J. Yue, A. Nikitin, J. Patera, P. Vassiliou, ...

Moving Frame — Space Curves



tangent — normal — binormal

$$\mathbf{t} = \frac{dz}{ds} \qquad \mathbf{n} = \frac{d^2 z}{ds^2} \qquad \mathbf{b} = \mathbf{t} \times \mathbf{n}$$

s — arc length

Frénet–Serret equations

$$\frac{d\mathbf{t}}{ds} = \kappa\, \mathbf{n} \qquad \frac{d\mathbf{n}}{ds} = -\,\kappa\, \mathbf{t} + \tau\, \mathbf{b} \qquad \frac{d\mathbf{b}}{ds} = -\,\tau\, \mathbf{n}$$

κ — curvature $\qquad$ τ — torsion

Moving Frame — Space Curves

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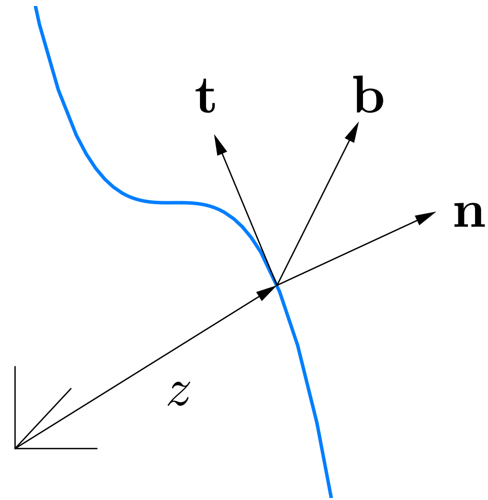

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κ — curvature $\qquad$ τ — torsion

"I did not quite understand how he [Cartan] does this in general, though in the examples he gives the procedure is clear."

"Nevertheless, I must admit I found the book, like most of Cartan's papers, hard reading."

— Hermann Weyl

"Cartan on groups and differential geometry"
Bull. Amer. Math. Soc. **44** (1938) 598–601

Applications of Moving Frames

- Differential geometry
- Equivalence
- Symmetry
- Differential invariants
- Rigidity
- Identities and syzygies
- Joint invariants and semi-differential invariants
- Invariant differential forms and tensors
- Integral invariants
- Classical invariant theory

- Computer vision
    - object recognition
    - symmetry detection
- Invariant variational problems
- Invariant numerical methods
- Mechanics, including DNA
- Poisson geometry & solitons
- Killing tensors in relativity
- Invariants of Lie algebras in quantum mechanics
- Control theory
- Lie pseudo-groups

The Basic Equivalence Problem

M — smooth m-dimensional manifold.

G — transformation group acting on M

- finite-dimensional Lie group
- infinite-dimensional Lie pseudo-group

Equivalence:

Determine when two p-dimensional submanifolds

$$N \quad \text{and} \quad \overline{N} \subset M$$

are *congruent*:

$$\overline{N} = g \cdot N \qquad \text{for} \qquad g \in G$$

Symmetry:

Find all symmetries,

i.e., self-equivalences or *self-congruences*:

$$N = g \cdot N$$

Classical Geometry — *F. Klein*

- **Euclidean group:** $$G = \begin{cases} \mathrm{SE}(m) = \mathrm{SO}(m) \ltimes \mathbb{R}^m \\ \mathrm{E}(m) = \mathrm{O}(m) \ltimes \mathbb{R}^m \end{cases}$$

  $$\boxed{z \longmapsto A \cdot z + b} \qquad A \in \mathrm{SO}(m) \text{ or } \mathrm{O}(m), \quad b \in \mathbb{R}^m, \quad z \in \mathbb{R}^m$$

  $\Rightarrow$ isometries: rotations, translations , (reflections)

- **Equi-affine group:** $G = \mathrm{SA}(m) = \mathrm{SL}(m) \ltimes \mathbb{R}^m$

  $A \in \mathrm{SL}(m)$ — volume-preserving

- **Affine group:** $G = \mathrm{A}(m) = \mathrm{GL}(m) \ltimes \mathbb{R}^m$

  $A \in \mathrm{GL}(m)$

- **Projective group:** $G = \mathrm{PSL}(m+1)$

  acting on $\mathbb{R}^m \subset \mathbb{R}\boldsymbol{\xi}^m$

$\Longrightarrow$ Applications in computer vision

Tennis, Anyone?

Classical Invariant Theory

Binary form:

$$Q(x) = \sum_{k=0}^{n} \binom{n}{k} a_k\, x^k$$

Equivalence of polynomials (binary forms):

$$Q(x) = (\gamma x + \delta)^n\, \overline{Q}\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right) \qquad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2)$$

- multiplier representation of GL(2)
- modular forms

$$Q(x) = (\gamma x + \delta)^n \, \overline{Q}\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right)$$

Transformation group:

$$g : (x, u) \longmapsto \left(\frac{\alpha x + \beta}{\gamma x + \delta}, \; \frac{u}{(\gamma x + \delta)^n}\right)$$

Equivalence of functions $\iff$ equivalence of graphs

$$\Gamma_Q = \{ \, (x, u) = (x, Q(x)) \, \} \subset \mathbb{C}^2$$

Moving Frames

Definition.

A moving frame is a G-equivariant map

$$\rho : M \longrightarrow G$$

Equivariance:

$$\rho(g \cdot z) = \begin{cases} g \cdot \rho(z) & \text{left moving frame} \\ \rho(z) \cdot g^{-1} & \text{right moving frame} \end{cases}$$

$$\rho_{left}(z) = \rho_{right}(z)^{-1}$$

The Main Result

Theorem. A moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.

Isotropy & Freeness

Isotropy subgroup: $G_z = \{ g \mid g \cdot z = z \}$ for $z \in M$

- free — the only group element $g \in G$ which fixes *one* point $z \in M$ is the identity: $\Longrightarrow$ $G_z = \{e\}$ for all $z \in M$.
- locally free — the orbits all have the same dimension as G: $\Longrightarrow$ G_z is a discrete subgroup of G.
- regular — all orbits have the same dimension and intersect sufficiently small coordinate charts only once

  $\not\approx$ irrational flow on the torus
- effective — the only group element which fixes *every* point in M is the identity: $g \cdot z = z$ for all $z \in M$ iff $g = e$:

$$G_M^* = \bigcap_{z \in M} G_z = \{e\}$$

Proof of the Main Theorem

Necessity: Let $\rho : M \to G$ be a left moving frame.

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Regularity: Suppose $z_n = g_n \cdot z \longrightarrow z$ as $n \to \infty$.

By continuity, $\rho(z_n) = \rho(g_n \cdot z) = g_n \cdot \rho(z) \longrightarrow \rho(z)$.
Hence $g_n \longrightarrow e$ in G.

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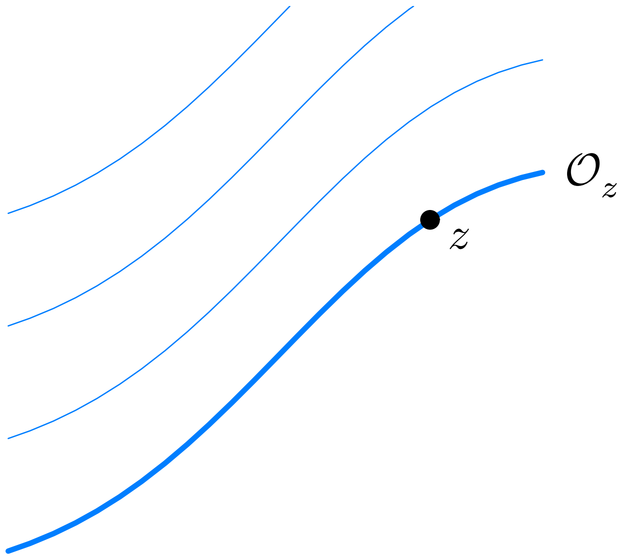
Sufficiency: By direct construction — "normalization".

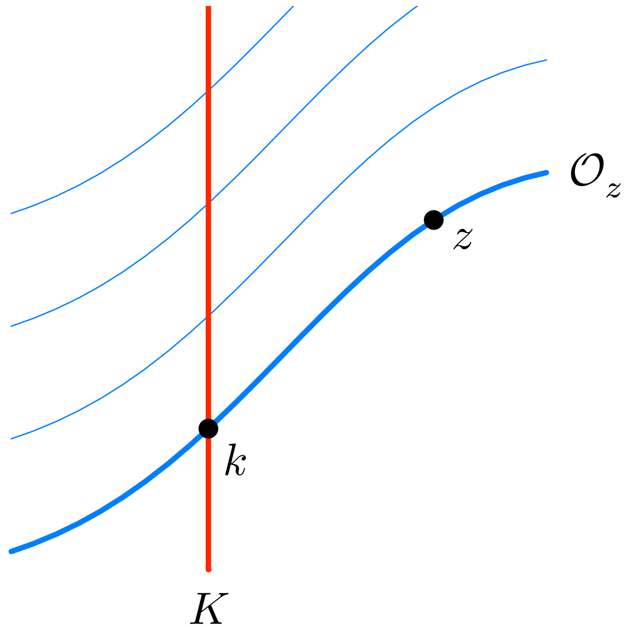
Q.E.D.

Geometric Construction



Normalization = choice of cross-section to the group orbits

Geometric Construction



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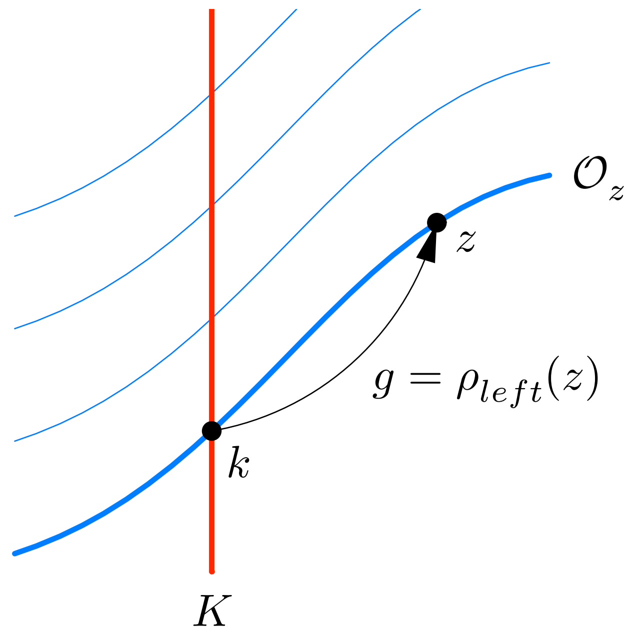

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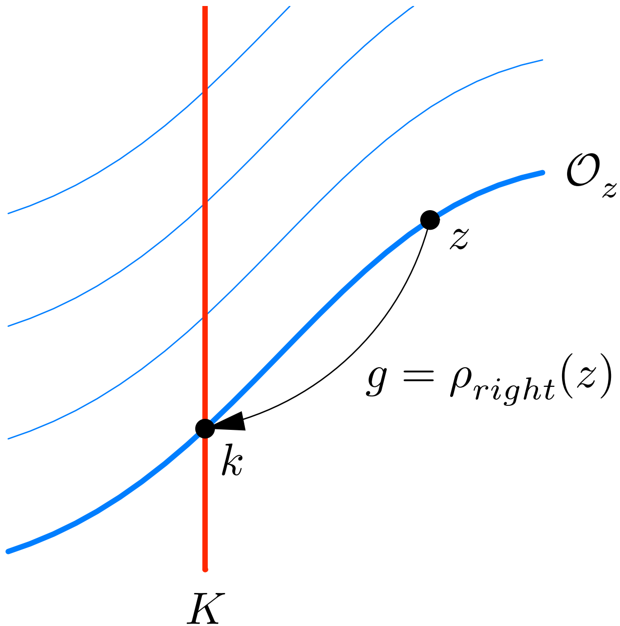

Normalization = choice of cross-section to the group orbits

K — cross-section to the group orbits

$\mathcal{O}_z$ — orbit through $z \in M$

$k \in K \cap \mathcal{O}_z$ — unique point in the intersection

- k is the *canonical* or *normal form* of z
- the (nonconstant) coordinates of k are the fundamental invariants

$g \in G$ — *unique* group element mapping k to z

$\Longrightarrow$ freeness

$\rho(z) = g$ left moving frame $\rho(h \cdot z) = h \cdot \rho(z)$

$$k = \rho^{-1}(z) \cdot z = \rho_{right}(z) \cdot z$$

Algebraic Construction

$$r = \dim G \;\leq\; m = \dim M$$

Coordinate cross-section

$$K = \{\, z_1 = c_1, \;\ldots\; , z_r = c_r \,\}$$

left	right
$w(g, z) = g^{-1} \cdot z$	$w(g, z) = g \cdot z$

$g = (g_1, \ldots, g_r)$ — group parameters

$z = (z_1, \ldots, z_m)$ — coordinates on M

Choose $r = \dim G$ components to *normalize*:

$$w_1(g, z) = c_1 \qquad \ldots \qquad w_r(g, z) = c_r$$

Solve for the group parameters $g = (g_1, \ldots, g_r)$

$\Longrightarrow$ Implicit Function Theorem

The solution

$$g = \rho(z)$$

is a (local) moving frame.

The Fundamental Invariants

Substituting the moving frame formulae

$$g = \rho(z)$$

into the unnormalized components of $w(g, z)$ produces the fundamental invariants

$$I_1(z) = w_{r+1}(\rho(z), z) \quad \ldots \quad I_{m-r}(z) = w_m(\rho(z), z)$$

Theorem. Every invariant $I(z)$ can be (locally) uniquely written as a function of the fundamental invariants:

$$I(z) = H(I_1(z), \ \ldots \ , I_{m-r}(z))$$

Prolongation

Most interesting group actions (Euclidean, affine, projective, etc.) are *not* free!

Freeness typically fails because the dimension of the underlying manifold is not large enough, i.e., $m < r = \dim G$.

Thus, to make the action free, we must increase the dimension of the space via some natural prolongation procedure.

- An effective action can usually be made free by:

- Prolonging to derivatives (jet space)

$$G^{(n)} : \mathrm{J}^n(M,p) \longrightarrow \mathrm{J}^n(M,p)$$

$\Longrightarrow$ differential invariants

- Prolonging to Cartesian product actions

$$G^{\times n} : M \times \cdots \times M \longrightarrow M \times \cdots \times M$$

$\Longrightarrow$ joint invariants

- Prolonging to "multi-space"

$$G^{(n)} : M^{(n)} \longrightarrow M^{(n)}$$

$\Longrightarrow$ joint or semi-differential invariants
$\Longrightarrow$ invariant numerical approximations

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Euclidean Plane Curves

Special Euclidean group: $G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$

acts on $M = \mathbb{R}^2$ via rigid motions: $w = R\,z + c$

To obtain the classical (left) moving frame we invert the group transformations:

$$\left.\begin{aligned} y &= \quad \cos\phi\,(x-a) + \sin\phi\,(u-b) \\ v &= -\sin\phi\,(x-a) + \cos\phi\,(u-b) \end{aligned}\right\} \qquad w = R^{-1}(z-c)$$

Assume for simplicity the curve is (locally) a graph:

$$\mathcal{C} = \{u = f(x)\}$$

$\Longrightarrow$ extensions to parametrized curves are straightforward

Prolong the action to J^n via implicit differentiation:

$$y = \cos\phi\,(x-a) + \sin\phi\,(u-b)$$

$$v = -\sin\phi\,(x-a) + \cos\phi\,(u-b)$$

$$v_y = \frac{-\sin\phi + u_x\cos\phi}{\cos\phi + u_x\sin\phi}$$

$$v_{yy} = \frac{u_{xx}}{(\cos\phi + u_x\sin\phi)^3}$$

$$v_{yyy} = \frac{(\cos\phi + u_x\sin\phi)u_{xxx} - 3u_{xx}^2\sin\phi}{(\cos\phi + u_x\sin\phi)^5}$$

$$\vdots$$

Choose a cross-section, or, equivalently a set of
$r = \dim G = 3$ normalization equations:

$$y = \cos\phi\,(x-a) + \sin\phi\,(u-b) = 0$$

$$v = -\sin\phi\,(x-a) + \cos\phi\,(u-b) = 0$$

$$v_y = \frac{-\sin\phi + u_x\cos\phi}{\cos\phi + u_x\sin\phi} = 0$$

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$$\vdots$$

Solve the normalization equations for the group parameters:

$$y = \cos\phi\,(x - a) + \sin\phi\,(u - b) = 0$$

$$v = -\sin\phi\,(x - a) + \cos\phi\,(u - b) = 0$$

$$v_y = \frac{-\sin\phi + u_x \cos\phi}{\cos\phi + u_x \sin\phi} = 0$$

The result is the left moving frame $\rho : \mathrm{J}^1 \longrightarrow \mathrm{SE}(2)$

$$a = x \qquad b = u \qquad \phi = \tan^{-1} u_x$$

$$a = x \qquad b = u \qquad \phi = \tan^{-1} u_x$$

Substitute into the moving frame formulas for the group parameters into the remaining prolonged transformation formulae to produce the basic differential invariants:

$$v_{yy} = \frac{u_{xx}}{(\cos\phi + u_x \sin\phi)^3} \longmapsto \kappa = \frac{u_{xx}}{(1 + u_x^2)^{3/2}}$$

$$v_{yyy} = \cdots \qquad \longmapsto \frac{d\kappa}{ds} = \frac{(1 + u_x^2)u_{xxx} - 3u_x u_{xx}^2}{(1 + u_x^2)^3}$$

$$v_{yyyy} = \cdots \qquad \longmapsto \frac{d^2\kappa}{ds^2} - 3\kappa^3 = \cdots$$

Theorem. All differential invariants are functions of the derivatives of curvature with respect to arc length:

$$\kappa \qquad \frac{d\kappa}{ds} \qquad \frac{d^2\kappa}{ds^2} \qquad \cdots$$

The invariant differential operators and invariant differential forms are also substituting the moving frame formulas for the group parameters:

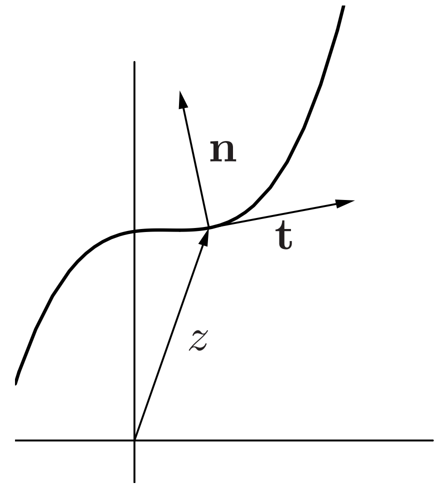
(Contact-)invariant one-form — arc length element

$$dy = (\cos\phi + u_x \sin\phi)\, dx \quad \longmapsto \quad ds = \sqrt{1 + u_x^2}\; dx$$

Invariant differential operator — arc length derivative

$$\frac{d}{dy} = \frac{1}{\cos\phi + u_x \sin\phi}\,\frac{d}{dx} \quad \longmapsto \quad \frac{d}{ds} = \frac{1}{\sqrt{1 + u_x^2}}\,\frac{d}{dx}$$

The Classical Picture:



Moving frame $\quad \rho : (x, u, u_x) \longmapsto (R, \mathbf{a}) \in \mathrm{SE}(2)$

$$R = \frac{1}{\sqrt{1+u_x^2}} \begin{pmatrix} 1 & -u_x \\ u_x & 1 \end{pmatrix} = (\,\mathbf{t},\ \mathbf{n}\,) \qquad \mathbf{a} = \begin{pmatrix} x \\ u \end{pmatrix}$$

Equi-affine Curves $\qquad G = \mathrm{SA}(2)$

$$z \longmapsto A\,z + \mathbf{b} \qquad A \in \mathrm{SL}(2), \qquad \mathbf{b} \in \mathbb{R}^2$$

Invert for left moving frame:

$$\left.\begin{aligned} y &= \delta\,(x-a) - \beta\,(u-b) \\ v &= -\,\gamma\,(x-a) + \alpha\,(u-b) \end{aligned}\right\} \qquad w = A^{-1}(z-b)$$

$$\alpha\,\delta - \beta\,\gamma = 1$$

Prolong to J^3 via implicit differentiation

$$dy = (\delta - \beta\,u_x)\;dx \qquad D_y = \frac{1}{\delta - \beta\,u_x}\,D_x$$

Prolongation:

$$y = \delta\,(x-a) - \beta\,(u-b)$$

$$v = -\,\gamma\,(x-a) + \alpha\,(u-b)$$

$$v_y = -\,\frac{\gamma - \alpha\,u_x}{\delta - \beta\,u_x}$$

$$v_{yy} = -\,\frac{u_{xx}}{(\delta - \beta\,u_x)^3}$$

$$v_{yyy} = -\,\frac{(\delta - \beta\,u_x)\,u_{xxx} + 3\,\beta\,u_{xx}^2}{(\delta - \beta\,u_x)^5}$$

$$v_{yyyy} = -\,\frac{u_{xxxx}(\delta - \beta\,u_x)^2 + 10\,\beta\,(\delta - \beta\,u_x)\,u_{xx}\,u_{xxx} + 15\,\beta^2\,u_{xx}^3}{(\delta - \beta\,u_x)^7}$$

$$v_{yyyyy} = \;\ldots$$

Normalization: $\qquad r = \dim G = 5$

$$y = \delta\,(x-a) - \beta\,(u-b) = 0$$

$$v = -\,\gamma\,(x-a) + \alpha\,(u-b) = 0$$

$$v_y = -\,\frac{\gamma - \alpha\,u_x}{\delta - \beta\,u_x} = 0$$

$$v_{yy} = -\,\frac{u_{xx}}{(\delta - \beta\,u_x)^3} = 1$$

$$v_{yyy} = -\,\frac{(\delta - \beta\,u_x)\,u_{xxx} + 3\,\beta\,u_{xx}^2}{(\delta - \beta\,u_x)^5} = 0$$

$$v_{yyyy} = -\,\frac{u_{xxxx}(\delta - \beta\,u_x)^2 + 10\,\beta\,(\delta - \beta\,u_x)\,u_{xx}\,u_{xxx} + 15\,\beta^2\,u_{xx}^3}{(\delta - \beta\,u_x)^7}$$

$$v_{yyyyy} = \;\ldots$$

Equi-affine Moving Frame

$$\rho : \ (x, u, u_x, u_{xx}, u_{xxx}) \ \longmapsto \ (A, \mathbf{b}) \in \mathrm{SA}(2)$$

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \sqrt[3]{u_{xx}} & -\frac{1}{3}\, u_{xx}^{-5/3}\, u_{xxx} \\ u_x \sqrt[3]{u_{xx}} & u_{xx}^{-1/3} - \frac{1}{3}\, u_x\, u_{xx}^{-5/3}\, u_{xxx} \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x \\ u \end{pmatrix}$$

Nondegeneracy condition: $\qquad u_{xx} \neq 0.$

Equi-affine arc length

$$dy = (\delta - \beta\, u_x)\ dx \qquad \longmapsto \qquad ds = \sqrt[3]{u_{xx}}\ dx$$

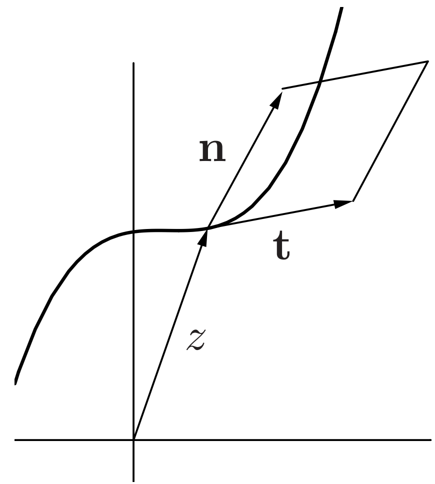
Equi-affine curvature

$$v_{yyyy} \qquad \longmapsto \qquad \kappa \;=\; \frac{5\, u_{xx} u_{xxxx} - 3\, u_{xxx}^2}{9\, u_{xx}^{8/3}}$$

$$v_{yyyyy} \qquad \longmapsto \qquad \frac{d\kappa}{ds}$$

$$v_{yyyyyy} \qquad \longmapsto \qquad \frac{d^2\kappa}{ds^2} - 5\kappa^2$$

The Classical Picture:



$$A = \begin{pmatrix} \sqrt[3]{u_{xx}} & -\frac{1}{3}\, u_{xx}^{-5/3}\, u_{xxx} \\ u_x \sqrt[3]{u_{xx}} & u_{xx}^{-1/3} - \frac{1}{3}\, u_{xx}^{-5/3}\, u_{xxx} \end{pmatrix} = (\,\mathbf{t},\ \mathbf{n}\,) \qquad \mathbf{b} = \begin{pmatrix} x \\ u \end{pmatrix}$$

Equivalence & Invariants

- Equivalent submanifolds $N \approx \overline{N}$
  must have the same invariants: $I = \bar{I}$.

Constant invariants provide immediate information:

$$\text{e.g.} \qquad \kappa = 2 \quad \Longleftrightarrow \quad \overline{\kappa} = 2$$

Non-constant invariants are not useful in isolation, because an equivalence map can drastically alter the dependence on the submanifold parameters:

$$\text{e.g.} \qquad \kappa = x^3 \qquad \text{versus} \qquad \overline{\kappa} = \sinh x$$

Syzygies

However, a functional dependency or syzygy among the invariants *is* intrinsic:

$$\text{e.g.} \qquad \kappa_s = \kappa^3 - 1 \quad \Longleftrightarrow \quad \overline{\kappa}_{\overline{s}} = \overline{\kappa}^3 - 1$$

- Universal syzygies — Gauss–Codazzi
- Distinguishing syzygies.

Equivalence & Syzygies

Theorem. (Cartan) Two smooth submanifolds are (locally) equivalent if and only if they have identical syzygies among *all* their differential invariants.

Proof:

Cartan's technique of the graph:

Construct the graph of the equivalence map as the solution to a (Frobenius) integrable differential system, which can be integrated by solving ordinary differential equations.

Finiteness of Generators and Syzygies

♠ There are, in general, an infinite number of differential invariants and hence an infinite number of syzygies must be compared to establish equivalence.

♡ But the higher order syzygies are all consequences of a finite number of low order syzygies!

Example — Plane Curves

If non-constant, both κ and κ_s depend on a single parameter, and so, locally, are subject to a syzygy:

$$\kappa_s = H(\kappa) \qquad (*)$$

But then

$$\kappa_{ss} = \frac{d}{ds} H(\kappa) = H'(\kappa)\,\kappa_s = H'(\kappa)\,H(\kappa)$$

and similarly for κ_{sss}, etc.

Consequently, all the higher order syzygies are generated by the fundamental first order syzygy $(*)$.

Thus, for Euclidean (or equi-affine or projective or . . .) plane curves we need only know a single syzygy between κ and κ_s in order to establish equivalence!

The Signature Map

The generating syzygies are encoded by the signature map

$$\Sigma : N \longrightarrow \mathcal{S}$$

of the submanifold N, which is parametrized by the fundamental differential invariants:

$$\Sigma(x) = (I_1(x), \ldots, I_m(x))$$

The image

$$\mathcal{S} = \operatorname{Im} \Sigma$$

is the signature subset (or submanifold) of N.

Equivalence & Signature

Theorem. Two smooth submanifolds are equivalent

$$\overline{N} = g \cdot N$$

if and only if their signatures are identical

$$\overline{\mathcal{S}} = \mathcal{S}$$

Signature Curves

Definition. The *signature curve* $\mathcal{S} \subset \mathbb{R}^2$ of a curve $\mathcal{C} \subset \mathbb{R}^2$ is parametrized by the two lowest order differential invariants

$$\mathcal{S} = \left\{ \left(\kappa \, , \, \frac{d\kappa}{ds} \right) \right\} \quad \subset \quad \mathbb{R}^2$$

Equivalence & Signature Curves

Theorem. Two smooth curves $\mathcal{C}$ and $\overline{\mathcal{C}}$ are equivalent:

$$\overline{\mathcal{C}} = g \cdot \mathcal{C}$$

if and only if their signature curves are identical:

$$\overline{\mathcal{S}} = \mathcal{S}$$

$\Longrightarrow$ object recognition

Symmetry and Signature

Theorem. The dimension of the symmetry group

$$G_N = \{ g \mid g \cdot N \subset N \}$$

of a nonsingular submanifold $N \subset M$ equals the codimension of its signature:

$$\dim G_N = \dim N - \dim \mathcal{S}$$

Corollary. For a nonsingular submanifold $N \subset M$,

$$0 \leq \dim G_N \leq \dim N$$

$\Longrightarrow$ Only totally singular submanifolds can have larger symmetry groups!

Maximally Symmetric Submanifolds

Theorem. The following are equivalent:

- The submanifold N has a p-dimensional symmetry group
- The signature $\mathcal{S}$ degenerates to a point: $\dim \mathcal{S} = 0$
- The submanifold has all constant differential invariants
- $N = H \cdot \{ z_0 \}$ is the orbit of a p-dimensional subgroup $H \subset G$

$\Longrightarrow$ Euclidean geometry: circles, lines, helices, spheres, cylinders, planes, . .

$\Longrightarrow$ Equi-affine plane geometry: conic sections.

$\Longrightarrow$ Projective plane geometry: W curves (*Lie & Klein*)

Discrete Symmetries

Definition. The index of a submanifold N equals the number of points in N which map to a generic point of its signature:

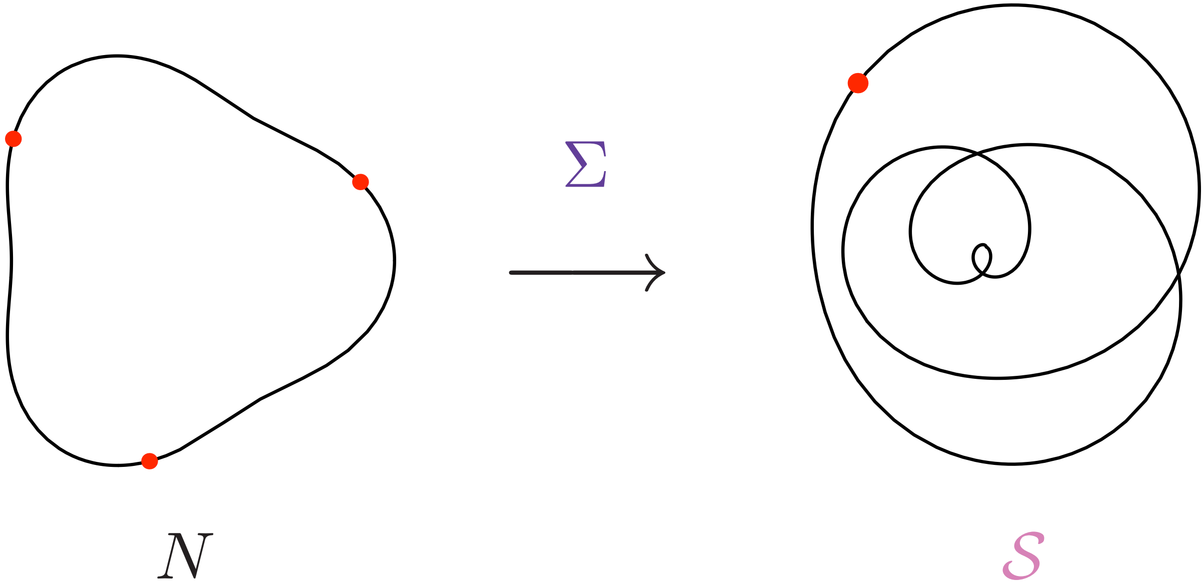
$$\iota_N = \min \left\{ \# \, \Sigma^{-1}\{w\} \;\middle|\; w \in \mathcal{S} \right\}$$

$\Longrightarrow$ Self–intersections

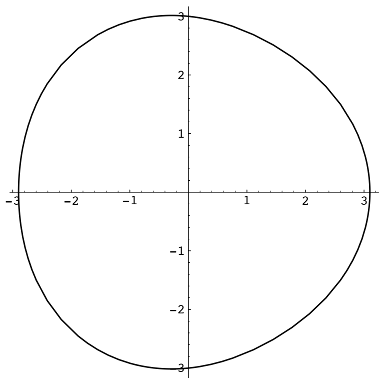
Theorem. The cardinality of the symmetry group of a submanifold N equals its index ι_N.

$\Longrightarrow$ Approximate symmetries

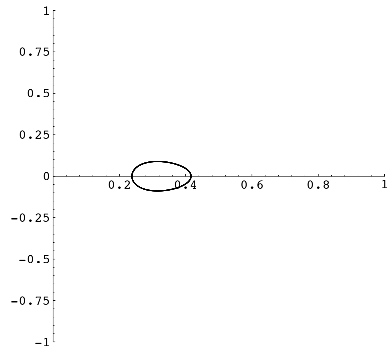
The Index



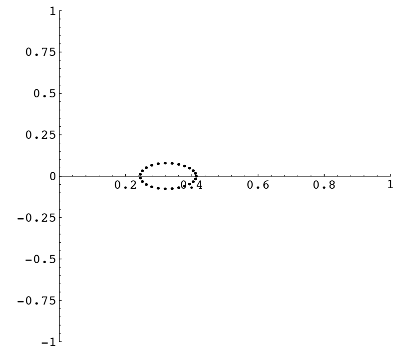
The polar curve $r = 3 + \frac{1}{10}\cos 3\theta$



The Original Curve

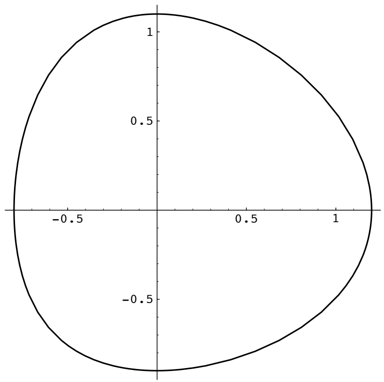


Euclidean Signature

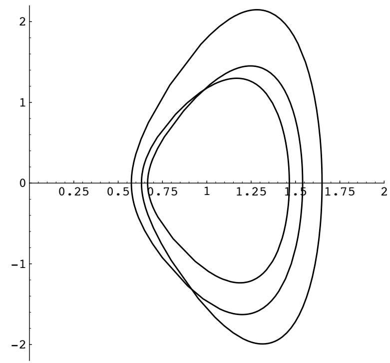


Numerical Signature

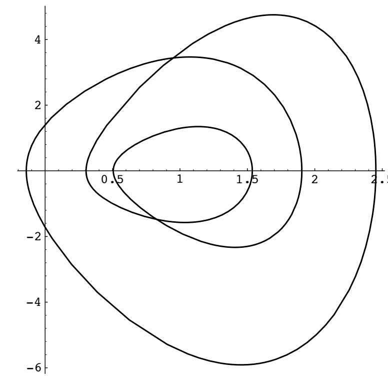
The Curve $x = \cos t + \frac{1}{5}\cos^2 t, \quad y = \sin t + \frac{1}{10}\sin^2 t$



The Original Curve

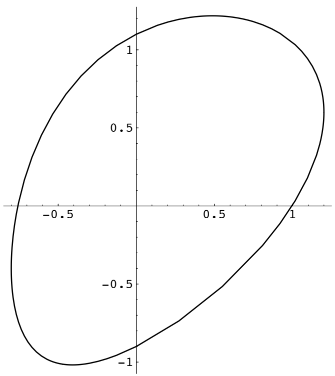


Euclidean Signature

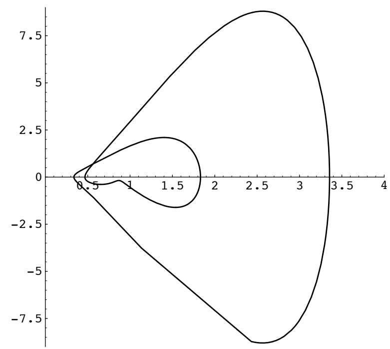


Affine Signature

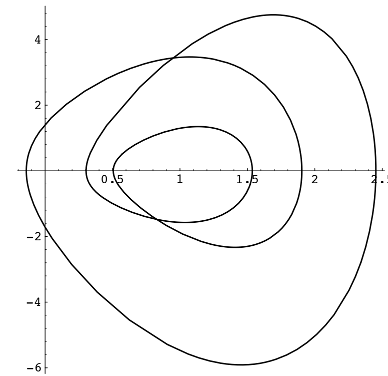
The Curve $x = \cos t + \frac{1}{5}\cos^2 t, \quad y = \frac{1}{2}\,x + \sin t + \frac{1}{10}\sin^2 t$



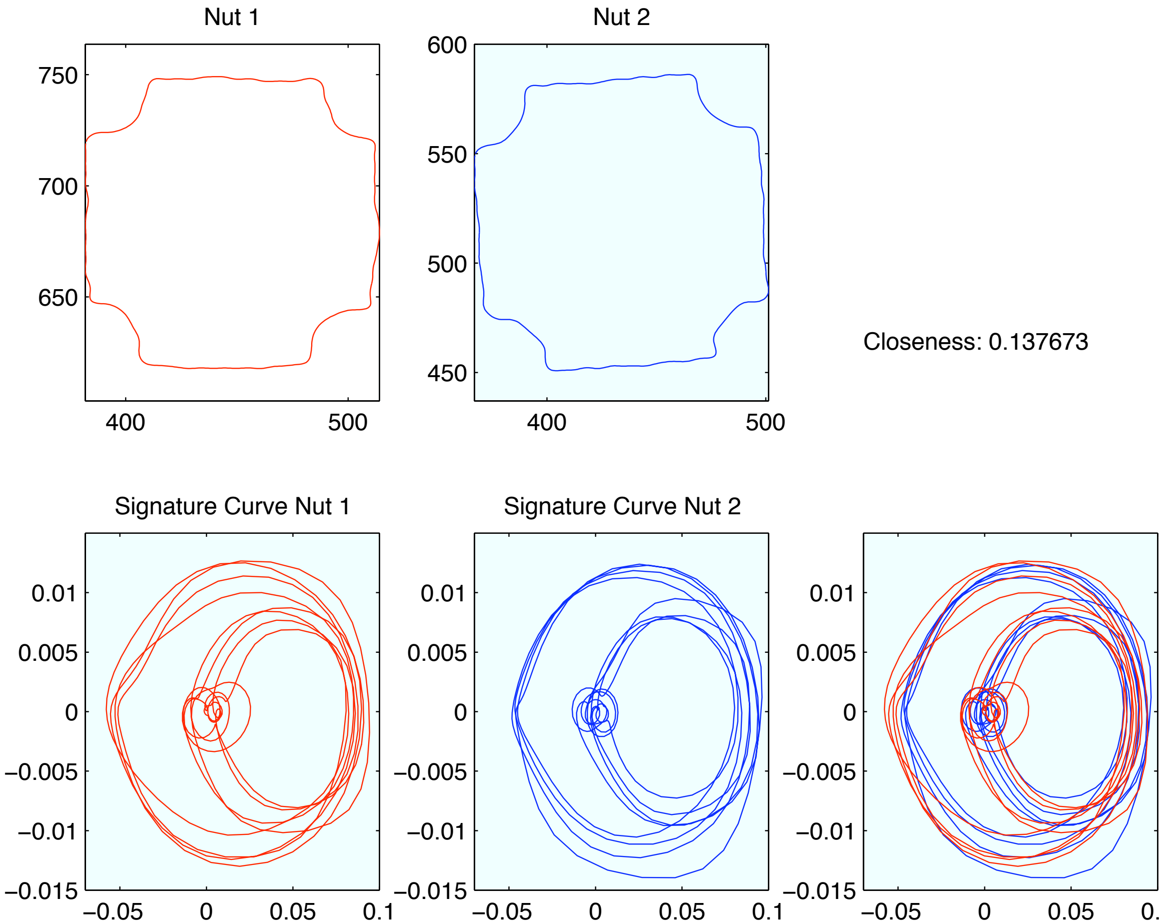
The Original Curve

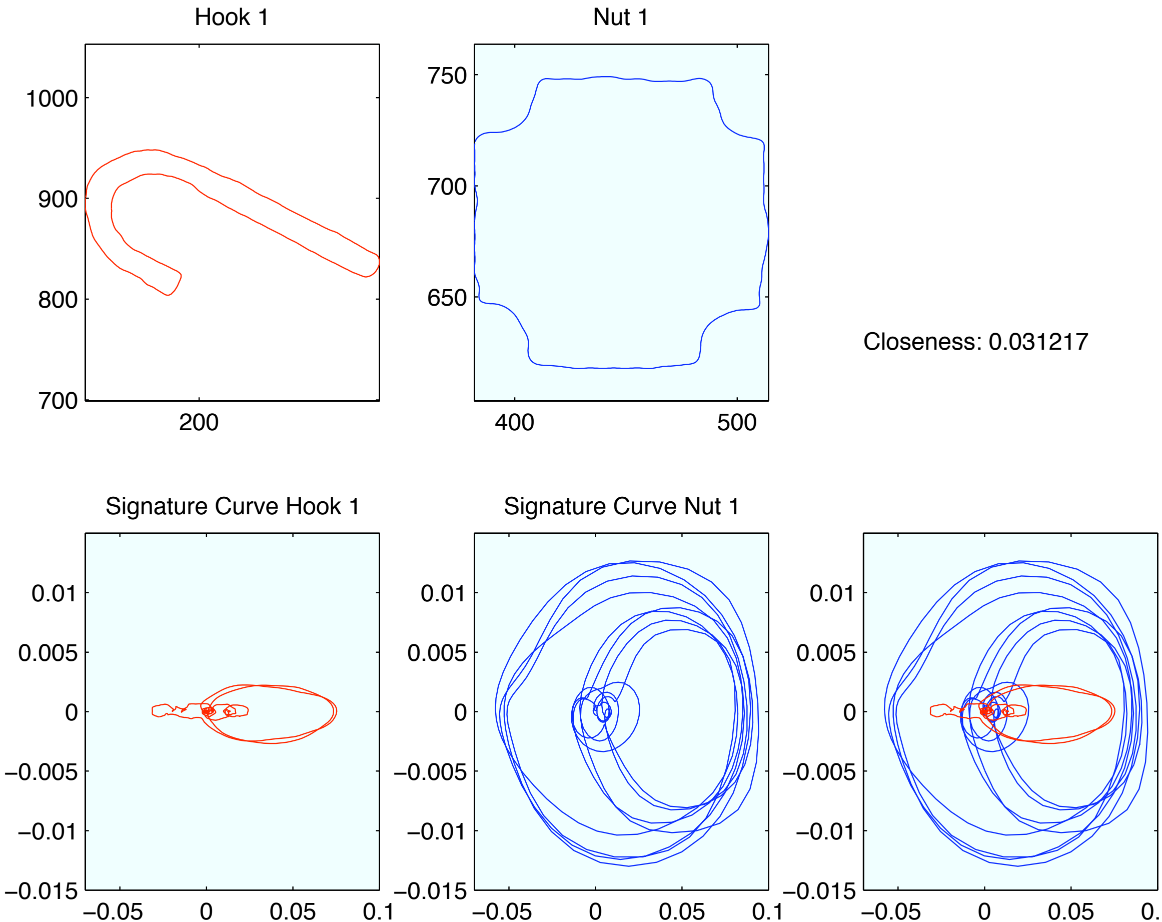


Euclidean Signature



Affine Signature


Nut 1
Nut 2
Closeness: 0.137673
Signature Curve Nut 1
Signature Curve Nut 2
750
700
650
400
500
600
550
500
450
400
500
0.01
0.005
0
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1


Hook 1
Nut 1
Closeness: 0.031217
Signature Curve Hook 1
Signature Curve Nut 1
1000
900
800
700
200
750
700
650
400
500
0.01
0.005
0
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1

Basic Jet Space Notation

M — $m = p + q$ dimensional manifold

$z = (x, u)$ — local coordinates on M

$x = (x^1, \ldots, x^p)$ — independent variables

$u = (u^1, \ldots, u^q)$ — dependent variables

$\mathrm{J}^n = \mathrm{J}^n(M, p)$ — jet space of p-dimensional submanifolds

$u^\alpha_J = \partial_J u^\alpha$ — partial derivatives (jet coordinates)

$F(x, u^{(n)}) = F(\ \ldots \ x^k \ \ldots \ u^\alpha_J \ \ldots \ )$ — differential function
$F : \mathrm{J}^n \to \mathbb{R}$

Invariantization

The process of replacing group parameters in transformation rules by their moving frame formulae is known as invariantization:

$$\iota : \begin{cases} \text{Functions} & \longrightarrow & \text{Invariants} \\ \text{Forms} & \longrightarrow & \text{Invariant Forms} \\ \begin{array}{c}\text{Differential}\\ \text{Operators}\end{array} & \longrightarrow & \begin{array}{c}\text{Invariant Differential}\\ \text{Operators}\end{array} \\ \vdots & & \vdots \end{cases}$$

- The invariantization $I = \iota(F)$ is the unique invariant function that agrees with F on the cross-section: $I \mid K = F \mid K$.
- Invariantization defines an (exterior) algebra morphism.
- Invariantization does not affect invariants: $\iota(I) = I$

The Fundamental Differential Invariants

Invariantized jet coordinate functions:

$$H^i(x, u^{(n)}) = \iota(x^i) \qquad I^\alpha_J(x, u^{(l)}) = \iota(u^\alpha_J)$$

- The constant differential invariants, as dictated by the moving frame normalizations, are known as the phantom invariants.
- The remaining non-constant differential invariants are the basic invariants and form a complete system of functionally independent differential invariants for the prolonged group action.

Invariantization of general differential functions:

$$\iota\,[\,F(\;\ldots\;x^i\;\ldots\;u^\alpha_J\;\ldots\;)\,] = F(\;\ldots\;H^i\;\ldots\;I^\alpha_J\;\ldots\;)$$

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The Replacement Theorem:

If $I(x, u^{(n)})$ is any differential invariant, then $\iota(I) = I$.

$$I(\;\ldots\;x^i\;\ldots\;u^\alpha_J\;\ldots\;) = I(\;\ldots\;H^i\;\ldots\;I^\alpha_J\;\ldots\;)$$

Invariantization of general differential functions:

$$\iota\,[\,F(\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ )\,] = F(\ \ldots\ H^i\ \ldots\ I^\alpha_J\ \ldots\ )$$

The Replacement Theorem:

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$$I(\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ ) = I(\ \ldots\ H^i\ \ldots\ I^\alpha_J\ \ldots\ )$$

Key fact: Invariantization and differentiation do not commute:

$$\iota(D_i F) \ \neq\ \mathcal{D}_i \iota(F)$$

⋆ ⋆ Recurrence Formulae ⋆ ⋆

The Differential Invariant Algebra

Differential invariants:

$$I(g^{(n)} \cdot (x, u^{(n)})) = I(x, u^{(n)})$$

$$\Longrightarrow \quad \text{curvature, torsion, } \ldots$$

Invariant differential operators:

$$\mathcal{D}_1, \ldots, \mathcal{D}_p \qquad \Longrightarrow \quad \text{arc length derivative}$$

- If I is a differential invariant, so is $\mathcal{D}_j I$.

$\mathcal{I}(G)$ — the algebra of differential invariants

Applications

- Equivalence and signatures of submanifolds
- Characterization of moduli spaces
- Invariant differential equations:

$$H(\ \ldots\ \mathcal{D}_J I_\kappa\ \ldots\ ) = 0$$

- Group splitting of PDEs and explicit solutions
- Invariant variational problems:

$$\int L(\ \ldots\ \mathcal{D}_J I_\kappa\ \ldots\ )\, \boldsymbol{\omega}$$

- Invariant geometric flows

The Basis Theorem

Theorem. The differential invariant algebra $\mathcal{I}(G)$ is locally generated by a finite number of differential invariants

$$I_1, \ \ldots \ , I_\ell$$

and $p = \dim S$ invariant differential operators

$$\mathcal{D}_1, \ \ldots \ , \mathcal{D}_p$$

meaning that *every* differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives:

$$\mathcal{D}_J I_\kappa = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_n} I_\kappa .$$

$\Longrightarrow$ Lie groups: *Lie, Ovsiannikov*

$\Longrightarrow$ Lie pseudo-groups: *Tresse, Kumpera, Kruglikov–Lychagin, Muñoz–Muriel–Rodríguez, Pohjanpelto–O*

Key Issues

- Minimal basis of generating invariants: $I_1, \ldots, I_\ell$
- Commutation formulae

for the invariant differential operators:

$$[\,\mathcal{D}_j, \mathcal{D}_k\,] = \sum_{i=1}^{p} Y^i_{jk}\, \mathcal{D}_i$$

$\Longrightarrow$ Non-commutative differential algebra

- Universal Syzygies (functional relations)

among the differentiated invariants:

$$\Phi(\ \ldots\ \mathcal{D}_J I_\kappa\ \ldots\ ) \equiv 0$$

$\Longrightarrow$ Codazzi relations

Computing Differential Invariants

♠ The infinitesimal method:

$$\mathbf{v}(I) = 0 \qquad \text{for every infinitesimal generator} \qquad \mathbf{v} \in \mathfrak{g}$$

$\Longrightarrow$ Requires solving differential equations.

♡ Moving frames.

- Completely algebraic.
- Can be adapted to arbitrary group and pseudo-group actions.
- Describes the complete structure of the differential invariant algebra $\mathcal{I}(G)$ — using only linear algebra & differentiation!
- Prescribes differential invariant signatures for equivalence and symmetry detection.

Infinitesimal Generators

Infinitesimal generators of action of G on M:

$$\mathbf{v}_\kappa = \sum_{i=1}^{p} \xi^i_\kappa(x,u)\,\frac{\partial}{\partial x^i} \;+\; \sum_{\alpha=1}^{q} \varphi^\alpha_\kappa(x,u)\,\frac{\partial}{\partial u^\alpha} \qquad \kappa = 1,\ldots,r$$

Prolonged infinitesimal generators on J^n:

$$\mathbf{v}^{(n)}_\kappa = \mathbf{v}_\kappa \;+\; \sum_{\alpha=1}^{q} \; \sum_{j=\#J=1}^{n} \varphi^\alpha_{J,\kappa}(x,u^{(j)})\;\frac{\partial}{\partial u^\alpha_J}$$

Prolongation formula:

$$\varphi^\alpha_{J,\kappa} = D_K\left(\varphi^\alpha_\kappa - \sum_{i=1}^{p} u^\alpha_i\,\xi^i_\kappa\right) + \sum_{i=1}^{p} u^\alpha_{J,i}\,\xi^i_\kappa$$

$D_1,\ldots,D_p$ — total derivatives

Recurrence Formulae

$$\mathcal{D}_j\,\iota(F) = \iota(D_j F) \;+\; \sum_{\kappa=1}^{r} R_j^\kappa\,\iota(\mathbf{v}_\kappa^{(n)}(F))$$

$\omega^i = \iota(dx^i)$ — invariant coframe

$\mathcal{D}_i = \iota(D_{x^i})$ — dual invariant differential operators

R_j^κ — Maurer–Cartan invariants

$\mathbf{v}_1, \ \ldots \ \mathbf{v}_r \in \mathfrak{g}$ — infinitesimal generators

$\mu^1, \ \ldots \ \mu^r \in \mathfrak{g}^*$ — dual Maurer–Cartan forms

The Maurer–Cartan Invariants

Invariantized Maurer–Cartan forms:

$$\gamma^\kappa = \rho^*(\mu^\kappa) \equiv \sum_{j=1}^{p} R_j^\kappa \, \omega^j$$

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Remark: When $G \subset \mathrm{GL}(N)$, the Maurer–Cartan invariants R_j^κ are the entries of the Frenet matrices

$$\mathcal{D}_i \, \rho(x, u^{(n)}) \cdot \rho(x, u^{(n)})^{-1}$$

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Theorem. (*E. Hubert*) The Maurer–Cartan invariants and, in the intransitive case, the order zero invariants serve to generate the differential invariant algebra $\mathcal{I}(G)$.

Recurrence Formulae

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♠ If $\iota(F) = c$ is a phantom differential invariant, then the left hand side of the recurrence formula is zero. The collection of all such phantom recurrence formulae form a linear algebraic system of equations that can be uniquely solved for the Maurer–Cartan invariants R_j^κ!

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♡ Once the Maurer–Cartan invariants are replaced by their explicit formulae, the induced recurrence relations completely determine the structure of the differential invariant algebra $\mathcal{I}(G)$!

The Differential Invariant Algebra

Thus, remarkably, the structure of $\mathcal{I}(G)$ can be completely determined without knowing the explicit formulae for either the moving frame, or the differential invariants, or the invariant differential operators!

The only required ingredients are the specification of the cross-section, and the standard formulae for the prolonged infinitesimal generators.

Theorem. If G acts transitively on M, or if the infinitesimal generator coefficients depend rationally in the coordinates, then all recurrence formulae are rational in the basic differential invariants and so $\mathcal{I}(G)$ is a rational, non-commutative differential algebra.

Euclidean Surfaces

$$M = \mathbb{R}^3 \qquad G = \mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3 \qquad \dim G = 6.$$

$$g \cdot z = R\,z + b, \qquad R^T R = I, \qquad z = \begin{pmatrix} x \\ y \\ u \end{pmatrix} \in \mathbb{R}^3.$$

Assume (for simplicity) that $S \subset \mathbb{R}^3$ is the graph of a function:

$$u = f(x, y)$$

Classical cross-section to the prolonged action on J^2:

$$x = y = u = u_x = u_y = u_{xy} = 0, \qquad u_{xx} \neq u_{yy}.$$

Invariantization — differential invariants: $\quad I_{jk} = \iota(u_{jk})$

Phantom differential invariants:

$$\iota(x) = \iota(y) = \iota(u) = \iota(u_x) = \iota(u_y) = \iota(u_{xy}) = 0$$

$$H_1 = H_2 = I_{00} = I_{10} = I_{01} = I_{11} = 0$$

Principal curvatures:

$$\kappa_1 = I_{20} = \iota(u_{xx}), \qquad \kappa_2 = I_{02} = \iota(u_{yy}),$$

$\star\star$ non-umbilic point: $\kappa_1 \neq \kappa_2$ $\star\star$

Mean and Gauss curvatures:

$$H = \tfrac{1}{2}(\kappa_1 + \kappa_2), \qquad K = \kappa_1 \kappa_2.$$

Invariant differential operators (diagonalizing Frenet frame):

$$\mathcal{D}_1 = \iota(D_x), \qquad \mathcal{D}_2 = \iota(D_y).$$

To obtain the recurrence formulae for the higher order differential invariants, we need the infinitesimal generators of $\mathfrak{g} = \mathfrak{se}(3)$:

$$\mathbf{v}_1 = -\,y\,\partial_x + x\,\partial_y$$

$$\mathbf{v}_2 = -\,u\,\partial_x + x\,\partial_u,$$

$$\mathbf{v}_3 = -\,u\,\partial_y + y\,\partial_u$$

$$\mathbf{w}_1 = \partial_x \qquad \mathbf{w}_2 = \partial_y \qquad \mathbf{w}_3 = \partial_u$$

- The translations will be ignored, as they play no role in the higher order recurrence formulae.

Recurrence formulae

$$\mathcal{D}_i \iota(u_{jk}) = \iota(D_i u_{jk}) + \sum_{\nu=1}^{3} \iota[\,\varphi_\nu^{jk}(x,y,u^{(j+k)})\,]\,R_i^\nu, \qquad j+k \geq 1$$

$$\mathcal{D}_1 I_{jk} = I_{j+1,k} + \sum_{\nu=1}^{3} \varphi_\nu^{jk}(0,0,I^{(j+k)})\,R_1^\nu$$

$$\mathcal{D}_2 I_{jk} = I_{j,k+1} + \sum_{\nu=1}^{3} \varphi_\nu^{jk}(0,0,I^{(j+k)})\,R_2^\nu$$

$\varphi_\nu^{jk}(0,0,I^{(j+k)}) = \iota[\,\varphi_\nu^{jk}(x,y,u^{(j+k)})\,]$ — invariantized prolonged infinitesimal generator coefficients

R_i^ν — Maurer–Cartan invariants

Phantom recurrence formulae:

$$0 = \mathcal{D}_1 I_{10} = I_{20} + R_1^2 \qquad 0 = \mathcal{D}_2 I_{10} = R_2^2$$

$$0 = \mathcal{D}_1 I_{01} = R_1^3 \qquad 0 = \mathcal{D}_2 I_{01} = I_{02} + R_2^3$$

$$0 = \mathcal{D}_1 I_{11} = I_{21} + (I_{20} - I_{02}) R_1^1 \qquad 0 = \mathcal{D}_2 I_{11} = I_{12} + (I_{20} - I_{02}) R_2^1$$

Maurer–Cartan invariants:

$$R_1 = (Y_2, -\kappa_1, 0) \qquad R_2 = (-Y_1, 0, -\kappa_2)$$

where

$$Y_1 = \frac{I_{12}}{I_{20} - I_{02}} = \frac{\mathcal{D}_1 \kappa_2}{\kappa_1 - \kappa_2} \qquad Y_2 = \frac{I_{21}}{I_{02} - I_{20}} = \frac{\mathcal{D}_2 \kappa_1}{\kappa_2 - \kappa_1}$$

are also the commutator invariants:

$$[\,\mathcal{D}_1, \mathcal{D}_2\,] = \mathcal{D}_1 \mathcal{D}_2 - \mathcal{D}_2 \mathcal{D}_1 = Y_2 \mathcal{D}_1 - Y_1 \mathcal{D}_2 .$$

Third order recurrence formulae:

$$I_{30} = \mathcal{D}_1 I_{20} = \kappa_{1,1} \qquad I_{21} = \mathcal{D}_2 I_{20} = \kappa_{1,2}$$

$$I_{12} = \mathcal{D}_1 I_{02} = \kappa_{2,1} \qquad I_{03} = \mathcal{D}_2 I_{02} = \kappa_{2,2}$$

The fourth order recurrence formulae

$$\mathcal{D}_2 I_{21} + \frac{I_{30} I_{12} - 2\, I_{12}^2}{\kappa_1 - \kappa_2} + \kappa_1 \kappa_2^2 = I_{22} = \mathcal{D}_1 I_{12} - \frac{I_{21} I_{03} - 2\, I_{21}^2}{\kappa_1 - \kappa_2} + \kappa_1^2 \kappa_2$$

lead to the Codazzi syzygy

$$\kappa_{1,22} - \kappa_{2,11} + \frac{\kappa_{1,1}\kappa_{2,1} + \kappa_{1,2}\kappa_{2,2} - 2\,\kappa_{2,1}^2 - 2\,\kappa_{1,2}^2}{\kappa_1 - \kappa_2} - \kappa_1 \kappa_2 (\kappa_1 - \kappa_2) = 0$$

- The principal curvatures κ_1, κ_2, or, equivalently, the Gauss and mean curvatures H, K, form a generating system for the differential invariant algebra.

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- The principal curvatures κ_1, κ_2, or, equivalently, the Gauss and mean curvatures H, K, form a generating system for the differential invariant algebra.

★ ★ Neither is a minimal generating set! ★ ★

Codazzi syzygy:

$$K = \kappa_1 \kappa_2 = -(\mathcal{D}_1 + Y_1)Y_1 - (\mathcal{D}_2 + Y_2)Y_2$$

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Gauss' Theorema Egregium

The Gauss curvature is intrinsic.

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Proof: The Frenet frame is intrinsic, hence so are the invariant differentiations and also commutator invariants. *Q.E.D.*

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Gauss' Theorema Egregium

The Gauss curvature is intrinsic.

Proof: The Frenet frame is intrinsic, hence so are the invariant differentiations and also commutator invariants. *Q.E.D.*

Theorem. For suitably nondegenerate surfaces, the mean curvature H is a generating differential invariant, i.e., all other Euclidean surface differential invariants can be expressed as functions of H and its invariant derivatives.

Proof: Since H, K generate the differential invariant algebra, it suffices to express the Gauss curvature K as a function of H and its derivatives. For this, the Codazzi syzygy implies that we need only express the commutator invariants in terms of H.

The commutator identity can be applied to any differential invariant. In particular,

$$\begin{aligned} \mathcal{D}_1 \mathcal{D}_2 H - \mathcal{D}_2 \mathcal{D}_1 H &= Y_2\, \mathcal{D}_1 H - Y_1\, \mathcal{D}_2 H \\ \mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_j H - \mathcal{D}_2 \mathcal{D}_1 \mathcal{D}_j H &= Y_2\, \mathcal{D}_1 \mathcal{D}_j H - Y_1\, \mathcal{D}_2 \mathcal{D}_j H \end{aligned} \qquad (*)$$

Provided the nondegeneracy condition

$$(\mathcal{D}_1 H)(\mathcal{D}_2 \mathcal{D}_j H) \neq (\mathcal{D}_2 H)(\mathcal{D}_1 \mathcal{D}_j H), \qquad \text{for } j = 1 \text{ or } 2$$

holds, we can solve $(*)$ for the commutator invariants as rational functions of invariant derivatives of H. *Q.E.D.*

Note: Constant Mean Curvature surfaces are degenerate.
Are there others?

Theorem. $G = \mathrm{SA}(3) = \mathrm{SL}(3) \ltimes \mathbb{R}^3$ acts on $S \subset M = \mathbb{R}^3$:

The algebra of differential invariants of generic equiaffine surfaces is generated by a single third order invariant, the Pick invariant.

Theorem. $G = \mathrm{SO}(4,1)$ acts on $S \subset M = \mathbb{R}^3$:

The algebra of differential invariants of generic conformal surfaces is generated by a single third order invariant.

Theorem. $G = \mathrm{PSL}(4)$ acts on $S \subset M = \mathbb{R}^3$:

The algebra of differential invariants of generic projective surfaces is generated by a single fourth order invariant.

Variational Problems

$\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x}$ — variational problem

$L(x, u^{(n)})$ — Lagrangian

Variational derivative — Euler-Lagrange equations: $\mathbf{E}(L) = 0$

components: $\mathbf{E}_\alpha(L) = \sum_J (-D)^J \dfrac{\partial L}{\partial u^\alpha_J}$

$$D_k F = \frac{\partial F}{\partial x^k} + \sum_{\alpha, J} u^\alpha_{J,k} \frac{\partial F}{\partial u^\alpha_J}$$

— total derivative of F with respect to x^k

Invariant Variational Problems

According to Lie, any G–invariant variational problem can be written in terms of the differential invariants:

$$\boxed{\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x} = \int P(\ \ldots\ \mathcal{D}_K I^\alpha\ \ldots\ )\, \boldsymbol{\omega}}$$

$I^1, \ldots, I^\ell$ — fundamental differential invariants

$\mathcal{D}_1, \ldots, \mathcal{D}_p$ — invariant differential operators

$\mathcal{D}_K I^\alpha$ — differentiated invariants

$\boldsymbol{\omega} = \omega^1 \wedge \cdots \wedge \omega^p$ — invariant volume form

If the variational problem is G-invariant, so

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x} = \int P(\ \ldots \ \mathcal{D}_K I^\alpha \ \ \ldots \ ) \ \boldsymbol{\omega}$$

then its Euler–Lagrange equations admit G as a symmetry group, and hence can also be expressed in terms of the differential invariants:

$$\mathbf{E}(L) \simeq F(\ \ldots \ \mathcal{D}_K I^\alpha \ \ \ldots \ ) = 0$$

Main Problem:

Construct F directly from P.

(P. Griffiths, I. Anderson)

Planar Euclidean group $G = \mathrm{SE}(2)$

$$\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}} \qquad \text{— curvature (differential invariant)}$$

$$ds = \sqrt{1+u_x^2}\, dx \qquad \text{— arc length}$$

$$\mathcal{D} = \frac{d}{ds} = \frac{1}{\sqrt{1+u_x^2}}\,\frac{d}{dx} \qquad \text{— arc length derivative}$$

Euclidean–invariant variational problem

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

Euler-Lagrange equations

$$\mathbf{E}(L) \simeq F(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ ) = 0$$

Euclidean Curve Examples

Minimal curves (geodesics):

$$\mathcal{I}[u] = \int ds = \int \sqrt{1 + u_x^2}\ dx$$

$$\mathbf{E}(L) = -\kappa = 0$$

$\Longrightarrow$ straight lines

The Elastica (Euler):

$$\mathcal{I}[u] = \int \tfrac{1}{2}\,\kappa^2\, ds = \int \frac{u_{xx}^2\, dx}{(1 + u_x^2)^{5/2}}$$

$$\mathbf{E}(L) = \kappa_{ss} + \tfrac{1}{2}\,\kappa^3 = 0$$

$\Longrightarrow$ elliptic functions

General Euclidean–invariant variational problem

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

General Euclidean–invariant variational problem

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

Invariantized Euler–Lagrange expression

$$\mathcal{E}(P) = \sum_{n=0}^{\infty} (-\mathcal{D})^n \frac{\partial P}{\partial \kappa_n} \qquad\qquad \mathcal{D} = \frac{d}{ds}$$

General Euclidean–invariant variational problem

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

Invariantized Euler–Lagrange expression

$$\mathcal{E}(P) = \sum_{n=0}^{\infty} (-\mathcal{D})^n \frac{\partial P}{\partial \kappa_n} \qquad\qquad \mathcal{D} = \frac{d}{ds}$$

Invariantized Hamiltonian

$$\mathcal{H}(P) = \sum_{i>j} \kappa_{i-j}\, (-\mathcal{D})^j \frac{\partial P}{\partial \kappa_i} - P$$

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

Euclidean–invariant Euler-Lagrange formula

$$\boxed{\mathbf{E}(L) = (\mathcal{D}^2 + \kappa^2)\ \mathcal{E}(P) + \kappa\, \mathcal{H}(P) = 0}$$

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

Euclidean–invariant Euler-Lagrange formula

$$\boxed{\mathbf{E}(L) = (\mathcal{D}^2 + \kappa^2)\ \mathcal{E}(P) + \kappa\, \mathcal{H}(P) = 0}$$

The Elastica: $\qquad \mathcal{I}[u] = \int \frac{1}{2}\, \kappa^2\, ds \qquad P = \frac{1}{2}\, \kappa^2$

$$\mathcal{E}(P) = \kappa \qquad \mathcal{H}(P) = -\, P = -\, \frac{1}{2}\, \kappa^2$$

$$\boxed{\begin{aligned} \mathbf{E}(L) &= (\mathcal{D}^2 + \kappa^2)\ \kappa + \kappa\, (\, -\, \tfrac{1}{2}\, \kappa^2\,) \\ &= \kappa_{ss} + \tfrac{1}{2}\, \kappa^3 = 0 \end{aligned}}$$

The shape of a Möbius strip

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The Möbius strip, obtained by taking a rectangular strip of plastic or paper, twisting one end through 180°, and then joining the ends, is the canonical example of a one-sided surface. Finding its characteristic developable shape has been an open problem ever since its first formulation in refs 1,2. Here we use the invariant variational bicomplex formalism to derive the first equilibrium equations for a wide developable strip undergoing large deformations, thereby giving the first non-trivial demonstration of the potential of this approach. We then formulate the boundary-value problem for the Möbius strip and solve it numerically. Solutions for increasing width show the formation of creases bounding nearly flat triangular regions, a feature also familiar from fabric draping[3] and paper crumpling[4,5]. This could give new insight into energy localization phenomena in unstretchable sheets[6], which might help to predict points of onset of tearing. It could also aid our understanding of the relationship between geometry and physical properties of nano- and microscopic Möbius strip structures[7–9].

It is fair to say that the Möbius strip is one of the few icons of mathematics that have been absorbed into wider culture. It has mathematical beauty and inspired artists such as Escher[10]. In engineering, pulley belts are often used in the form of Möbius strips to wear 'both' sides equally. At a much smaller scale, Möbius strips have recently been formed in ribbon-shaped $NbSe_3$ crystals under certain growth conditions involving a large temperature gradient[7,8].

Figure 1 Photo of a paper Möbius strip of aspect ratio 2π. The strip adopts a characteristic shape. Inextensibility of the material causes the surface to be developable. Its straight generators are drawn and the colouring varies according to the bending energy density.

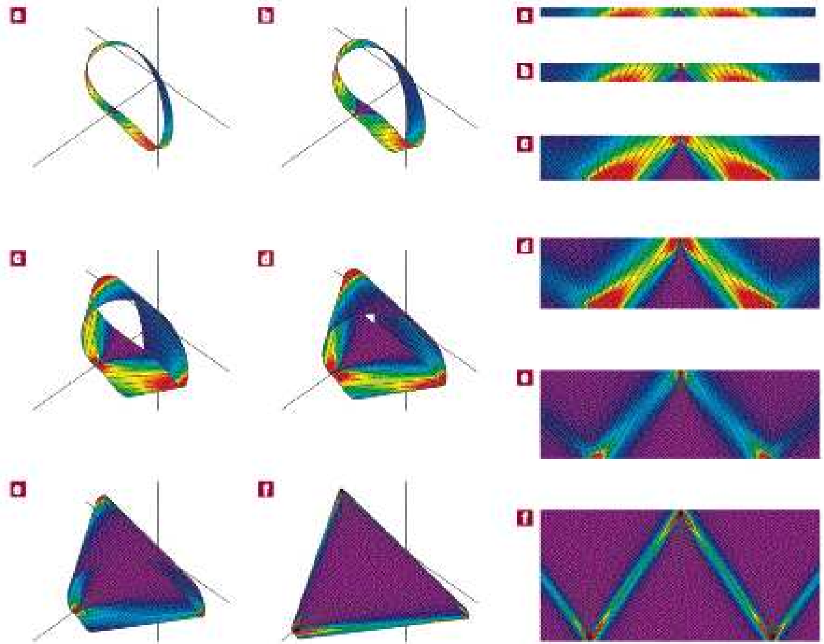


Figure 2 Computed Möbius strips. The left panel shows their three-dimensional shapes for $w = 0.1$ (**a**), 0.2 (**b**), 0.5 (**c**), 0.8 (**d**), 1.0 (**e**) and 1.5 (**f**), and the right panel the corresponding developments on the plane. The colouring changes according to the local bending energy density, from violet for regions of low bending to red for regions of high bending (scales are individually adjusted). Solution **c** may be compared with the paper model in Fig. 1 on which the generator field and density colouring have been printed.

The Infinite Jet Bundle

Jet bundles

$$M = \mathrm{J}^0 \longleftarrow \mathrm{J}^1 \longleftarrow \mathrm{J}^2 \longleftarrow \cdots$$

Inverse limit

$$\mathrm{J}^\infty = \lim_{n \to \infty} \mathrm{J}^n$$

Local coordinates

$$z^{(\infty)} = (x, u^{(\infty)}) = (\ \ldots \ x^i \ \ldots \ u^\alpha_J \ \ldots \ )$$

$\Longrightarrow$ Taylor series

Differential Forms

Coframe — basis for the cotangent space $T^*\mathrm{J}^\infty$:

- Horizontal one-forms

$$dx^1, \ldots, dx^p$$

- Contact (vertical) one-forms

$$\theta_J^\alpha = du_J^\alpha - \sum_{i=1}^{p} u_{J,i}^\alpha \, dx^i$$

Intrinsic definition of contact form

$$\theta \mid \mathrm{j}_\infty N = 0 \qquad \Longleftrightarrow \qquad \theta = \sum A_J^\alpha \, \theta_J^\alpha$$

The Variational Bicomplex

$\Longrightarrow$ *Dedecker, Vinogradov, Tsujishita, I. Anderson, ...*

Bigrading of the differential forms on J^∞:

$$\Omega^* = \bigoplus_{r,s} \Omega^{r,s}$$

$r = \#$ horizontal forms

$s = \#$ contact forms

Vertical and Horizontal Differentials

$$d = d_H + d_V$$

$$d_H : \Omega^{r,s} \longrightarrow \Omega^{r+1,s}$$

$$d_V : \Omega^{r,s} \longrightarrow \Omega^{r,s+1}$$

Vertical and Horizontal Differentials

$$F(x, u^{(n)}) \quad \text{— differential function}$$

$$d_H F = \sum_{i=1}^{p} (D_i F)\, dx^i \quad \text{— total differential}$$

$$d_V F = \sum_{\alpha, J} \frac{\partial F}{\partial u_J^\alpha}\, \theta_J^\alpha \quad \text{— first variation}$$

$$d_H(dx^i) = d_V(dx^i) = 0,$$

$$d_H(\theta_J^\alpha) = \sum_{i=1}^{p} dx^i \wedge \theta_{J,i}^\alpha \qquad d_V(\theta_J^\alpha) = 0$$

The Simplest Example

$$(x, u) \in M = \mathbb{R}^2$$

x — independent variable

u — dependent variable

Horizontal form

$$dx$$

Contact (vertical) forms

$$\theta = du - u_x\, dx$$

$$\theta_x = du_x - u_{xx}\, dx$$

$$\theta_{xx} = du_{xx} - u_{xxx}\, dx$$

$$\vdots$$

$$\theta = du - u_x\, dx, \qquad \theta_x = du_x - u_{xx}\, dx, \qquad \theta_{xx} = du_{xx} - u_{xxx}\, dx$$

Differential:

$$\begin{aligned} dF &= \frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial u}\, du + \frac{\partial F}{\partial u_x}\, du_x + \frac{\partial F}{\partial u_{xx}}\, du_{xx} + \cdots \\ &= (D_x F)\, dx + \frac{\partial F}{\partial u}\theta + \frac{\partial F}{\partial u_x}\theta_x + \frac{\partial F}{\partial u_{xx}}\theta_{xx} + \cdots \\ &= d_H F + d_V F \end{aligned}$$

Total derivative:

$$D_x F = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial u}\, u_x + \frac{\partial F}{\partial u_x}\, u_{xx} + \frac{\partial F}{\partial u_{xx}}\, u_{xxx} + \cdots$$

The Variational Bicomplex

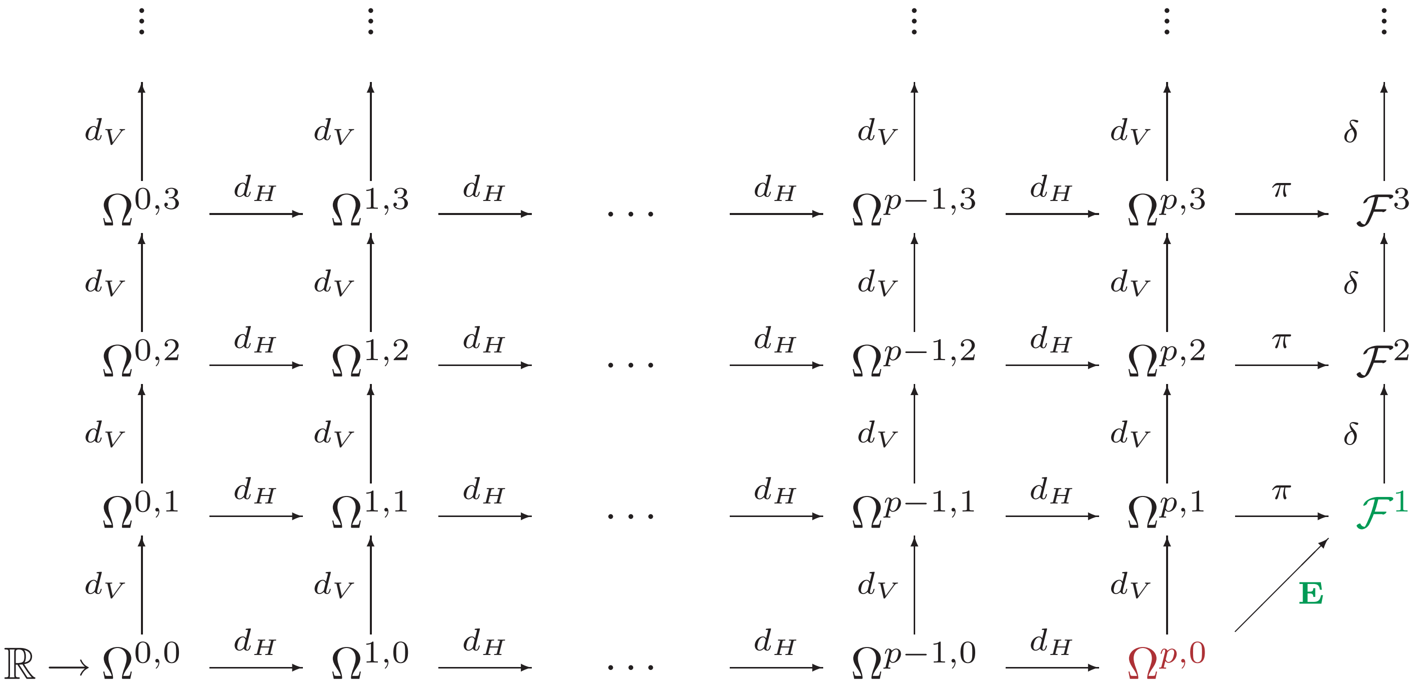
$$
\begin{array}{ccccccccccccc}
 & & \vdots & & \vdots & & & & \vdots & & \vdots & & \vdots \\
 & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & {\scriptstyle \delta}\uparrow \\
 & & \Omega^{0,3} & \xrightarrow{d_H} & \Omega^{1,3} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,3} & \xrightarrow{d_H} & \Omega^{p,3} & \xrightarrow{\pi} & \mathcal{F}^3 \\
 & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & {\scriptstyle \delta}\uparrow \\
 & & \Omega^{0,2} & \xrightarrow{d_H} & \Omega^{1,2} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,2} & \xrightarrow{d_H} & \Omega^{p,2} & \xrightarrow{\pi} & \mathcal{F}^2 \\
 & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & {\scriptstyle \delta}\uparrow \\
 & & \Omega^{0,1} & \xrightarrow{d_H} & \Omega^{1,1} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,1} & \xrightarrow{d_H} & \Omega^{p,1} & \xrightarrow{\pi} & \mathcal{F}^1 \\
 & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & \nearrow{\scriptstyle \mathbf{E}} & \\
\mathbb{R} & \longrightarrow & \Omega^{0,0} & \xrightarrow{d_H} & \Omega^{1,0} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,0} & \xrightarrow{d_H} & \Omega^{p,0} & &
\end{array}
$$

The Variational Bicomplex

$$
\begin{array}{ccccccccccccc}
 & & \vdots & & \vdots & & & & \vdots & & \vdots & & \vdots \\
 & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle \delta} \\
 & & \Omega^{0,3} & \xrightarrow{d_H} & \Omega^{1,3} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,3} & \xrightarrow{d_H} & \Omega^{p,3} & \xrightarrow{\pi} & \mathcal{F}^3 \\
 & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle \delta} \\
 & & \Omega^{0,2} & \xrightarrow{d_H} & \Omega^{1,2} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,2} & \xrightarrow{d_H} & \Omega^{p,2} & \xrightarrow{\pi} & \mathcal{F}^2 \\
 & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle \delta} \\
 & & \Omega^{0,1} & \xrightarrow{d_H} & \Omega^{1,1} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,1} & \xrightarrow{d_H} & \Omega^{p,1} & \xrightarrow{\pi} & \mathcal{F}^1 \\
 & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & \nearrow{\scriptstyle \mathbf{E}} & \\
\mathbb{R} & \longrightarrow & \Omega^{0,0} & \xrightarrow{d_H} & \Omega^{1,0} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,0} & \xrightarrow{d_H} & \Omega^{p,0} & &
\end{array}
$$

Lagrangians

The Variational Bicomplex



Lagrangians PDEs (Euler–Lagrange)

The Variational Bicomplex

$$
\begin{array}{ccccccccccccc}
 & & \vdots & & \vdots & & & & \vdots & & \vdots & & \vdots \\
 & & \uparrow d_V & & \uparrow d_V & & & & \uparrow d_V & & \uparrow d_V & & \uparrow \delta \\
 & & \Omega^{0,3} & \xrightarrow{d_H} & \Omega^{1,3} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,3} & \xrightarrow{d_H} & \Omega^{p,3} & \xrightarrow{\pi} & \mathcal{F}^3 \\
 & & \uparrow d_V & & \uparrow d_V & & & & \uparrow d_V & & \uparrow d_V & & \uparrow \delta \\
 & & \Omega^{0,2} & \xrightarrow{d_H} & \Omega^{1,2} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,2} & \xrightarrow{d_H} & \Omega^{p,2} & \xrightarrow{\pi} & \mathcal{F}^2 \\
 & & \uparrow d_V & & \uparrow d_V & & & & \uparrow d_V & & \uparrow d_V & & \uparrow \delta \\
 & & \Omega^{0,1} & \xrightarrow{d_H} & \Omega^{1,1} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,1} & \xrightarrow{d_H} & \Omega^{p,1} & \xrightarrow{\pi} & \mathcal{F}^1 \\
 & & \uparrow d_V & & \uparrow d_V & & & & \uparrow d_V & & \uparrow d_V & \nearrow \mathbf{E} & \\
\mathbb{R} & \longrightarrow & \Omega^{0,0} & \xrightarrow{d_H} & \Omega^{1,0} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,0} & \xrightarrow{d_H} & \Omega^{p,0} & &
\end{array}
$$

Lagrangians PDEs (Euler–Lagrange) Helmholtz conditions

The Variational Bicomplex

$$
\begin{array}{ccccccccccccc}
 & \vdots & & \vdots & & & & \vdots & & \vdots & & \vdots \\
 & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle \delta} \\
 & \Omega^{0,3} & \xrightarrow{d_H} & \Omega^{1,3} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,3} & \xrightarrow{d_H} & \Omega^{p,3} & \xrightarrow{\pi} & \mathcal{F}^3 \\
 & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle \delta} \\
 & \Omega^{0,2} & \xrightarrow{d_H} & \Omega^{1,2} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,2} & \xrightarrow{d_H} & \Omega^{p,2} & \xrightarrow{\pi} & \mathcal{F}^2 \\
 & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle \delta} \\
 & \Omega^{0,1} & \xrightarrow{d_H} & \Omega^{1,1} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,1} & \xrightarrow{d_H} & \Omega^{p,1} & \xrightarrow{\pi} & \mathcal{F}^1 \\
 & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & \nearrow{\scriptstyle \mathbf{E}} & \\
\mathbb{R} \longrightarrow & \Omega^{0,0} & \xrightarrow{d_H} & \Omega^{1,0} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,0} & \xrightarrow{d_H} & \Omega^{p,0} & &
\end{array}
$$

conservation laws Lagrangians PDEs (Euler–Lagrange) Helmholtz conditions

The Variational Derivative

$$\boxed{\mathbf{E} = \pi \circ d_V}$$

d_V — first variation

π — integration by parts = mod out by image of d_H

$$\Omega^{p,0} \xrightarrow{d_V} \Omega^{p,1} \xrightarrow{\pi} \mathcal{F}^1 = \Omega^{p,1} / d_H \Omega^{p-1,1}$$

$$\lambda = L\, d\mathbf{x} \longrightarrow \sum_{\alpha,J} \frac{\partial L}{\partial u_J^\alpha} \theta_J^\alpha \wedge d\mathbf{x} \longrightarrow \sum_{\alpha=1}^{q} \mathbf{E}_\alpha(L)\ \theta^\alpha \wedge d\mathbf{x}$$

Variational problem ⟶ First variation ⟶ Euler–Lagrange source form

The Simplest Example: $(x, u) \in M = \mathbb{R}^2$

Lagrangian form: $\quad \lambda = L(x, u^{(n)})\, dx \;\in\; \Omega^{1,0}$

The Simplest Example: $(x, u) \in M = \mathbb{R}^2$

Lagrangian form: $\lambda = L(x, u^{(n)})\, dx \ \in \ \Omega^{1,0}$

First variation — vertical derivative:

$$d\lambda = d_V\, \lambda = d_V\, L \wedge dx$$

$$= \left(\frac{\partial L}{\partial u}\, \theta + \frac{\partial L}{\partial u_x}\, \theta_x + \frac{\partial L}{\partial u_{xx}}\, \theta_{xx} + \cdots \right) \wedge\, dx \ \in \ \Omega^{1,1}$$

The Simplest Example: $(x,u) \in M = \mathbb{R}^2$

Lagrangian form: $\lambda = L(x, u^{(n)})\, dx \ \in \ \Omega^{1,0}$

First variation — vertical derivative:

$$d\lambda = d_V \lambda = d_V L \wedge dx$$

$$= \left(\frac{\partial L}{\partial u}\theta + \frac{\partial L}{\partial u_x}\theta_x + \frac{\partial L}{\partial u_{xx}}\theta_{xx} + \cdots \right) \wedge dx \ \in \ \Omega^{1,1}$$

Integration by parts — compute modulo im d_H :

$$d\lambda \ \sim \ \delta\lambda = \left(\frac{\partial L}{\partial u} - D_x \frac{\partial L}{\partial u_x} + D_x^2 \frac{\partial L}{\partial u_{xx}} - \cdots \right) \theta \wedge dx \ \in \ \mathcal{F}^1$$

$$= \mathbf{E}(L)\, \theta \wedge dx$$

$\Longrightarrow$ Euler-Lagrange source form.

To analyze invariant variational problems, invariant conservation laws, invariant flows, etc., we apply the moving frame invariantization process to the variational bicomplex:

Differential Invariants and Invariant Differential Forms

ι — invariantization associated with moving frame ρ.

- Fundamental differential invariants

$$H^i(x,u^{(n)}) = \iota(x^i) \qquad I^\alpha_K(x,u^{(n)}) = \iota(u^\alpha_K)$$

- Invariant horizontal forms

$$\varpi^i = \iota(dx^i)$$

- Invariant contact forms

$$\vartheta^\alpha_J = \iota(\theta^\alpha_J)$$

The Invariant "Quasi–Tricomplex"

Differential forms

$$\Omega^* = \bigoplus_{r,s} \widehat{\Omega}^{r,s}$$

Differential

$$d = d_{\mathcal{H}} + d_{\mathcal{V}} + d_{\mathcal{W}}$$

$$\begin{aligned} d_{\mathcal{H}} : \quad & \widehat{\Omega}^{r,s} \longrightarrow \widehat{\Omega}^{r+1,s} \\ d_{\mathcal{V}} : \quad & \widehat{\Omega}^{r,s} \longrightarrow \widehat{\Omega}^{r,s+1} \\ d_{\mathcal{W}} : \quad & \widehat{\Omega}^{r,s} \longrightarrow \widehat{\Omega}^{r-1,s+2} \end{aligned}$$

Key fact: invariantization and differentiation *do not commute*:

$$d\,\iota(\Omega) \neq \iota(d\Omega)$$

The Universal Recurrence Formula

$$d\,\iota(\Omega) = \iota(d\,\Omega) + \sum_{\kappa=1}^{r} \nu^{\kappa} \wedge \iota\,[\mathbf{v}_{\kappa}(\Omega)]$$

$\mathbf{v}_1, \ldots, \mathbf{v}_r$ — basis for $\mathfrak{g}$ — infinitesimal generators

$\nu^1, \ldots, \nu^r$ — invariantized dual Maurer–Cartan forms

The invariantized Maurer–Cartan forms are uniquely determined as solutions to the recurrence formulae for the phantom differential invariants.

$$\boxed{\boxed{d\,\iota(\Omega) = \iota(d\,\Omega) + \sum_{\kappa=1}^{r} \nu^{\kappa} \wedge \iota\,[\mathbf{v}_{\kappa}(\Omega)]}}$$

★ ★ ★ All identities, commutation formulae, syzygies, etc., among differential invariants and, more generally, the invariant variational bicomplex follow from this universal formula by letting Ω range over the basic functions and differential forms!

★ ★ ★ Moreover, determining the structure of the differential invariant algebra and invariant variational bicomplex requires only linear differential algebra, and not any explicit formulas for the moving frame, the differential invariants, the invariant differential forms, or the group transformations!

Euclidean plane curves

Fundamental normalized differential invariants

$$\left.\begin{aligned} \iota(x) &= H = 0 \\ \iota(u) &= I_0 = 0 \\ \iota(u_x) &= I_1 = 0 \end{aligned}\right\} \qquad \text{phantom diff. invs.}$$

$$\iota(u_{xx}) = I_2 = \kappa \qquad \iota(u_{xxx}) = I_3 = \kappa_s \qquad \iota(u_{xxxx}) = I_4 = \kappa_{ss} + 3\kappa^3$$

In general:

$$\iota(\, F(x, u, u_x, u_{xx}, u_{xxx}, u_{xxxx}, \ldots\,)) = F(0, 0, 0, \kappa, \kappa_s, \kappa_{ss} + 3\kappa^3, \ldots\,)$$

Left moving frame: $a = x \qquad b = u \qquad \phi = \tan^{-1} u_x$

$$dy = (\cos\phi + u_x \sin\phi)\, dx + (\sin\phi)\, \theta$$

Fully invariant arc length form:

$$\varpi = \iota(dx) = \omega + \eta$$

$$= \sqrt{1 + u_x^2}\, dx + \frac{u_x}{\sqrt{1 + u_x^2}}\, \theta$$

Invariant contact forms

$$\vartheta = \iota(\theta) = \frac{\theta}{\sqrt{1 + u_x^2}} \qquad \vartheta_1 = \iota(\theta_x) = \frac{(1 + u_x^2)\, \theta_x - u_x u_{xx} \theta}{(1 + u_x^2)^2}$$

Prolonged infinitesimal generators

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_u, \qquad \mathbf{v}_3 = -u\,\partial_x + x\,\partial_u + (1+u_x^2)\,\partial_{u_x} + 3\,u_x u_{xx}\,\partial_{u_{xx}} + \cdots$$

Basic recurrence formula

$$d\iota(F) = \iota(dF) + \iota(\mathbf{v}_1(F))\,\nu^1 + \iota(\mathbf{v}_2(F))\,\nu^2 + \iota(\mathbf{v}_3(F))\,\nu^3$$

Use phantom invariants

$$0 = dH = \iota(dx) + \iota(\mathbf{v}_1(x))\,\nu^1 + \iota(\mathbf{v}_2(x))\,\nu^2 + \iota(\mathbf{v}_3(x))\,\nu^3 = \varpi + \nu^1,$$

$$0 = dI_0 = \iota(du) + \iota(\mathbf{v}_1(u))\,\nu^1 + \iota(\mathbf{v}_2(u))\,\nu^2 + \iota(\mathbf{v}_3(u))\,\nu^3 = \vartheta + \nu^2,$$

$$0 = dI_1 = \iota(du_x) + \iota(\mathbf{v}_1(u_x))\,\nu^1 + \iota(\mathbf{v}_2(u_x))\,\nu^2 + \iota(\mathbf{v}_3(u_x))\,\nu^3 = \kappa\,\varpi + \vartheta_1 + \nu^3,$$

to solve for the Maurer–Cartan forms:

$$\boxed{\nu^1 = -\varpi, \qquad \nu^2 = -\vartheta, \qquad \nu^3 = -\kappa\,\varpi - \vartheta_1.}$$

$$\boxed{\nu^1 = -\varpi, \qquad \nu^2 = -\vartheta, \qquad \nu^3 = -\kappa\,\varpi - \vartheta_1.}$$

Recurrence formulae:

$$d\kappa = d\iota(u_{xx}) = \iota(du_{xx}) + \iota(\mathbf{v}_1(u_{xx}))\,\nu^1 + \iota(\mathbf{v}_2(u_{xx}))\,\nu^2 + \iota(\mathbf{v}_3(u_{xx}))\,\nu^3$$

$$= \iota(u_{xxx}\,dx + \theta_{xx}) - \iota(3\,u_x u_{xx})\,(\kappa\,\varpi + \vartheta_1) = I_3\varpi + \vartheta_2.$$

Therefore,

$$\mathcal{D}\kappa = \kappa_s = I_3, \qquad d_{\mathcal{V}}\,\kappa = \vartheta_2 = (\mathcal{D}^2 + \kappa^2)\,\vartheta$$

where the final formula follows from the contact form recurrence formulae

$$d\vartheta = d\iota(\theta_x) = \varpi \wedge \vartheta_1, \qquad d\vartheta_1 = d\iota(\theta) = \varpi \wedge (\vartheta_2 - \kappa^2\,\vartheta) - \kappa\,\vartheta_1 \wedge \vartheta$$

which imply

$$\vartheta_1 = \mathcal{D}\vartheta, \qquad \vartheta_2 = \mathcal{D}\vartheta_1 + \kappa^2\,\vartheta = (\mathcal{D}^2 + \kappa^2)\,\vartheta$$

Similarly,

$$\begin{aligned} d\varpi &= \iota(d^2x) + \nu^1 \wedge \iota(\mathbf{v}_1(dx)) + \nu^2 \wedge \iota(\mathbf{v}_2(dx)) + \nu^3 \wedge \iota(\mathbf{v}_3(dx)) \\ &= (\kappa\,\varpi + \vartheta_1) \wedge \iota(u_x\,dx + \theta) = \kappa\,\varpi \wedge \vartheta + \vartheta_1 \wedge \vartheta. \end{aligned}$$

In particular,

$$d_{\mathcal{V}}\,\varpi = -\,\kappa\,\vartheta \wedge \varpi$$

Key recurrence formulae:

$$\boxed{d_{\mathcal{V}}\,\kappa = (\mathcal{D}^2 + \kappa^2)\,\vartheta} \qquad \boxed{d_{\mathcal{V}}\,\varpi = -\,\kappa\,\vartheta \wedge \varpi}$$

Plane Curves

Invariant Lagrangian:

$$\widetilde{\lambda} = L(x, u^{(n)})\, dx = P(\kappa, \kappa_s, \ldots)\, \varpi$$

Euler–Lagrange form:

$$d_{\mathcal{V}}\, \widetilde{\lambda} \;\sim\; \mathbf{E}(L)\, \vartheta \wedge \varpi$$

Invariant Integration by Parts Formula

$$F\, d_{\mathcal{V}}\, (\mathcal{D}H) \wedge \varpi \;\sim\; -(\mathcal{D}F)\, d_{\mathcal{V}}\, H \wedge \varpi - (F \cdot \mathcal{D}H)\, d_{\mathcal{V}}\, \varpi$$

$$\begin{aligned} d_{\mathcal{V}}\, \widetilde{\lambda} &= d_{\mathcal{V}}\, P \wedge \varpi + P\, d_{\mathcal{V}}\, \varpi \\ &= \sum_n \frac{\partial P}{\partial \kappa_n}\; d_{\mathcal{V}}\, \kappa_n \wedge \varpi + P\, d_{\mathcal{V}}\, \varpi \\ &\sim\; \mathcal{E}(P)\; d_{\mathcal{V}}\, \kappa \wedge \varpi + \mathcal{H}(P)\; d_{\mathcal{V}}\, \varpi \end{aligned}$$

Vertical differentiation formulae

$$d_{\mathcal{V}}\, \kappa = \mathcal{A}(\vartheta) \qquad \text{— invariant variation of curvature}$$

$$d_{\mathcal{V}}\, \varpi = \mathcal{B}(\vartheta) \wedge \varpi \qquad \text{— invariant variation of arc length}$$

$$\begin{aligned} d_{\mathcal{V}}\, \widetilde{\lambda} \;&\sim\; \mathcal{E}(P)\, \mathcal{A}(\vartheta) \wedge \varpi + \mathcal{H}(P)\, \mathcal{B}(\vartheta) \wedge \varpi \\ &\sim\; \Big[\mathcal{A}^* \mathcal{E}(P) - \mathcal{B}^* \mathcal{H}(P) \Big]\, \vartheta \wedge \varpi \end{aligned}$$

Invariant Euler-Lagrange equation

$$\boxed{\mathcal{A}^* \mathcal{E}(P) - \mathcal{B}^* \mathcal{H}(P) = 0}$$

Euclidean Plane Curves

$$d_{\mathcal{V}}\,\kappa = (\mathcal{D}^2 + \kappa^2)\,\vartheta$$

Invariant variation of curvature:

$$\mathcal{A} = \mathcal{D}^2 + \kappa^2 \qquad\qquad \mathcal{A}^* = \mathcal{D}^2 + \kappa^2$$

$$d_{\mathcal{V}}\,\varpi = -\,\kappa\,\vartheta \wedge \varpi$$

Invariant variation of arc length:

$$\mathcal{B} = -\,\kappa \qquad\qquad \mathcal{B}^* = -\,\kappa$$

Euclidean–invariant Euler-Lagrange formula:

$$\mathbf{E}(L) = \mathcal{A}^*\mathcal{E}(P) - \mathcal{B}^*\mathcal{H}(P) = (\mathcal{D}^2 + \kappa^2)\;\mathcal{E}(P) + \kappa\,\mathcal{H}(P).$$

Invariant Plane Curve Flows

G — Lie group acting on $\mathbb{R}^2$

$C(t)$ — parametrized family of plane curves

G–invariant curve flow:

$$\frac{dC}{dt} = \mathbf{V} = I\,\mathbf{t} + J\,\mathbf{n}$$

- I, J — differential invariants
- $\mathbf{t}$ — "unit tangent"
- $\mathbf{n}$ — "unit normal"

$\mathbf{t}$, $\mathbf{n}$ — basis of the invariant vector fields dual to the invariant horizontal and order 0 contact one-forms:

$$\langle\, \mathbf{t}\,;\varpi\,\rangle = 1, \qquad \langle\, \mathbf{n}\,;\varpi\,\rangle = 0,$$

$$\langle\, \mathbf{t}\,;\vartheta\,\rangle = 0, \qquad \langle\, \mathbf{n}\,;\vartheta\,\rangle = 1.$$

$$C_t = \mathbf{V} = I\,\mathbf{t} + J\,\mathbf{n}$$

- The tangential component $I\,\mathbf{t}$ only affects the underlying parametrization of the curve. Thus, we can set I to be anything we like without affecting the curve evolution.
- There are two principal choices of tangential component:

Normal Curve Flows

$$C_t = J\,\mathbf{n}$$

Examples — Euclidean–invariant curve flows

- $C_t = \mathbf{n}$ — geometric optics or grassfire flow;
- $C_t = \kappa\,\mathbf{n}$ — curve shortening flow;
- $C_t = \kappa^{1/3}\,\mathbf{n}$ — equi-affine invariant curve shortening flow:
  $$C_t = \mathbf{n}_{\text{equi}-\text{affine}}\,;$$
- $C_t = \kappa_s\,\mathbf{n}$ — modified Korteweg–deVries flow;
- $C_t = \kappa_{ss}\,\mathbf{n}$ — thermal grooving of metals.

Intrinsic Curve Flows

Theorem. The curve flow generated by

$$\mathbf{v} = I\,\mathbf{t} + J\,\mathbf{n}$$

preserves arc length if and only if

$$\mathcal{B}(J) + \mathcal{D}\,I = 0.$$

$\mathcal{D}$ — invariant arc length derivative

$$d_{\mathcal{V}}\,\varpi = \mathcal{B}(\vartheta) \wedge \varpi$$

$\mathcal{B}$ — invariant variation of arc length operator

Normal Evolution of Differential Invariants

Theorem. Under a normal flow $C_t = J\,\mathbf{n}$,

$$\frac{\partial \kappa}{\partial t} = \mathcal{A}_\kappa(J), \qquad \frac{\partial \kappa_s}{\partial t} = \mathcal{A}_{\kappa_s}(J).$$

$$d_{\mathcal{V}}\,\kappa = \mathcal{A}_\kappa(\vartheta), \qquad d_{\mathcal{V}}\,\kappa_s = \mathcal{A}_{\kappa_s}(\vartheta).$$

$\mathcal{A}_\kappa = \mathcal{A}$ — invariant variation of κ

$\mathcal{A}_{\kappa_s} = \mathcal{D}\,\mathcal{A}_\kappa + \kappa\,\kappa_s$ — invariant variation of κ_s

Euclidean–invariant Curve Evolution

Normal flow: $C_t = J\,\mathbf{n}$

$$\frac{\partial \kappa}{\partial t} = \mathcal{A}_\kappa(J) = (\mathcal{D}^2 + \kappa^2)\,J,$$

$$\frac{\partial \kappa_s}{\partial t} = \mathcal{A}_{\kappa_s}(J) = (\mathcal{D}^3 + \kappa^2\mathcal{D} + 3\,\kappa\,\kappa_s)\,J.$$

Warning: For non-intrinsic flows, ∂_t and ∂_s *do not commute*!

Grassfire flow: $J = 1$

$$\frac{\partial \kappa}{\partial t} = \kappa^2, \qquad \frac{\partial \kappa_s}{\partial t} = 3\,\kappa\,\kappa_s, \qquad \ldots$$

$\Longrightarrow$ caustics

Euclidean Signature Evolution

Evolution of the Euclidean signature curve

$$\kappa_s = \Phi(t, \kappa).$$

Grassfire flow:

$$\frac{\partial \Phi}{\partial t} = 3\,\kappa\,\Phi - \kappa^2\,\frac{\partial \Phi}{\partial \kappa}\,.$$

Curve shortening flow:

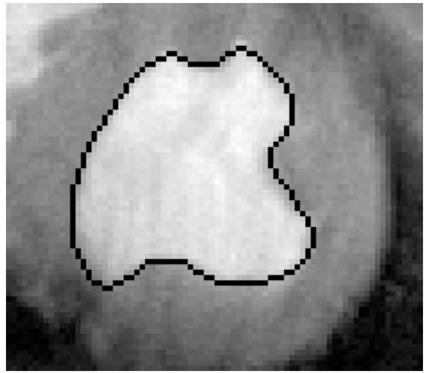
$$\frac{\partial \Phi}{\partial t} = \Phi^2\,\Phi_{\kappa\kappa} - \kappa^3 \Phi_\kappa + 4\,\kappa^2 \Phi.$$

Modified Korteweg-deVries flow:

$$\frac{\partial \Phi}{\partial t} = \Phi^3\,\Phi_{\kappa\kappa\kappa} + 3\,\Phi^2\,\Phi_\kappa\,\Phi_{\kappa\kappa} + 3\,\kappa\,\Phi^2.$$

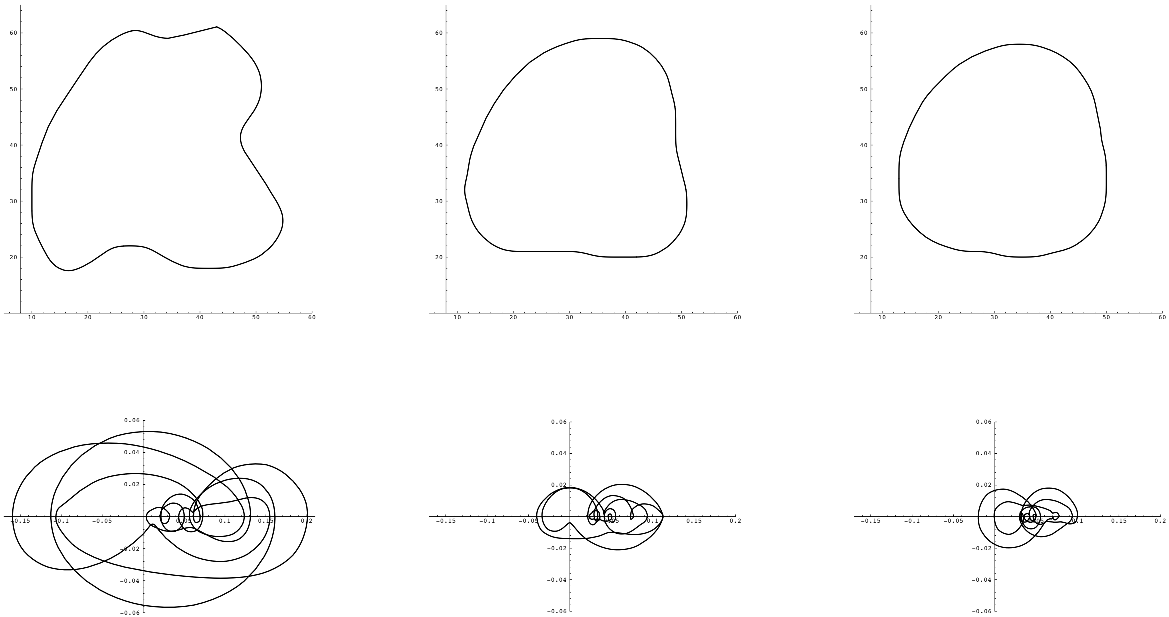
Canine Left Ventricle Signature

Original Canine Heart
MRI Image



Boundary of Left Ventricle

Smoothed Ventricle Signature



Intrinsic Evolution of Differential Invariants

Theorem.

Under an arc-length preserving flow,

$$\kappa_t = \mathcal{R}(J) \qquad \text{where} \qquad \mathcal{R} = \mathcal{A} - \kappa_s \mathcal{D}^{-1} \mathcal{B} \tag{$*$}$$

In surprisingly many situations, (*) is a well-known integrable evolution equation, and $\mathcal{R}$ is its recursion operator!

$\Longrightarrow$ Hasimoto

$\Longrightarrow$ Langer, Singer, Perline

$\Longrightarrow$ Marí–Beffa, Sanders, Wang

$\Longrightarrow$ Qu, Chou, and many more ...

Euclidean plane curves

$$G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$$

$$d_{\mathcal{V}}\, \kappa = (\mathcal{D}^2 + \kappa^2)\, \vartheta, \qquad d_{\mathcal{V}}\, \varpi = -\, \kappa\, \vartheta \wedge \varpi$$

$$\Longrightarrow \qquad \mathcal{A} = \mathcal{D}^2 + \kappa^2, \qquad \mathcal{B} = -\, \kappa$$

$$\mathcal{R} = \mathcal{A} - \kappa_s \mathcal{D}^{-1} \mathcal{B} = \mathcal{D}^2 + \kappa^2 + \kappa_s \mathcal{D}^{-1} \cdot \kappa$$

$$\kappa_t = \mathcal{R}(\kappa_s) = \kappa_{sss} + \tfrac{3}{2}\, \kappa^2 \kappa_s$$

$\Longrightarrow$ modified Korteweg-deVries equation

Equi-affine plane curves

$$G = \mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$$

$$d_{\mathcal{V}}\, \kappa = \mathcal{A}(\vartheta), \qquad d_{\mathcal{V}}\, \varpi = \mathcal{B}(\vartheta) \wedge \varpi$$

$$\mathcal{A} = \mathcal{D}^4 + \tfrac{5}{3}\, \kappa\, \mathcal{D}^2 + \tfrac{5}{3}\, \kappa_s \mathcal{D} + \tfrac{1}{3}\, \kappa_{ss} + \tfrac{4}{9}\, \kappa^2,$$

$$\mathcal{B} = \tfrac{1}{3}\, \mathcal{D}^2 - \tfrac{2}{9}\, \kappa,$$

$$\mathcal{R} = \mathcal{A} - \kappa_s \mathcal{D}^{-1} \mathcal{B}$$

$$= \mathcal{D}^4 + \tfrac{5}{3}\, \kappa\, \mathcal{D}^2 + \tfrac{4}{3}\, \kappa_s \mathcal{D} + \tfrac{1}{3}\, \kappa_{ss} + \tfrac{4}{9}\, \kappa^2 + \tfrac{2}{9}\, \kappa_s \mathcal{D}^{-1} \cdot \kappa$$

$$\kappa_t = \mathcal{R}(\kappa_s) = \kappa_{5s} + 2\, \kappa\, \kappa_{ss} + \tfrac{4}{3}\, \kappa_s^2 + \tfrac{5}{9}\, \kappa^2\, \kappa_s$$

$\Longrightarrow$ Sawada–Kotera equation

Euclidean space curves

$$G = \mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$$

$$\begin{pmatrix} d_{\mathcal{V}}\,\kappa \\ d_{\mathcal{V}}\,\tau \end{pmatrix} = \mathcal{A} \begin{pmatrix} \vartheta_1 \\ \vartheta_2 \end{pmatrix} \qquad d_{\mathcal{V}}\,\varpi = \mathcal{B} \begin{pmatrix} \vartheta_1 \\ \vartheta_2 \end{pmatrix} \wedge \varpi$$

$$\mathcal{A} = \begin{pmatrix} D_s^2 + (\kappa^2 - \tau^2) & -2\tau D_s - \tau_s \\ \frac{2\tau}{\kappa} D_s^2 + \frac{3\kappa\tau_s - 2\kappa_s\tau}{\kappa^2} D_s + \frac{\kappa\tau_{ss} - \kappa_s\tau_s + 2\kappa^3\tau}{\kappa^2} & \frac{1}{\kappa} D_s^3 - \frac{\kappa_s}{\kappa^2} D_s^2 + \frac{\kappa^2 - \tau^2}{\kappa} D_s + \frac{\kappa_s\tau^2 - 2\kappa\tau\tau_s}{\kappa^2} \end{pmatrix}$$

$$\mathcal{B} = \begin{pmatrix} \kappa & 0 \end{pmatrix}$$

Recursion operator:

$$\mathcal{R} = \mathcal{A} - \begin{pmatrix} \kappa_s \\ \tau_s \end{pmatrix} \mathcal{D}^{-1} \mathcal{B}$$

$$\begin{pmatrix} \kappa_t \\ \tau_t \end{pmatrix} = \mathcal{R} \begin{pmatrix} \kappa_s \\ \tau_s \end{pmatrix}$$

$\Longrightarrow$ vortex filament flow

$\Longrightarrow$ nonlinear Schrödinger equation (Hasimoto)