Reassembly of Broken Objects

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Symmetry → *Group Theory!*

Next to the concept of a function, which is the most important concept pervading the whole of mathematics, the concept of a group is of the greatest significance in the various branches of mathematics and its applications.

— P.S. Alexandroff

Groups

Definition. A group G is a set with a binary operation $g \cdot h$ satisfying

- Associativity: $g \cdot (h \cdot k) = (g \cdot h) \cdot k$
- Identity: $g \cdot e = g = e \cdot g$
- Inverse: $g \cdot g^{-1} = e = g^{-1} \cdot g$

$\Longrightarrow$ not necessarily commutative: $g \cdot h \neq h \cdot g$

Examples of groups

The integers

…, -4, -3, -2, -1, 0, 1, 2, 3, 4, …

Group operation: addition $3 + 5 = 8$

Identity: zero $3 + 0 = 3 = 0 + 3$

Inverse: negative $7 + (-7) = 0 = (-7) + 7$

Examples of groups

The rational numbers (fractions)

Group operation: addition $1/4 + 5/3 = 23/12$

Identity: zero $5/3 + 0 = 5/3 = 0 + 5/3$

Inverse: negative $7/2 + (-7/2) = 0 = (-7/2) + 7/2$

Examples of groups

The positive rational numbers

Group operation: multiplication $1/4 \times 5/3 = 5/12$

Identity: one $5/3 \times 1 = 5/3 = 1 \times 5/3$

Inverse: reciprocal $7/2 \times 2/7 = 1 = 2/7 \times 7/2$

Examples of groups

The positive real numbers

Group operation: multiplication

$\sqrt{2}$ x π $= \sqrt{2}\,\pi$ = 4.44288293815836624701588099006....

Identity: one $\qquad$ π x 1 = π = 1 x π

Inverse: reciprocal $\qquad$ π x 1/ π = 1 = 1/ π x π

Examples of groups

Non-singular 2 x 2 matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad h = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \qquad ad - bc \neq 0 \neq xw - yz$$

Group operation:

$$g{\cdot}h = \begin{pmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{pmatrix} \neq \begin{pmatrix} ax + cy & bx + dy \\ az + cw & bz + dw \end{pmatrix} = h{\cdot}g$$

Identity: $$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad e \cdot g = g = g \cdot e$$

Inverse: $$g^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad g \cdot g^{-1} = e^{-1} = g \cdot g$$

Symmetry Groups

A symmetry g of a geometric object S is a transformation that preserves it: $g \cdot S = S$

The set of symmetries of a geometric object forms a group

The group operation is composition: $g \cdot h$ = first do h, then do g

The composition of two symmetries is a symmetry

The identity (do nothing) is always a symmetry

The inverse of a symmetry (undo it) is a symmetry

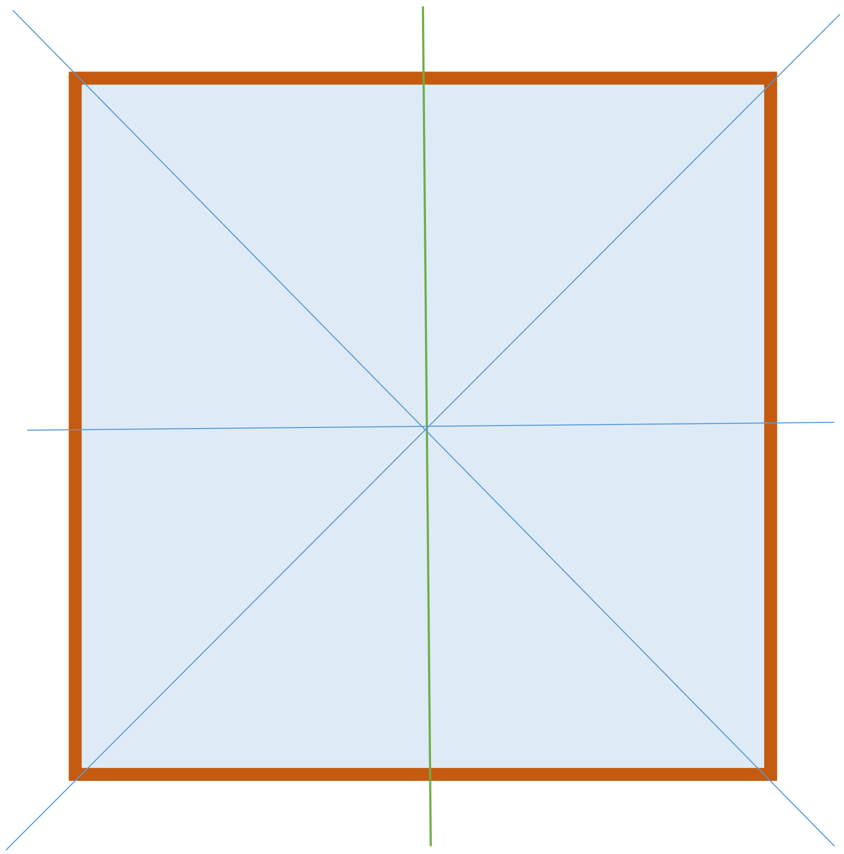
Symmetry

Definition. A symmetry of a set S is a transformation that preserves it:

$$g \cdot S = S$$

$\star\star$ The set of symmetries forms a group G_S, called the symmetry group of the set S.

Discrete Symmetry Group



Rotations by 90º, 180º, 270º

and 0º (identity)

… and 4 reflections
(mirror image)

Wallpaper patterns

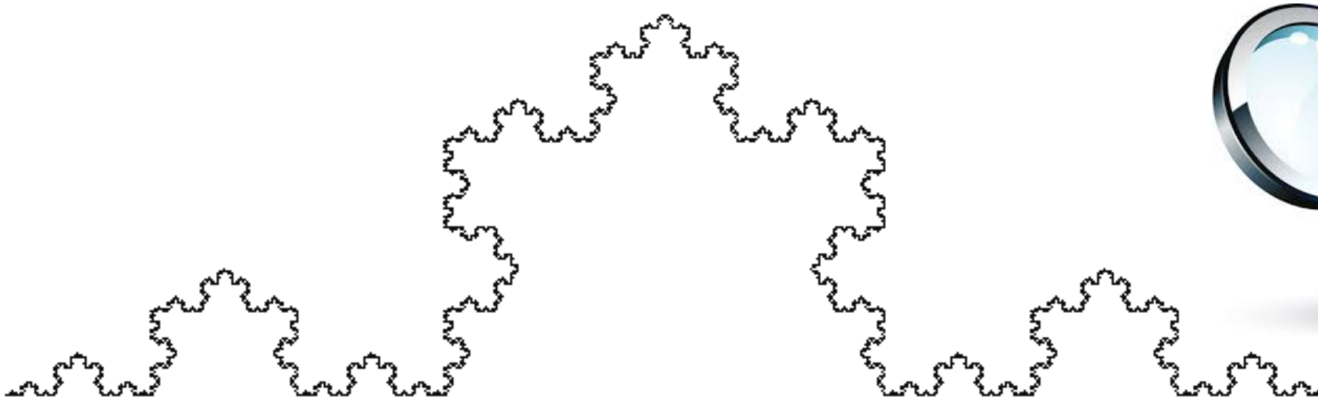
17 symmetry types

Tiling — The Alhambra, Spain

Tiling — The Alhambra, Spain

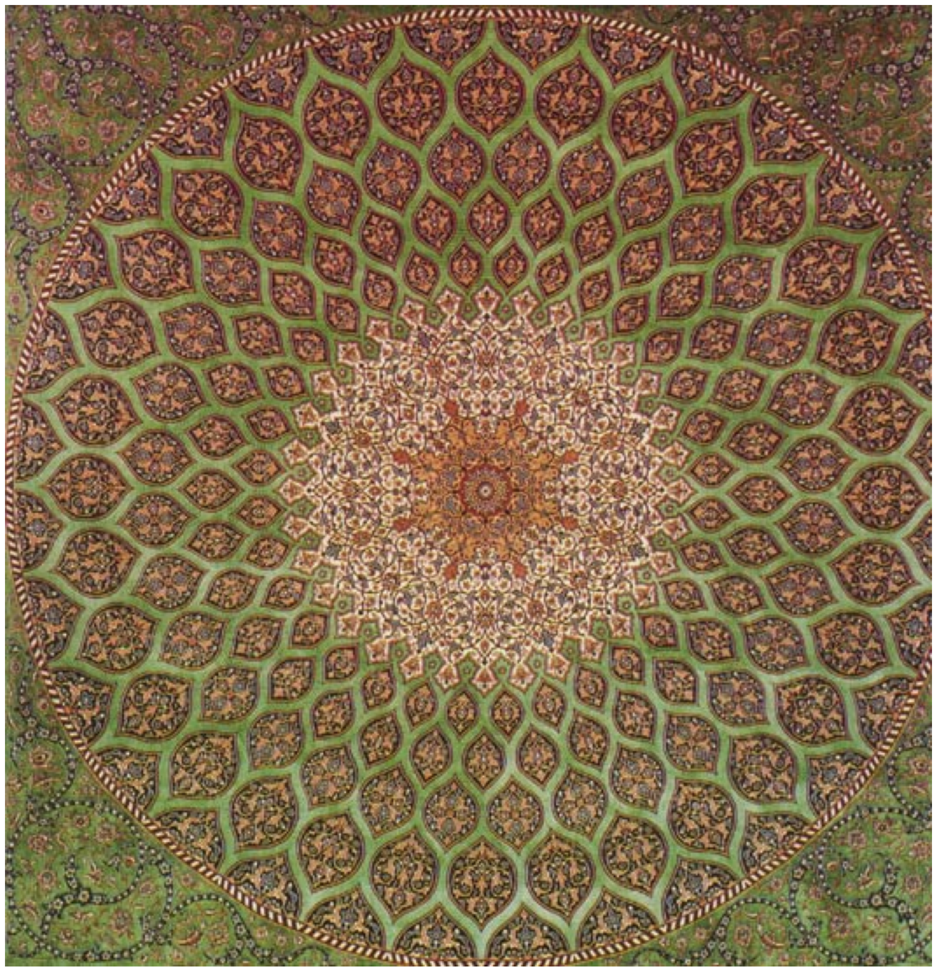

Crystallography
❄ 230 groups

The Koch snowflake — a fractal curve

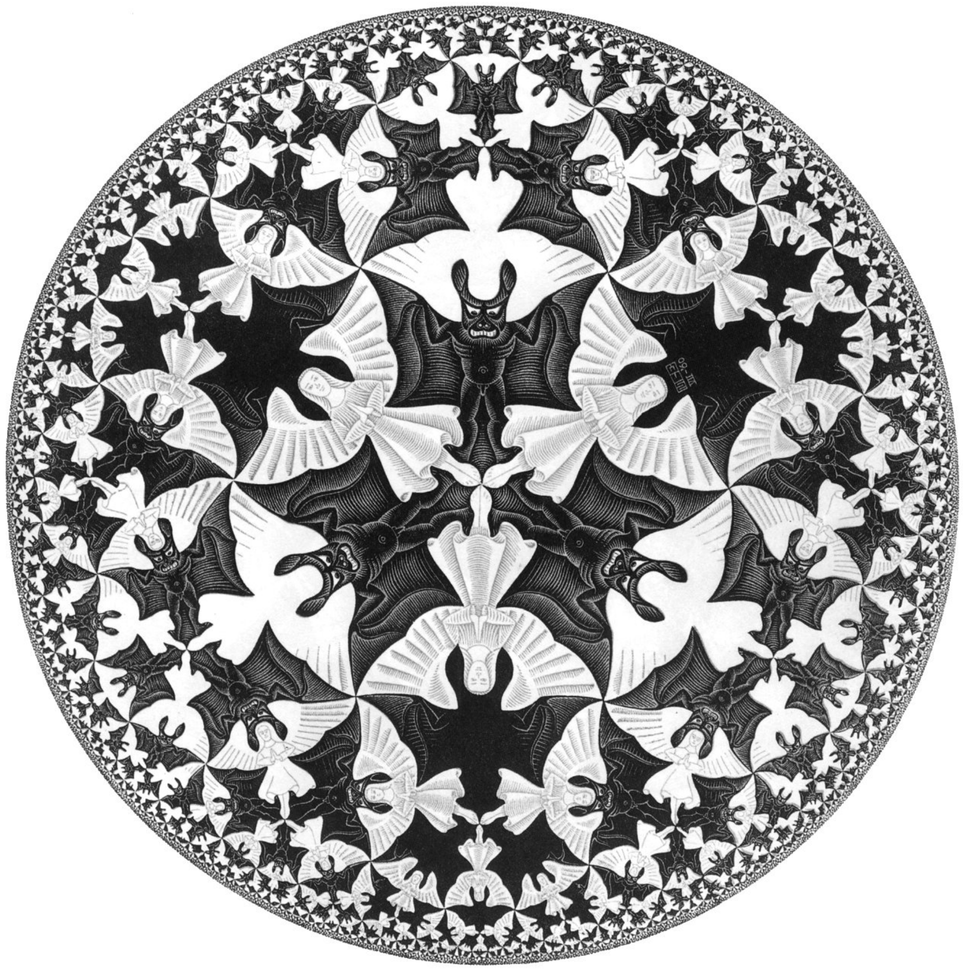


❋❋ Scaling symmetry

Dome of the Sheikh Lotfollah Mosque — Isfahan, Iran

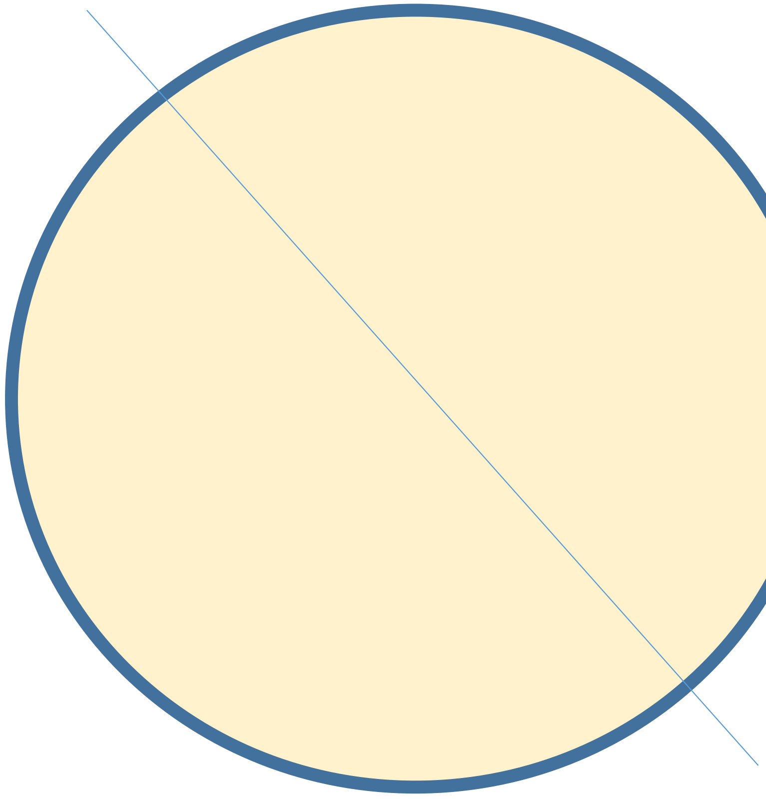


M.C. Escher — Circle Limit IV



❋❋ Conformal symmetry

Continuous Symmetry Group



Rotations through any angle

and reflections

and conformal inversions

$$\overline{x} = \frac{x}{x^2 + y^2} \qquad \overline{y} = \frac{y}{x^2 + y^2}$$

Continuous Symmetry Group = Lie Group

Rotations through any angle

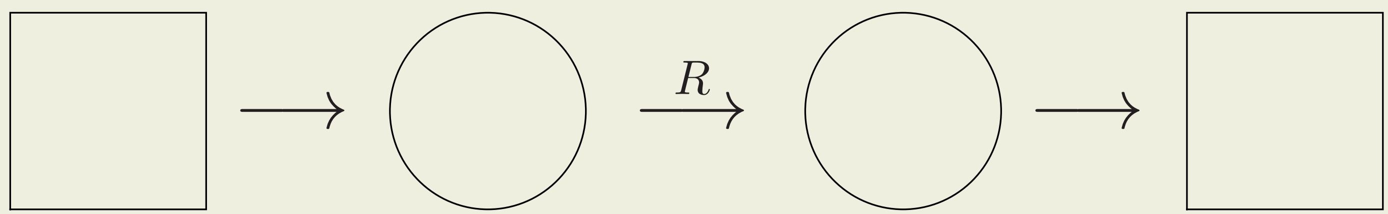
and reflections

and conformal inversions

$$\overline{x} = \frac{x}{x^2 + y^2} \qquad \overline{y} = \frac{y}{x^2 + y^2}$$

A continuous symmetry group is known as a Lie group in honor of the nineteenth century Norwegian mathematician Sophus Lie

Continuous Symmetries of a Square



Symmetry

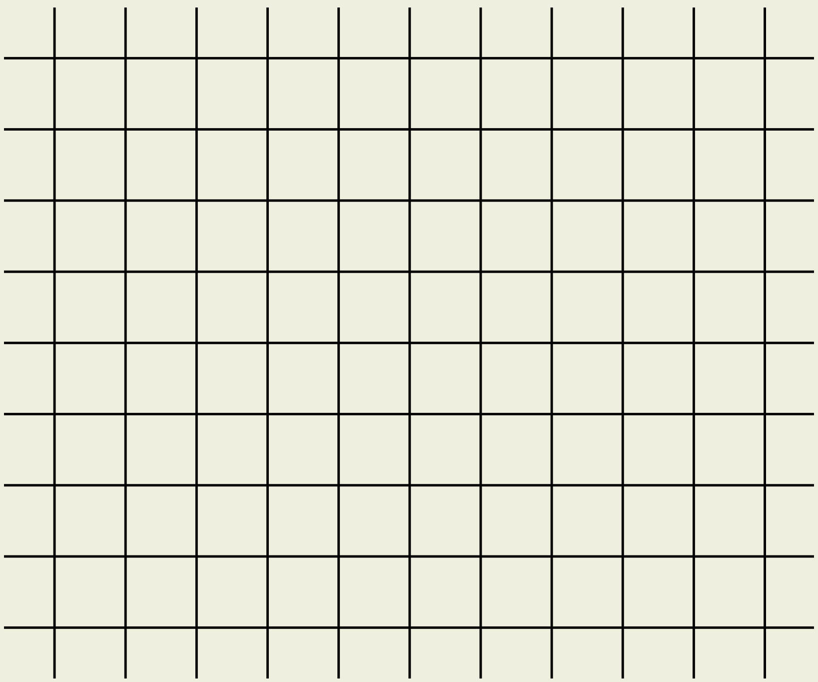
⋆ To define the set of symmetries requires a priori specification of the allowable transformations

G — transformation group containing all allowable transformations of the ambient space M

Definition. A symmetry of a subset $S \subset M$ is an allowable transformation $g \in G$ that preserves it:

$$g \cdot S = S$$

What is the Symmetry Group?



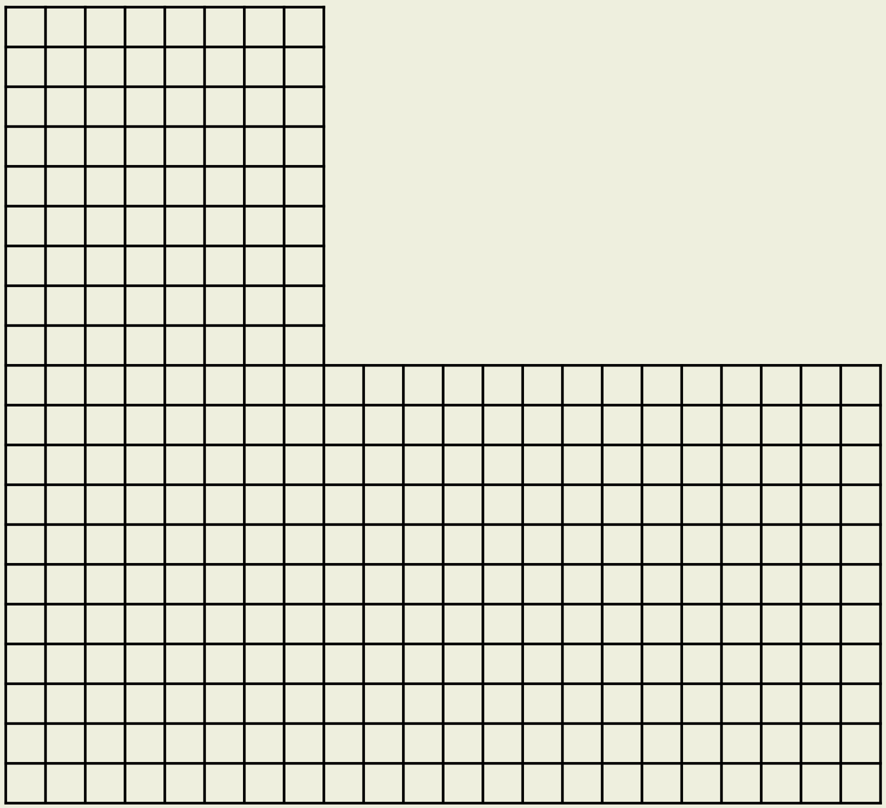
Allowable transformations:

Rigid motions

$G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$

$G_S = \mathbb{Z}_4 \ltimes \mathbb{Z}^2$

What is the Symmetry Group?



Allowable transformations:

Rigid motions

$G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$

$G_S = \{e\}$

Local Symmetries

Definition. $g \in G$ is a local symmetry of $S \subset M$ based at a point $z \in S$ if there is an open neighborhood $z \in U \subset M$ such that

$$g \cdot (S \cap U) = S \cap (g \cdot U)$$

$\star\star$ The set of all local symmetries forms a groupoid!

Definition. A groupoid is a small category such that every morphism has an inverse.

- ⋆ Groupoids form the appropriate framework for studying objects with variable symmetry.
- ⋆ Symmetry groupoids are not necessarily Lie groupoids

Groupoids

$\Longrightarrow$ In practice you are only allowed to multiply groupoid elements $g \cdot h$ when

$$\text{source (domain) of } g = \text{target (range) of } h$$

Similarly for inverses g^{-1} and the identities e.

A groupoid is a “collection of arrows”:

$$\begin{tikzcd} \bullet & \bullet \arrow[l, bend right, "g"] & \bullet \arrow[l, bend right, "h"] \arrow[ll, bend right, "g \cdot h"'] \end{tikzcd}$$

Jet Groupoids

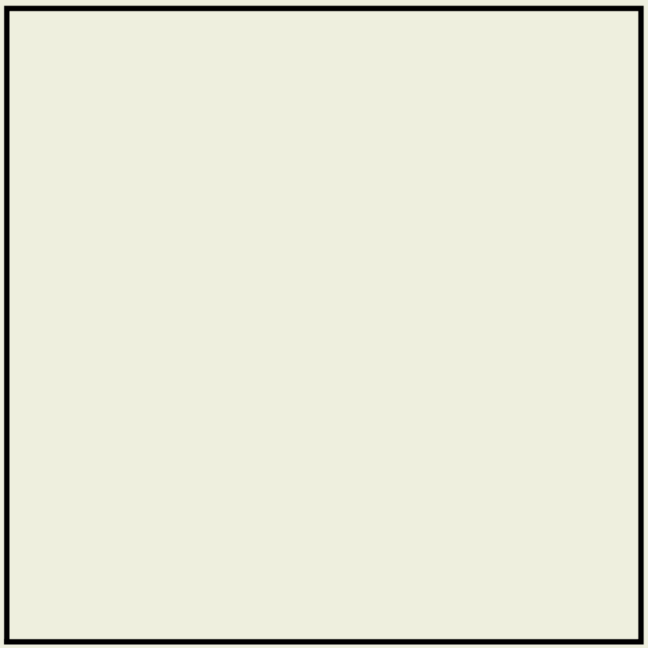
$\Longrightarrow$ Ehresmann

The set of Taylor polynomials of degree $\leq\ n$, or Taylor series ($n = \infty$) of local diffeomorphisms $\Psi : M \to M$ forms a groupoid.

$\diamondsuit$ Algebraic composition of Taylor polynomials/series is well-defined only when the source of the second matches the target of the first.

$\Longrightarrow$ Lie pseudo-groups

What is the Symmetry Groupoid?



$G = \mathrm{SE}(2)$

Corners:

$$G_z = G_S = \mathbb{Z}_4$$

Sides: G_z generated by

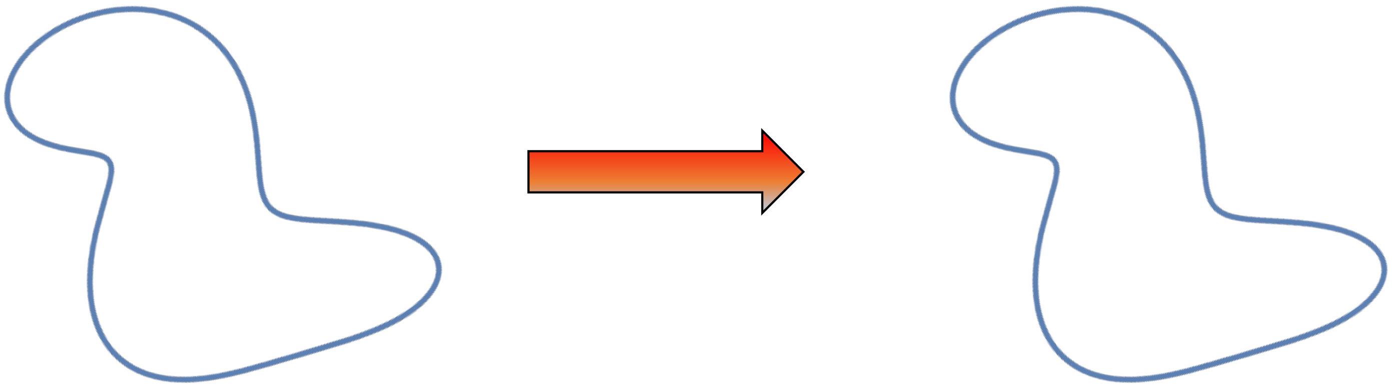
$G_S = \mathbb{Z}_4$

some translations

180° rotation around z

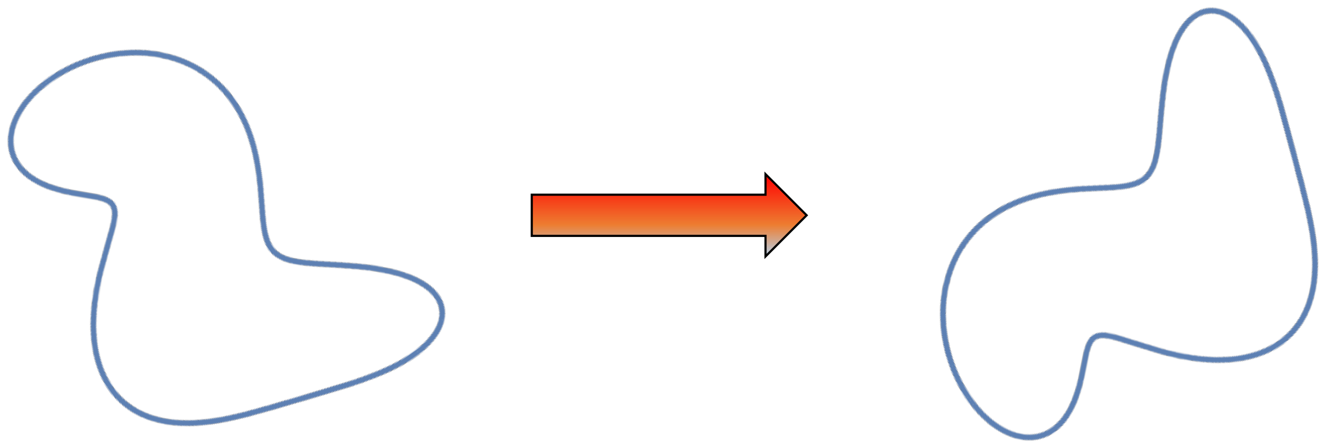
Transformation groups

Translations

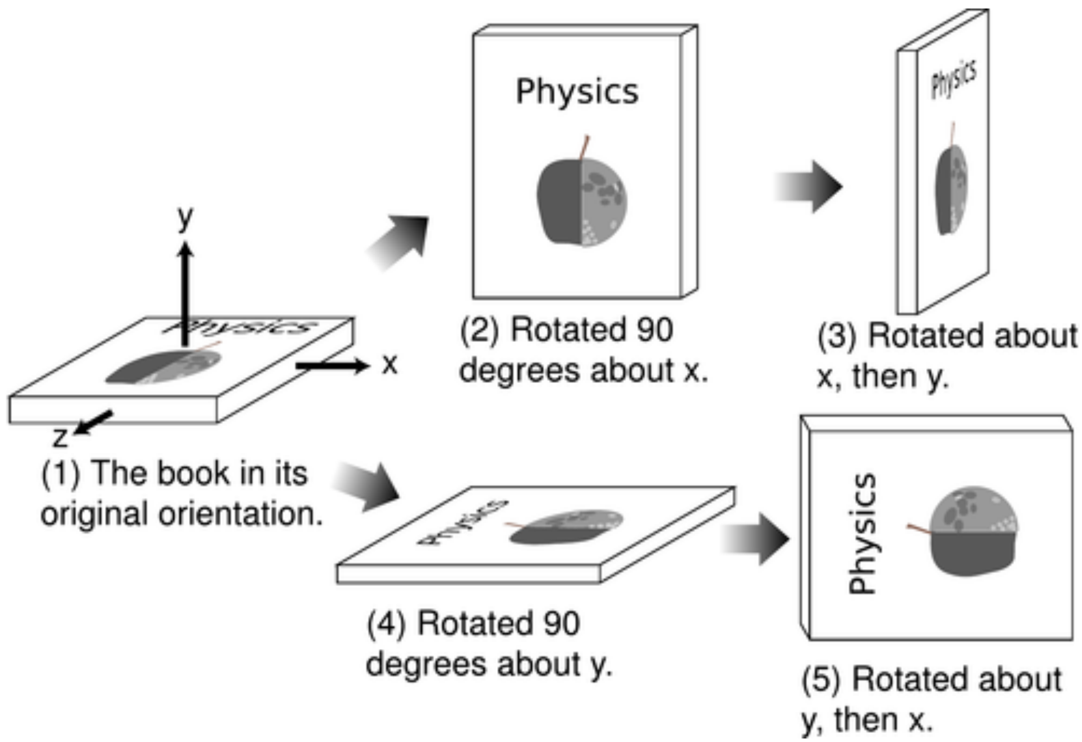


Transformation groups

Rotations

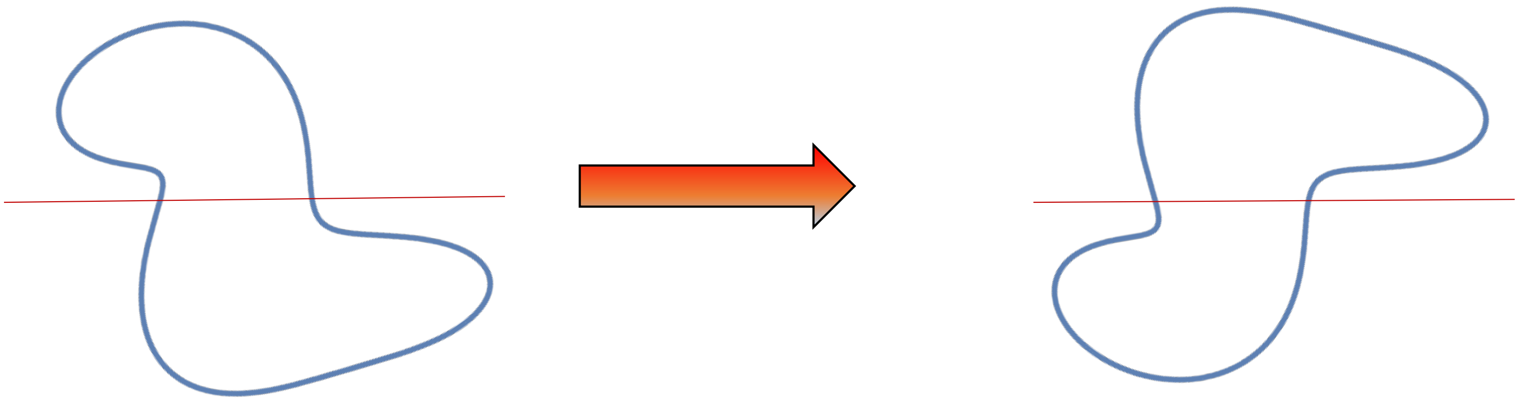


Noncommutativity of 3D rotations — order matters!



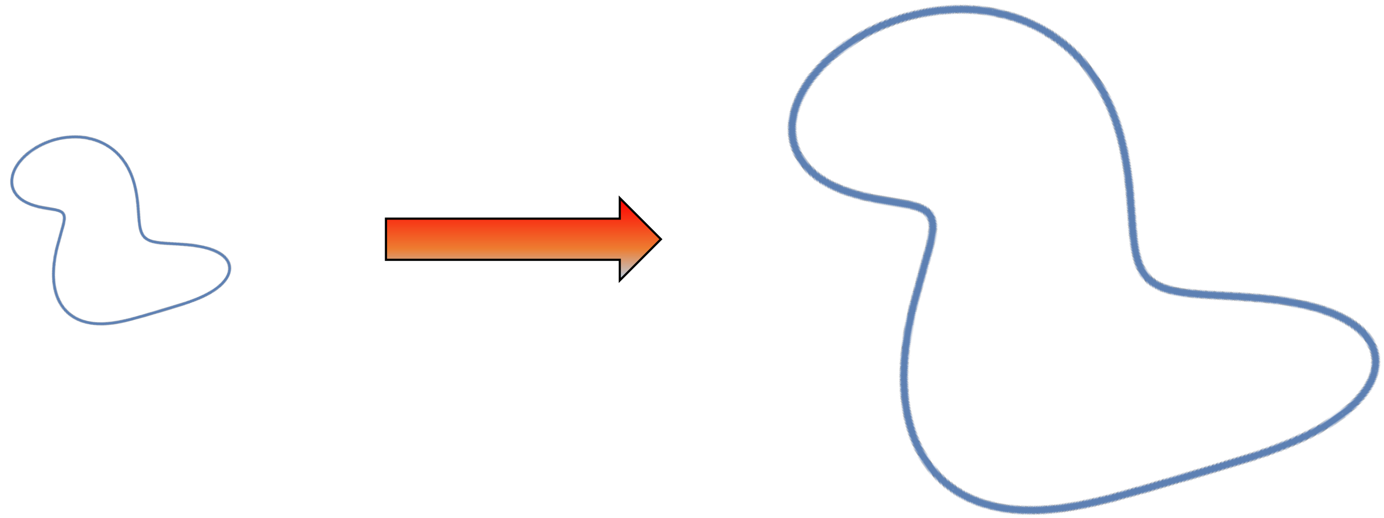
Transformation groups

Reflections



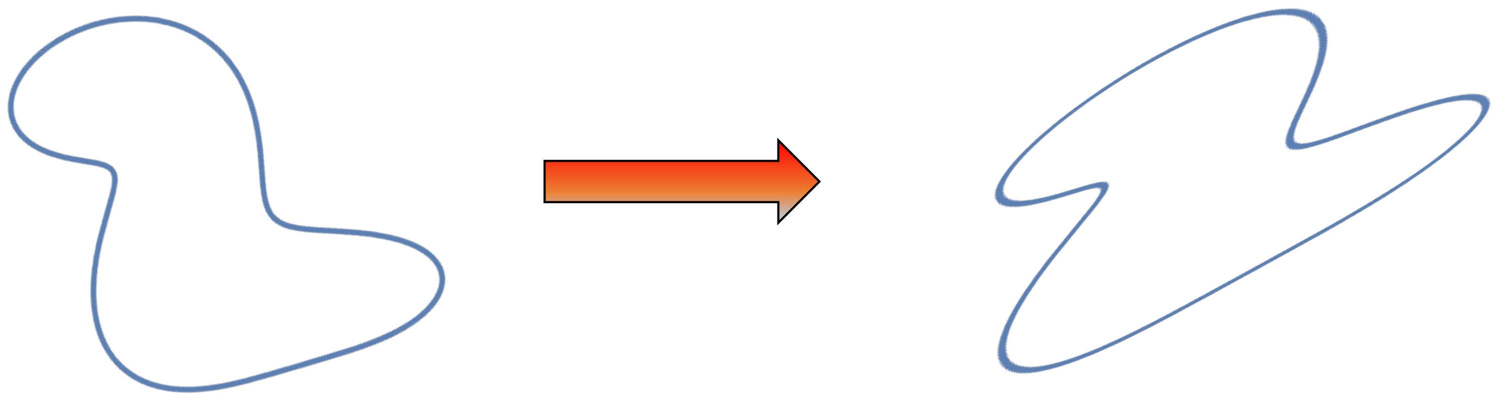
Transformation groups

Scaling (similarity)



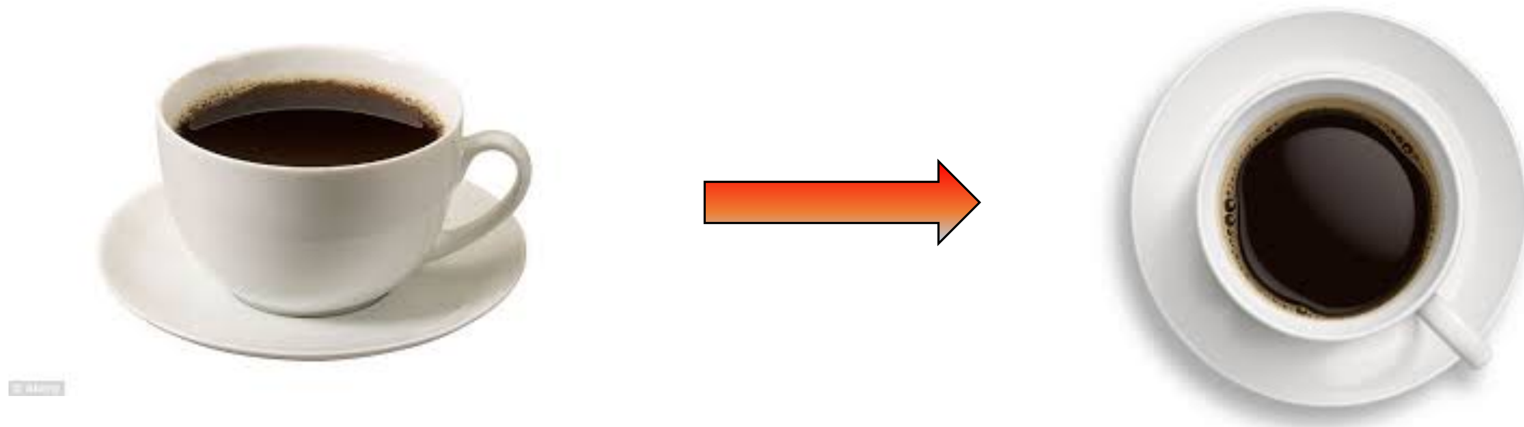
Transformation groups

Projective Transformation

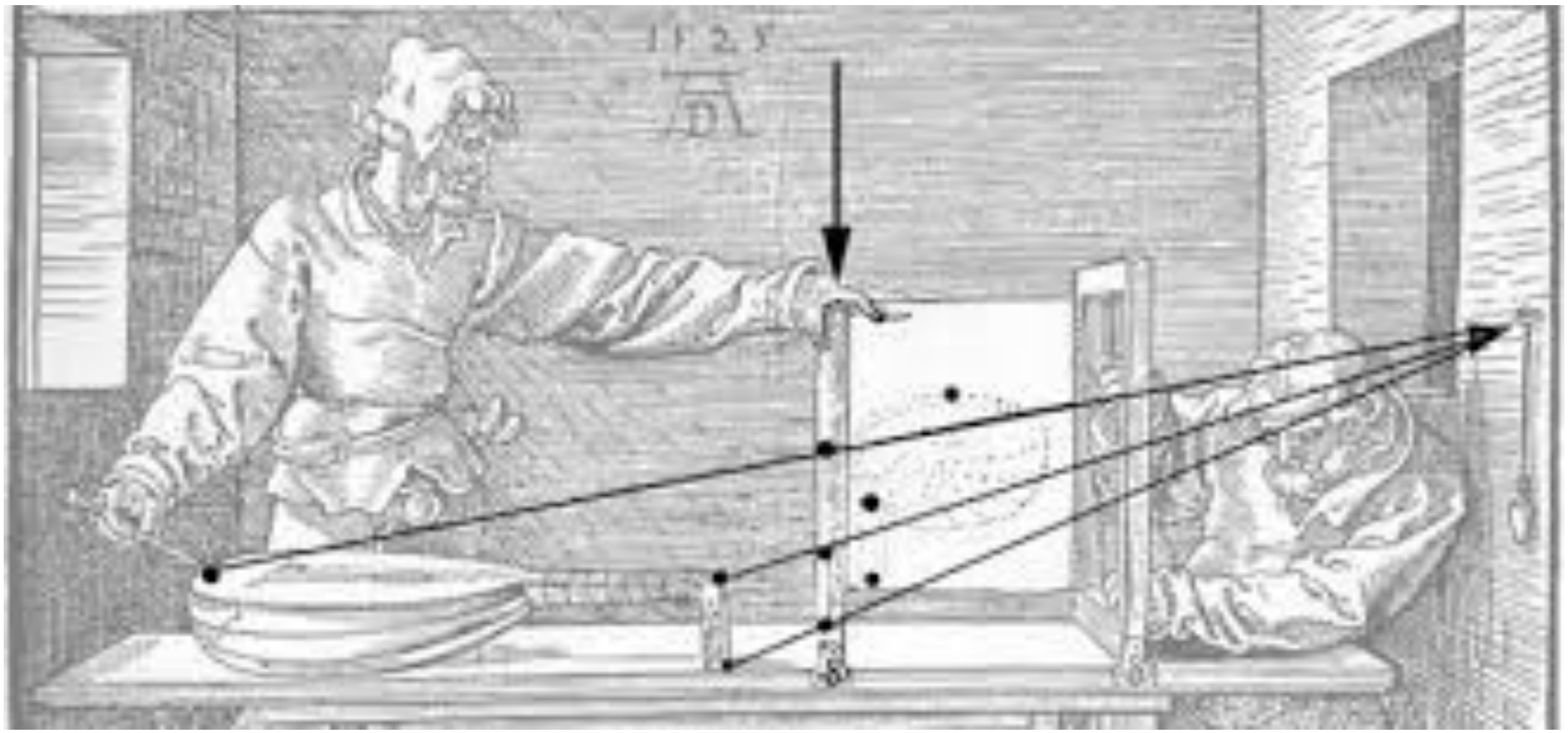


Transformation groups

Projective Transformation



Projective transformations in art and photography



Albrecht Durer — 1500

Geometry = Group Theory

Felix Klein's Erlanger Programm (1872):

Each type of geometry is founded on a corresponding transformation group.

Euclidean geometry:	rigid motions (translations and rotations)
"Mirror" geometry:	translations, rotations, and reflections
Similarity geometry:	translations, rotations, reflections, and scalings
Projective geometry:	all projective transformations

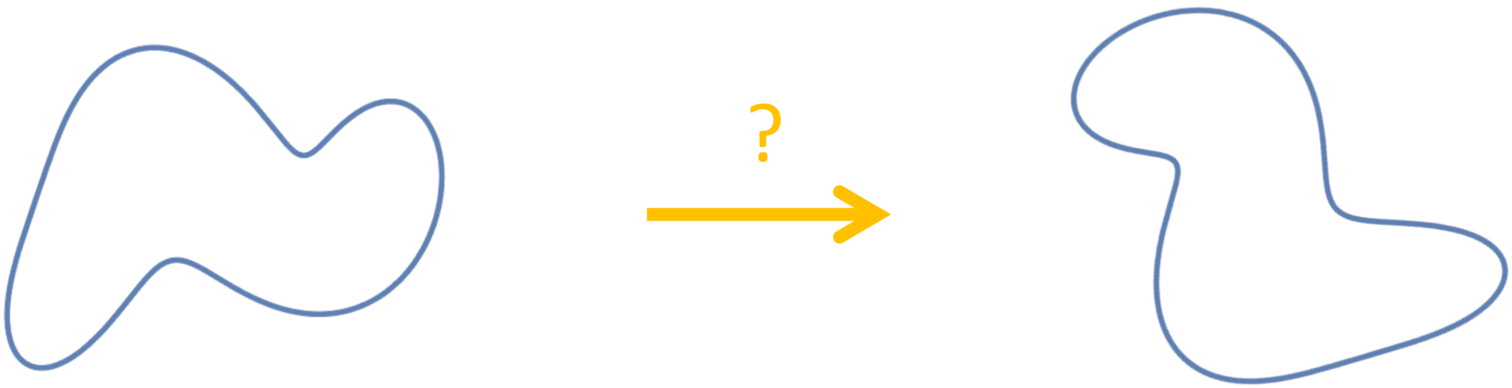
The Equivalence Problem

When are two shapes related by a group transformation?

- Rigid (Euclidean) equivalence
- Similarity equivalence
- Projective equivalence
- etc.

Rigid equivalence

When are two shapes related by a rigid motion?



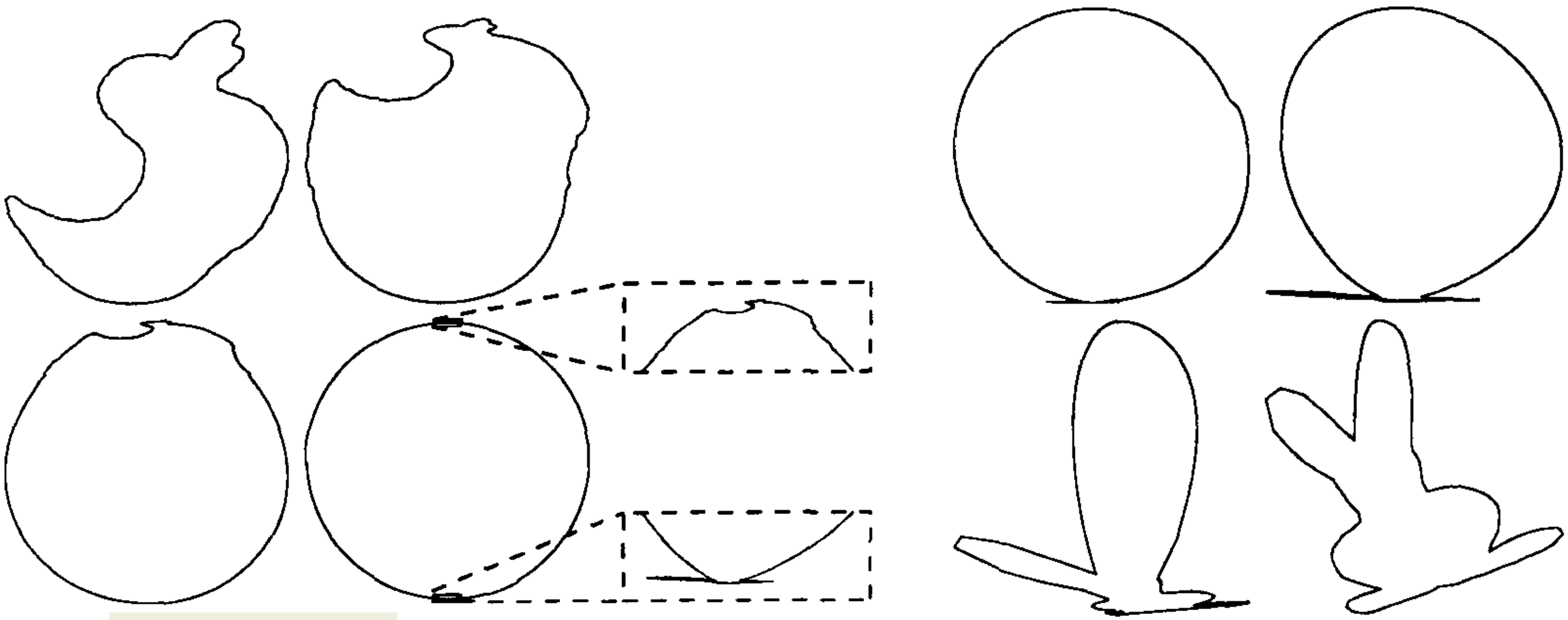
Tennis, anyone?

$\xrightarrow{?}$

☞ Projective equivalence & symmetry

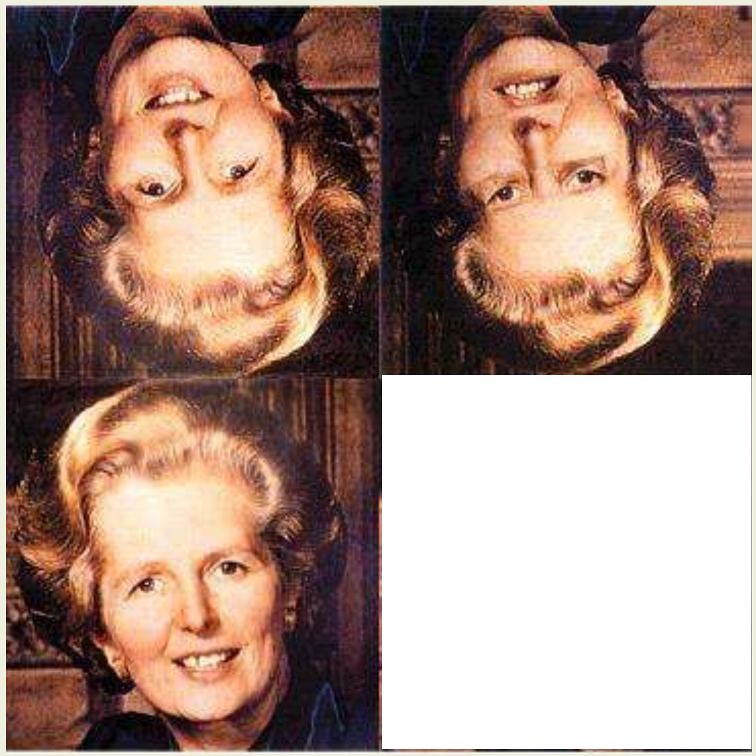
Duck = Rabbit?

Limitations of Projective Equivalence



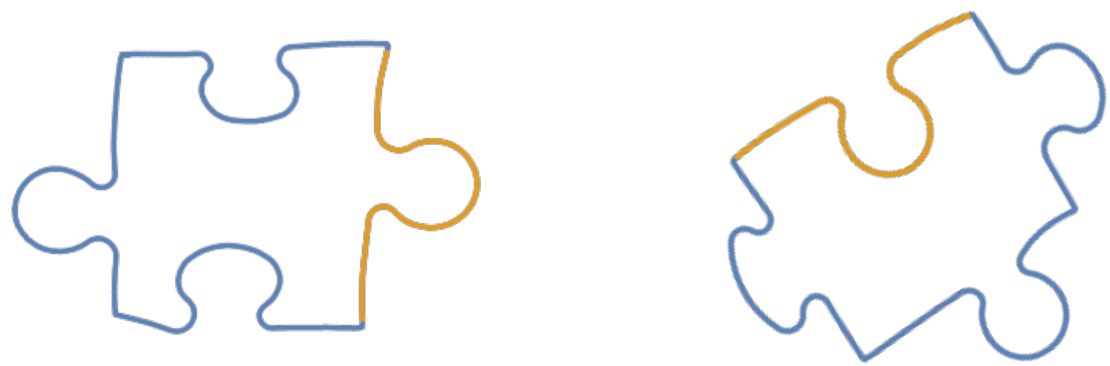
$\Longrightarrow$ K. Åström (1995)

Thatcher Illusion

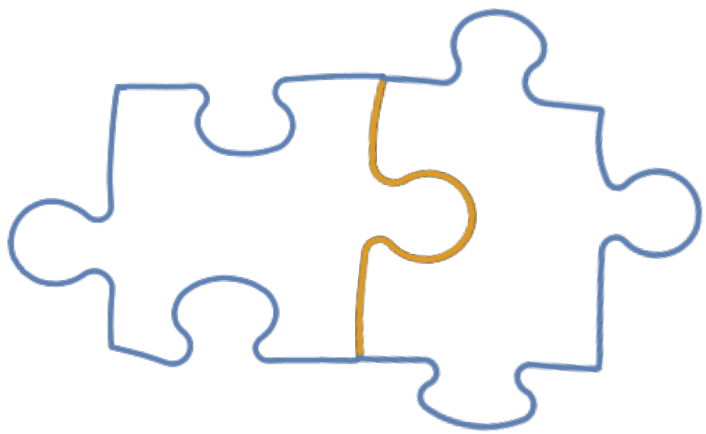


Thatcher Illusion

Local equivalence of puzzle pieces



Local equivalence of puzzle pieces



The Equivalence Problem

When are two shapes related by a group transformation?

Invariants

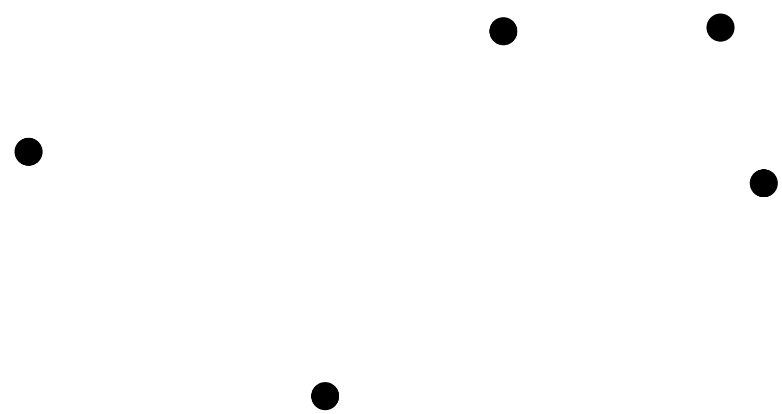
✮✮ Solving the equivalence problem requires knowing enough invariants

Invariants

Invariants are quantities that are unchanged by the group transformations

★ If two shapes are equivalent, they must have the same invariants.

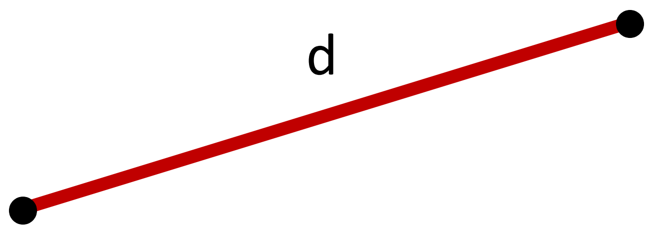
Joint invariants



An invariant that depends on several points is known as a joint invariant

Joint invariants

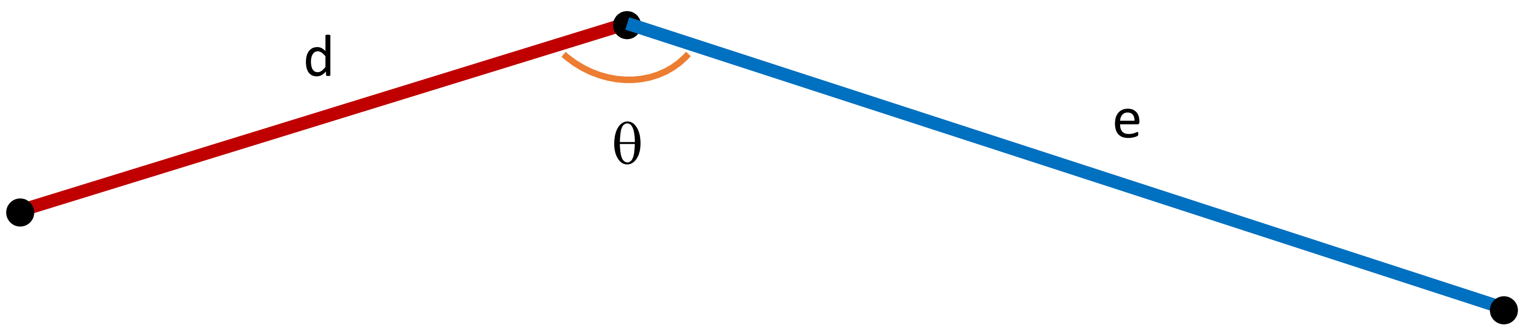
Rigid motions: distance between two points



Joint invariants

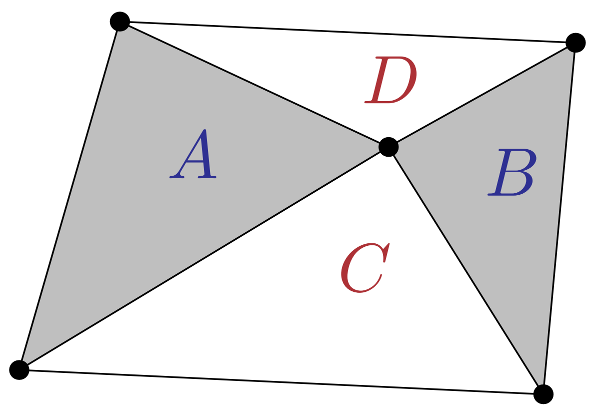
Similarity group:

ratios of distances $R = d/e$ and angles θ



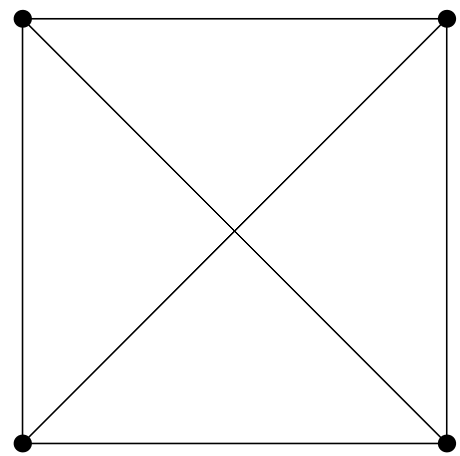
Joint invariants

Projective group: ratios of 4 areas



$$\frac{A\,B}{C\,D}$$

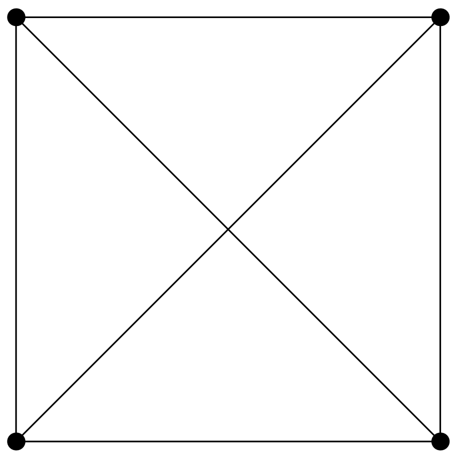
Distances between multiple points



$$1, \quad 1, \quad 1, \quad 1, \quad \sqrt{2}, \quad \sqrt{2}.$$

The Distance Histogram —

invariant under rigid motions



$$1,\quad 1,\quad 1,\quad 1,\quad \sqrt{2},\quad \sqrt{2}.$$

If two sets of points are equivalent up to rigid motion, they have the same distance histogram

---

Does the distance histogram uniquely determine a set of points up to rigid motion?

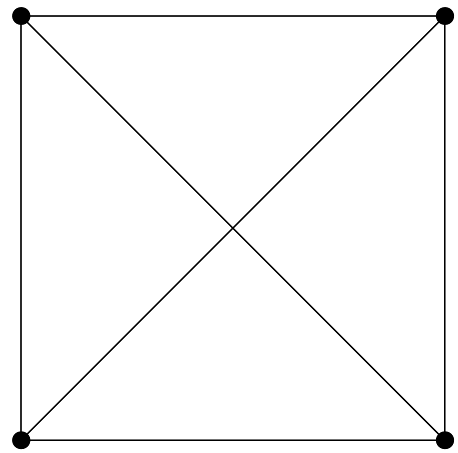
Does the distance histogram uniquely determine a set of points up to rigid motion?

Answer: Yes for most sets of points, but there are some exceptions!

☆☆ Mireille (Mimi) Boutin and Gregor Kemper (2004)

Does the distance histogram uniquely determine a set of points up to rigid motion?

Yes:

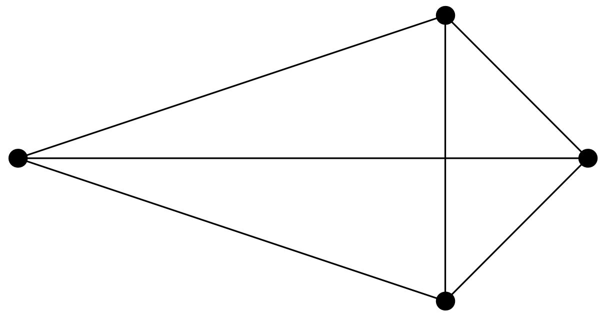


$$1, \quad 1, \quad 1, \quad 1, \quad \sqrt{2}, \quad \sqrt{2}.$$

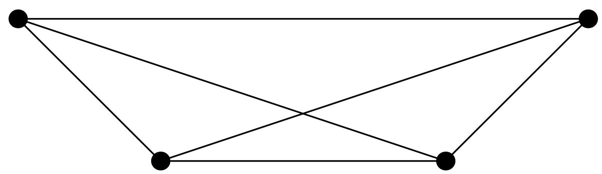
Does the distance histogram uniquely determine a set of points up to rigid motion?

No:

Kite



Trapezoid



$$\sqrt{2}, \quad \sqrt{2}, \quad 2, \quad \sqrt{10}, \quad \sqrt{10}, \quad 4.$$

Distance histogram for points on a line

Does the distance histogram uniquely determine a set of points on a line up to translation?

Distance histogram for points on a line

No:

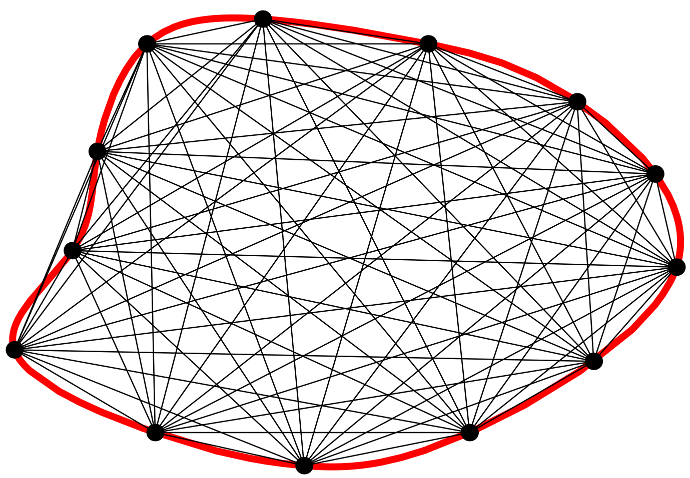
$$P = \{0, 1, 4, 10, 12, 17\}$$

$$Q = \{0, 1, 8, 11, 13, 17\}$$

$$\eta = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 17$$

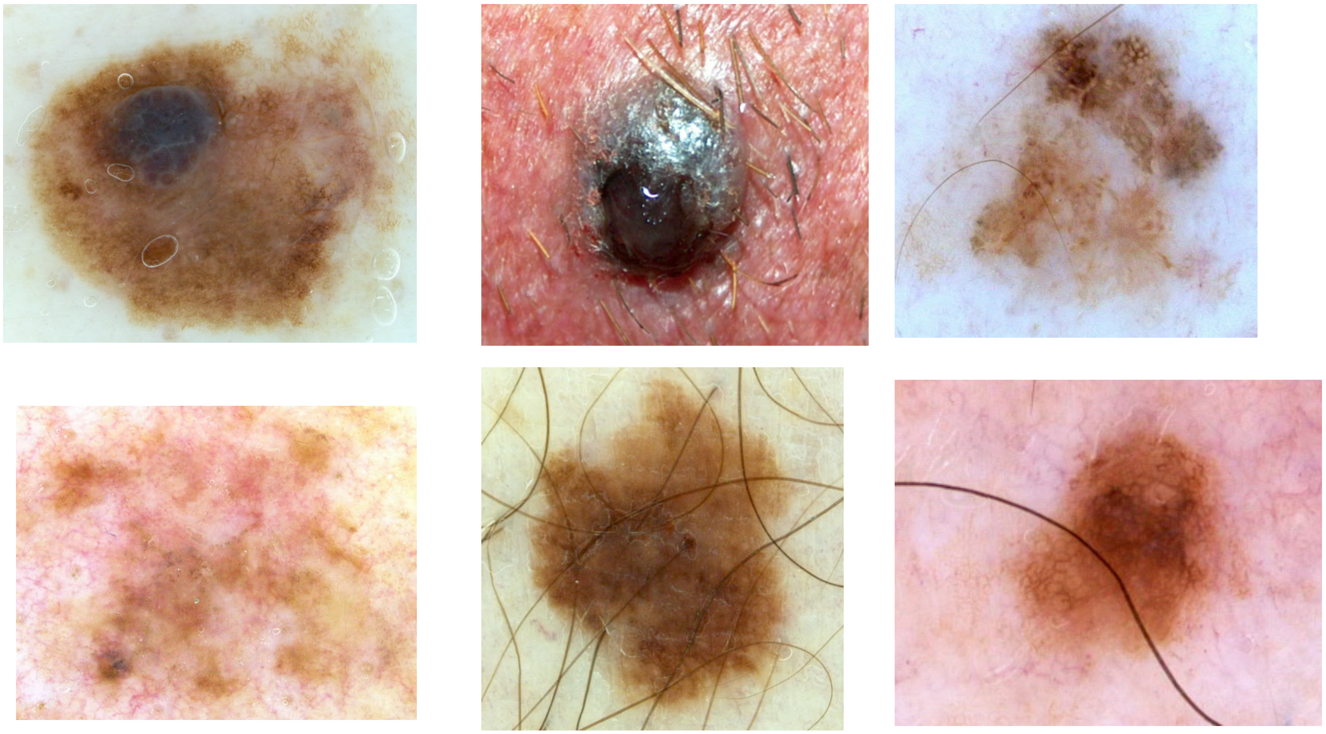
$\Longrightarrow$ G. Bloom, *J. Comb. Theory, Ser. A* **22** (1977) 378–379

Limiting Curve Histogram



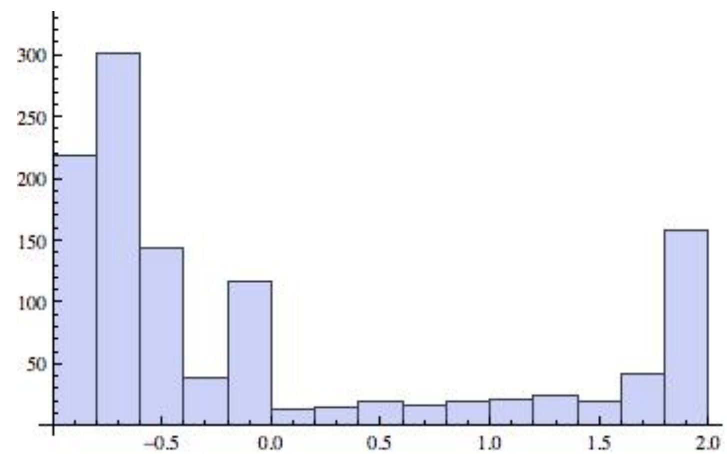
Brinkman, D., and Olver, P.J., Invariant histograms, Amer. Math. Monthly 119 (2012), 4-24

Distinguishing Moles from Melanomas

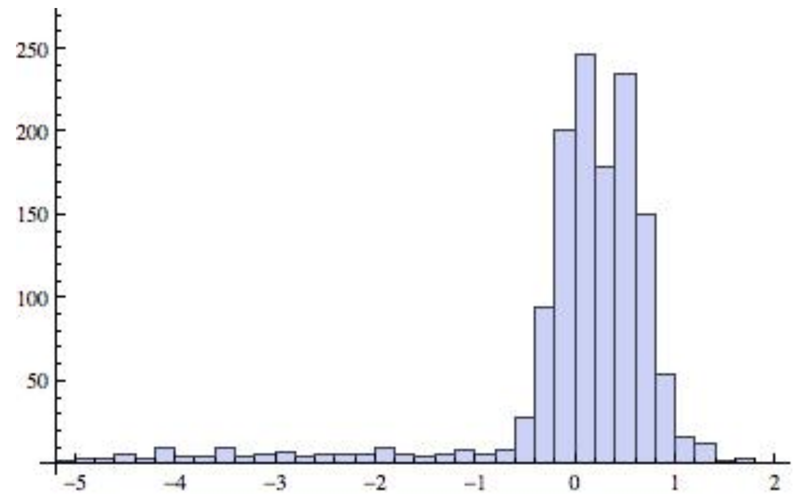


- Anna Grim and Cheri Shakiban, 2015

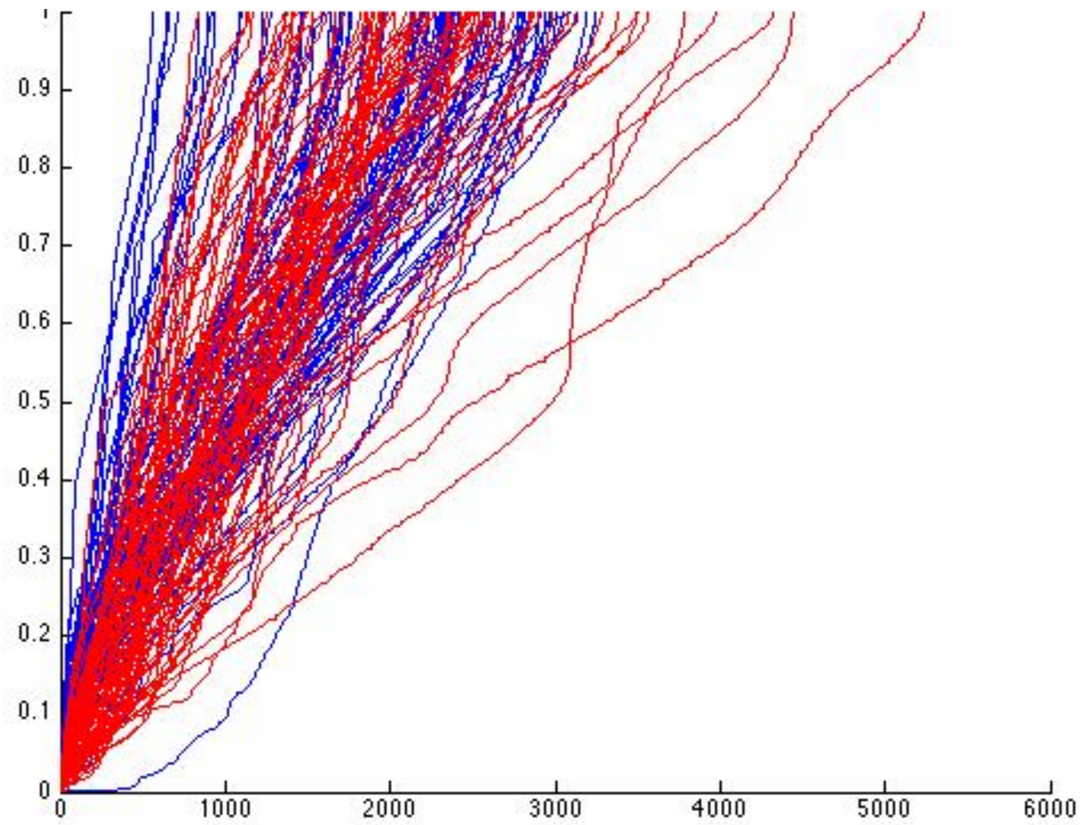
Distance Histogram — Melanoma



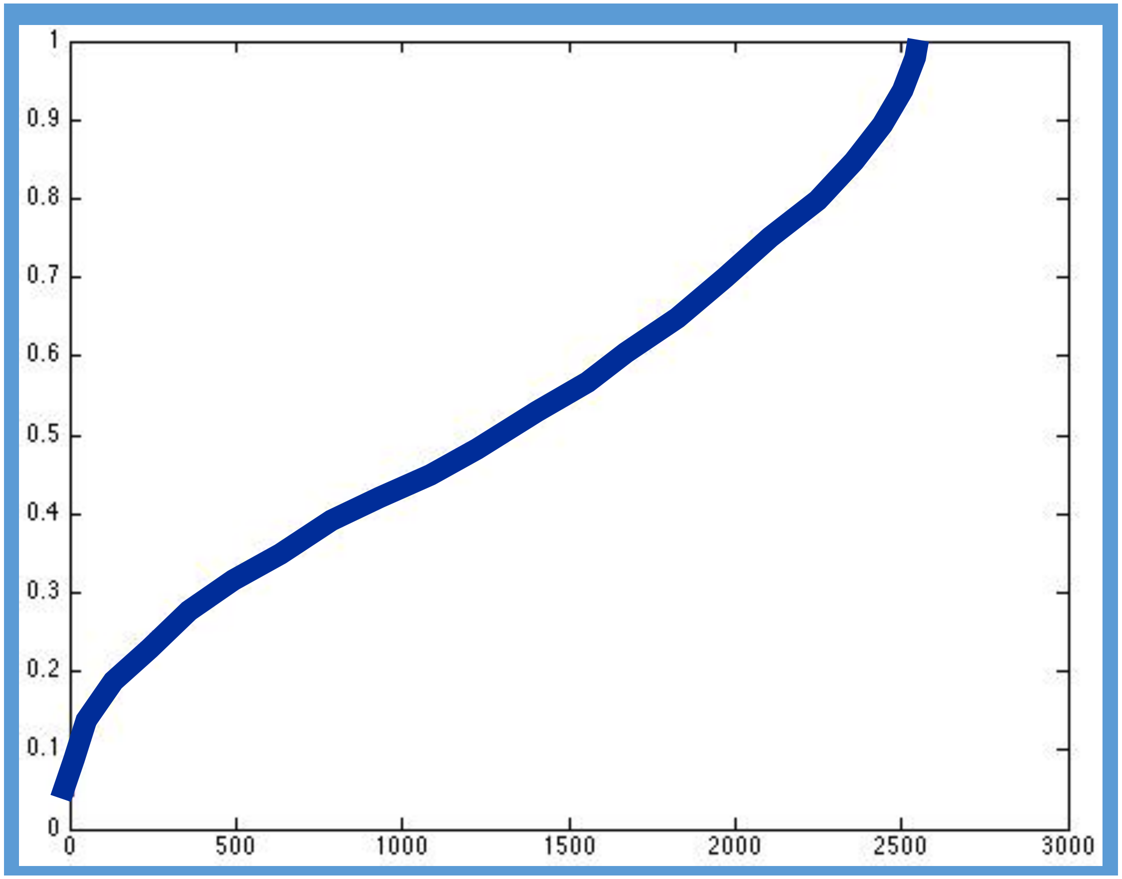
Distance Histogram — Mole



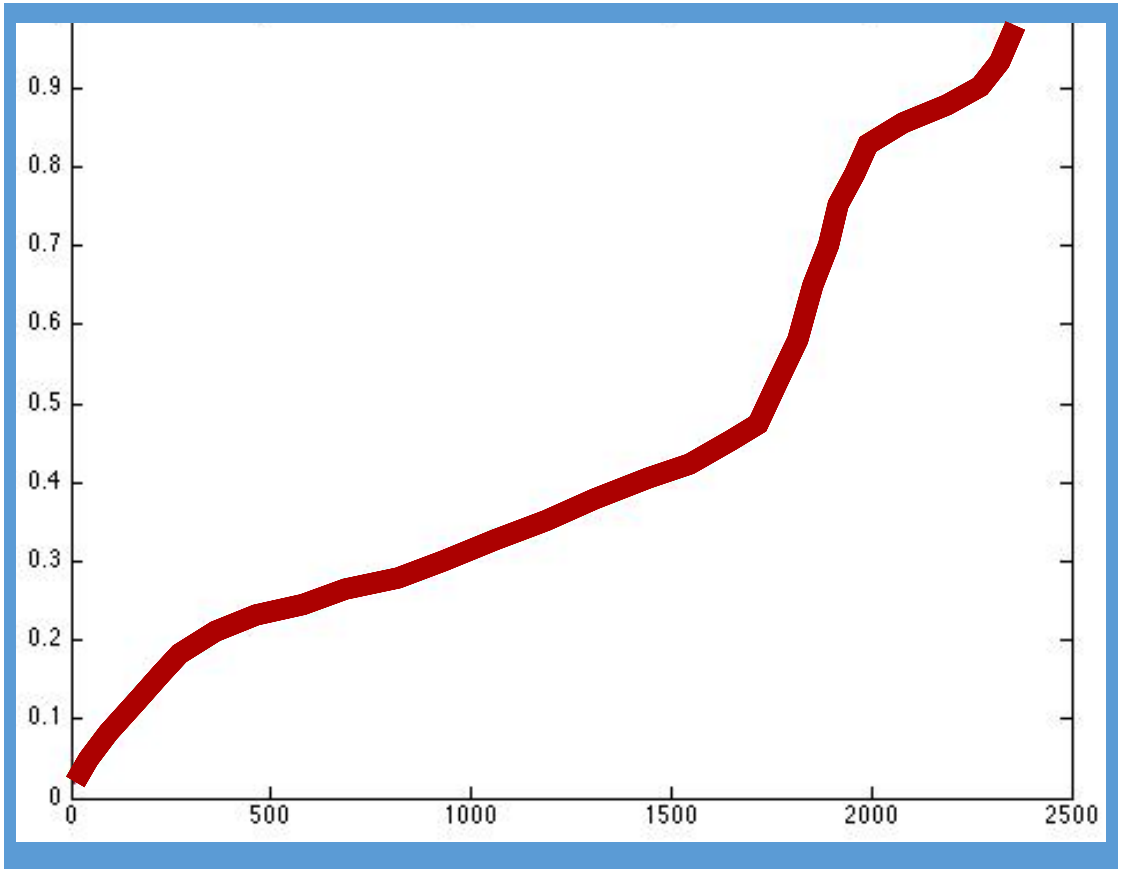
CUMULATIVE HISTOGRAM:
Mole versus Melanoma



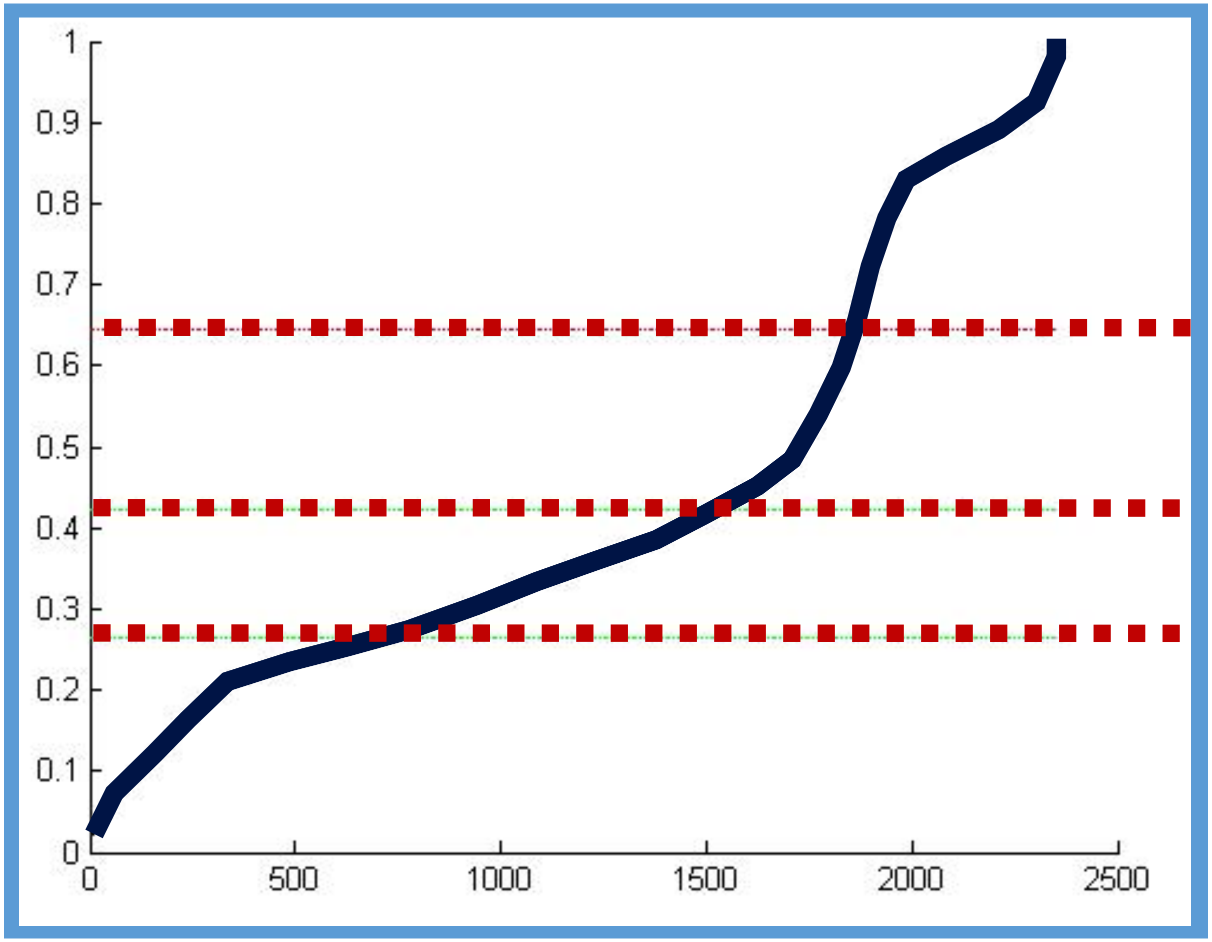
TYPICAL MOLE CUMULATIVE HISTOGRAM

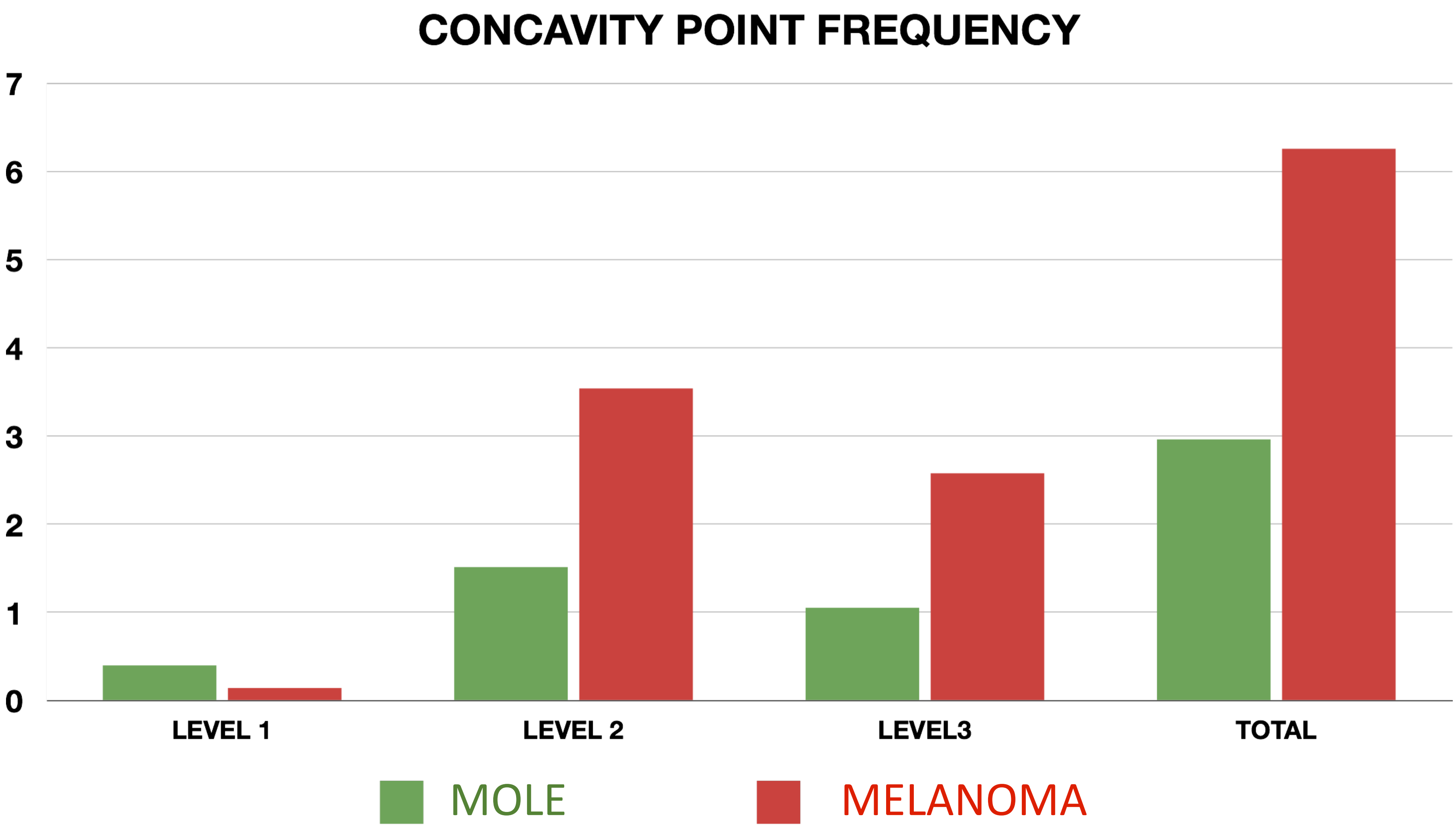


TYPICAL MELANOMA CUMULATIVE HISTOGRAM



CONCAVITY POINT ANALYSIS




CONCAVITY POINT FREQUENCY
7
6
5
4
3
2
1
0
LEVEL 1
LEVEL 2
LEVEL3
TOTAL
MOLE
MELANOMA

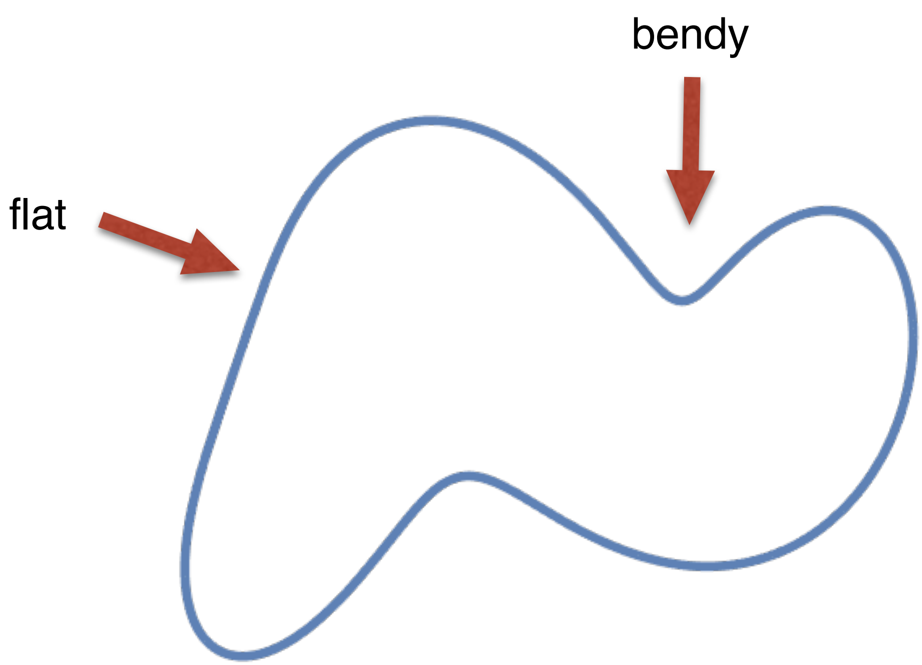
For smooth objects — curves, surfaces, etc.,

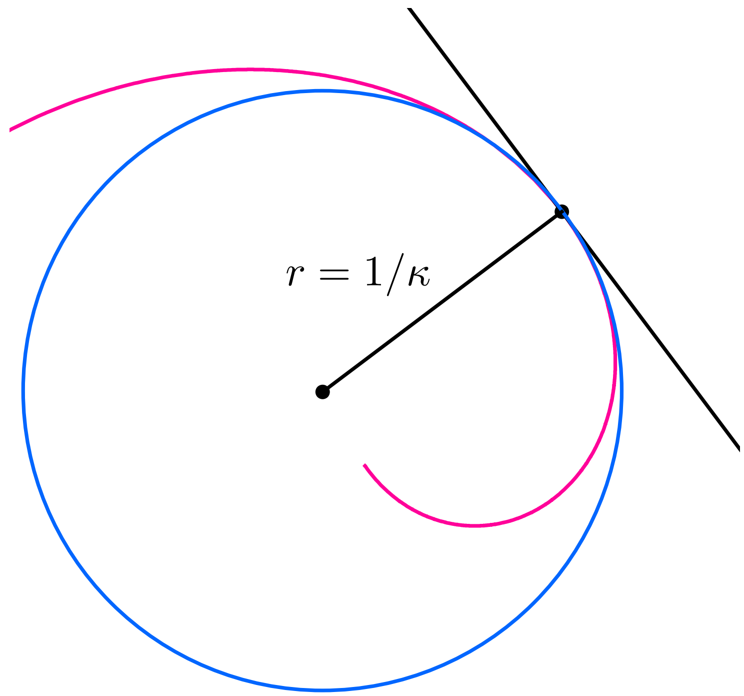
we need to use calculus to find

Differential Invariants

A Differential Invariant

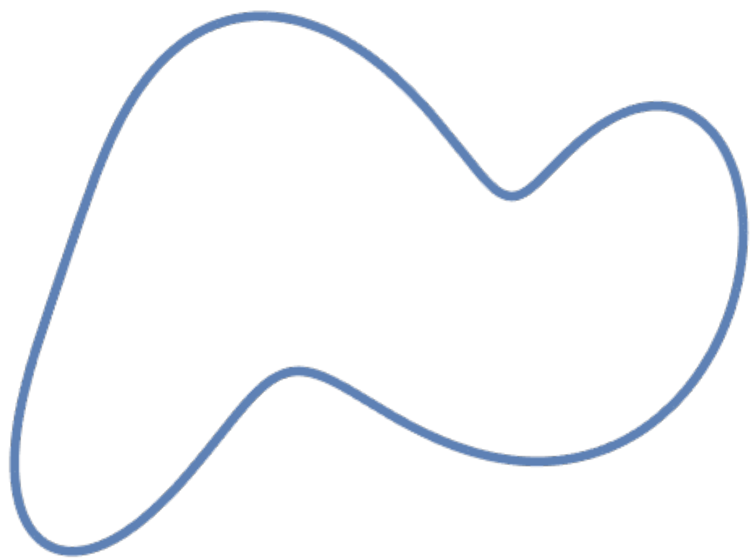
Curvature is a measure of “bendiness”.



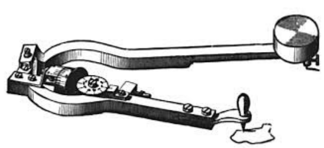


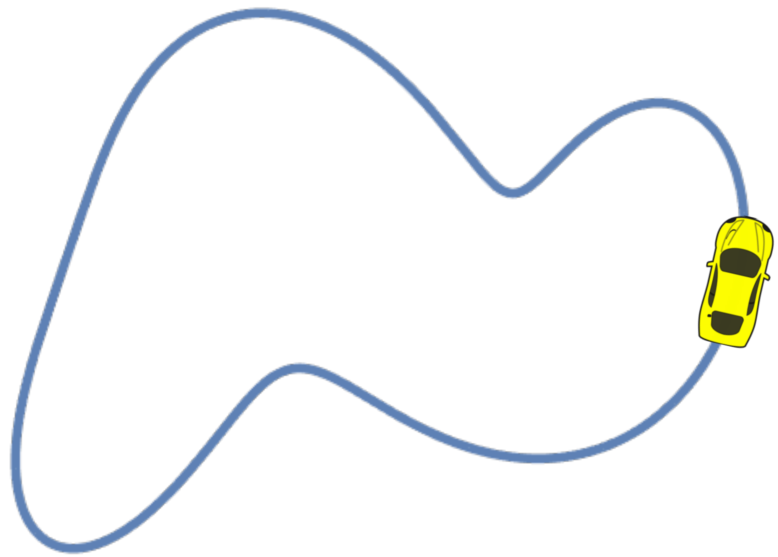
Curvature = reciprocal of radius of osculating circle

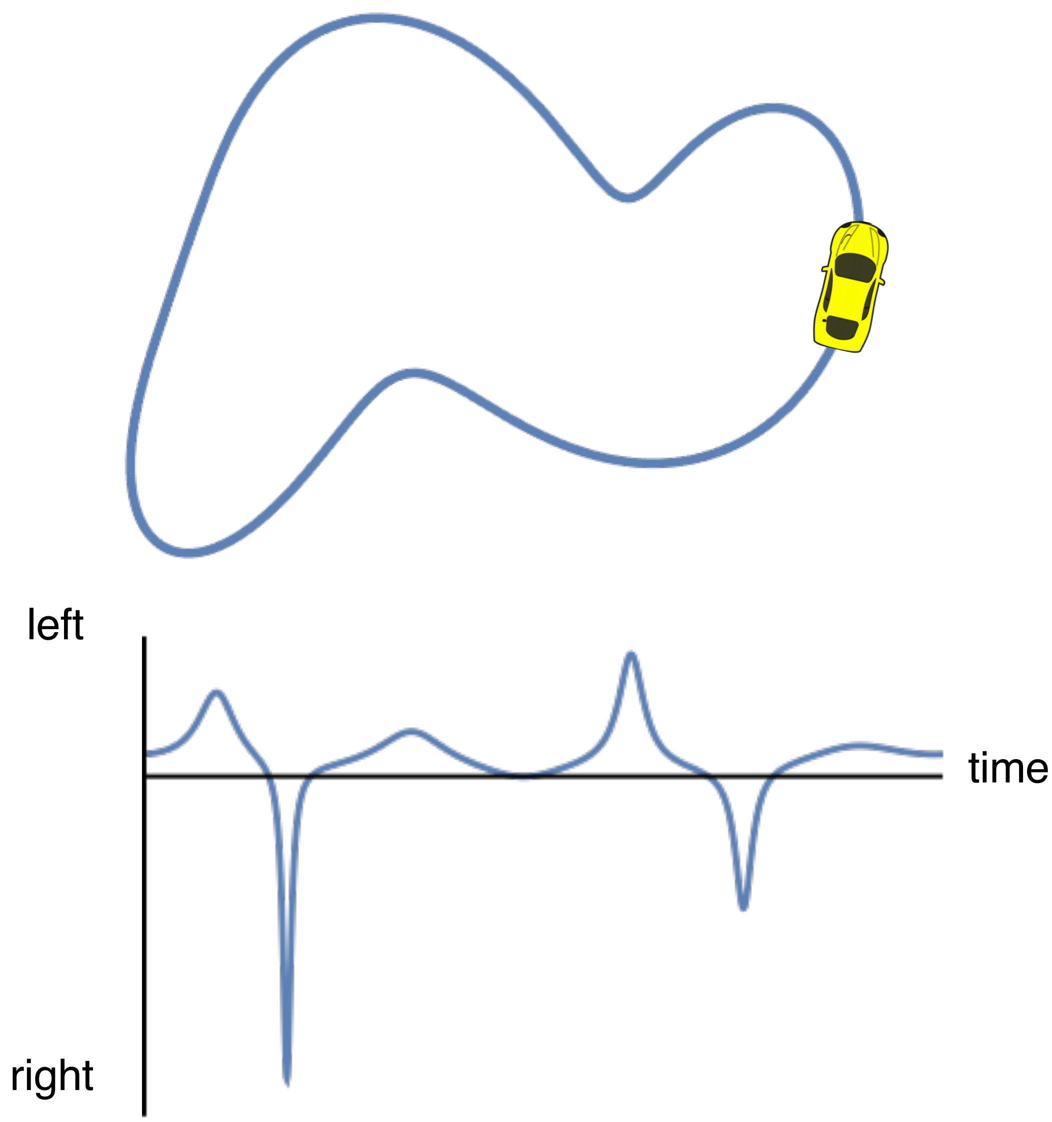
Curvature is a measure of “bendiness”.



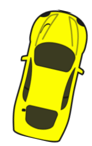
What everyday device can measure curvature?

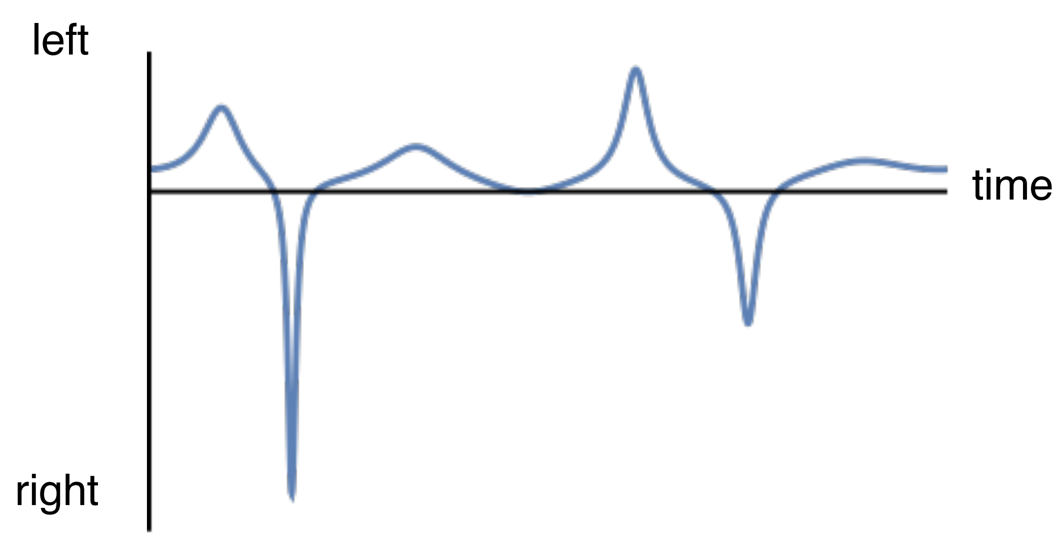




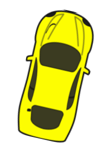

left
time
right

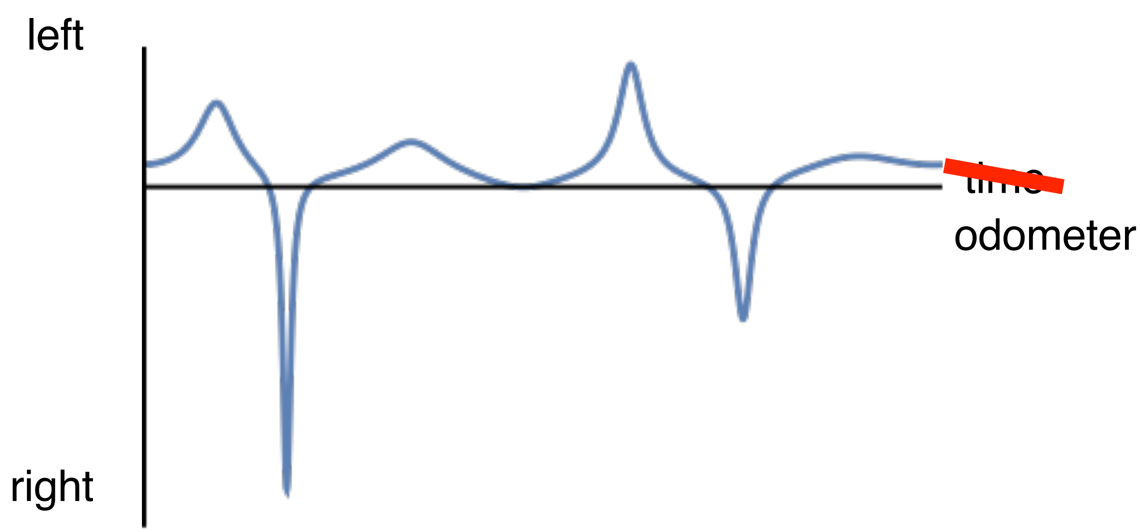
Can you reconstruct the racetrack?





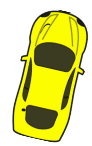
Can you reconstruct the racetrack?



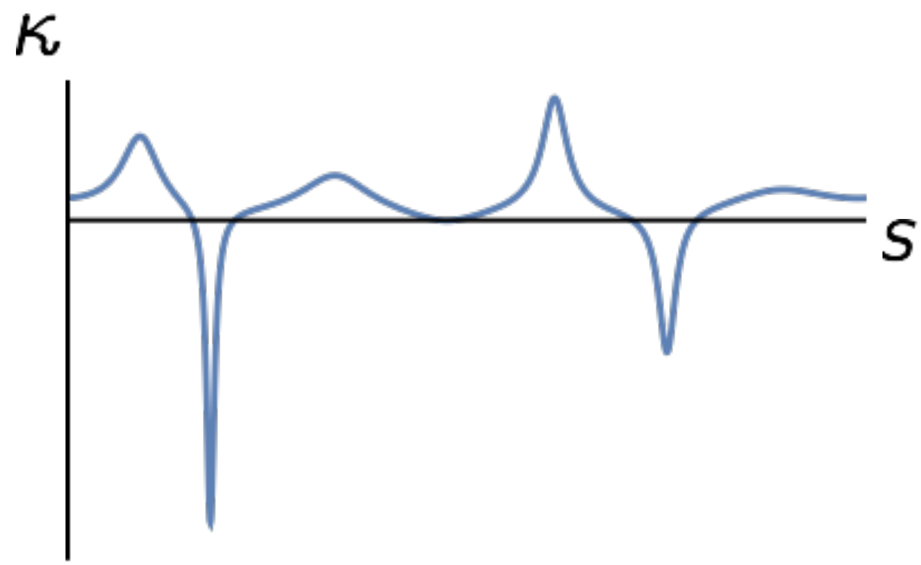


Can you reconstruct the racetrack?

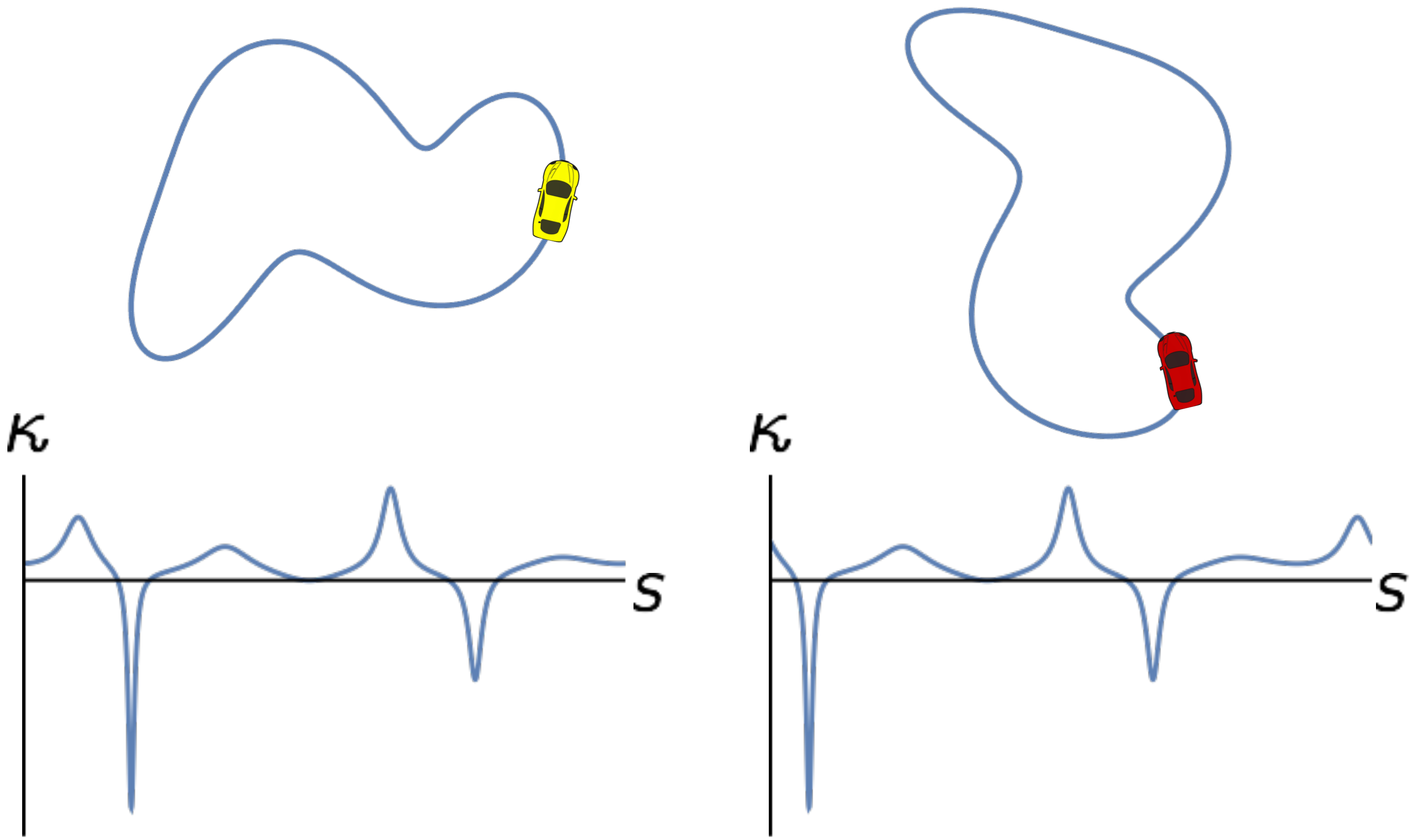
κ is (Euclidean) curvature



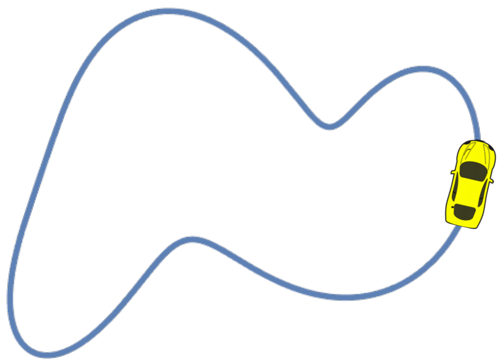
s is (Euclidean) arclength

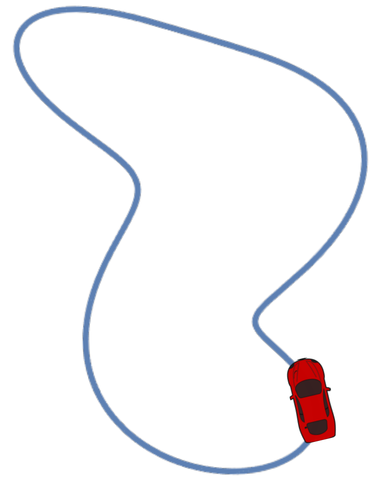


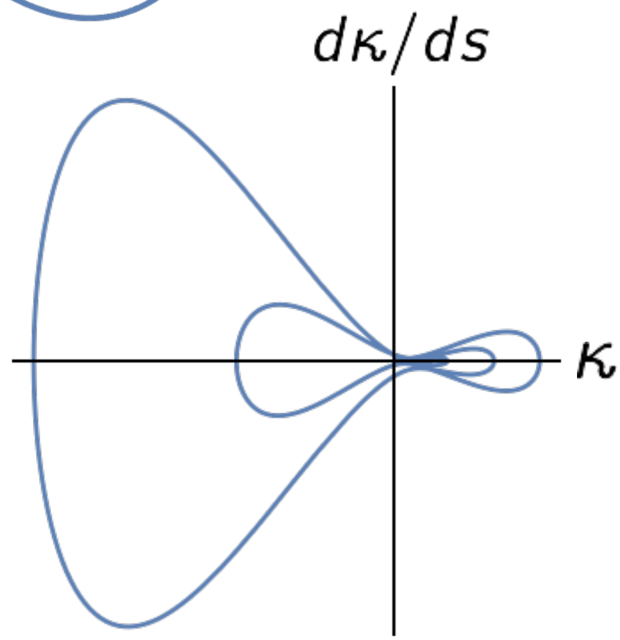
Racetrack comparison problem



Racetrack comparison problem





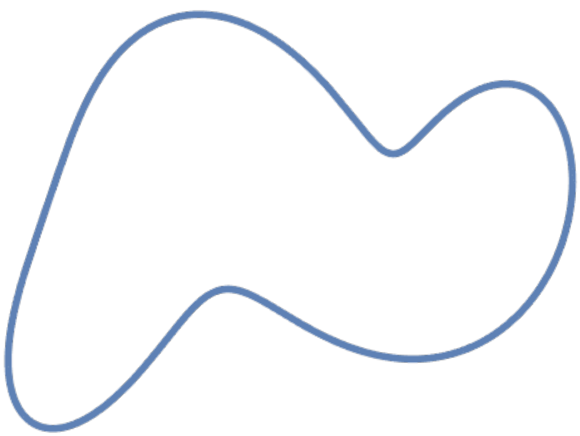


$d\kappa/ds$

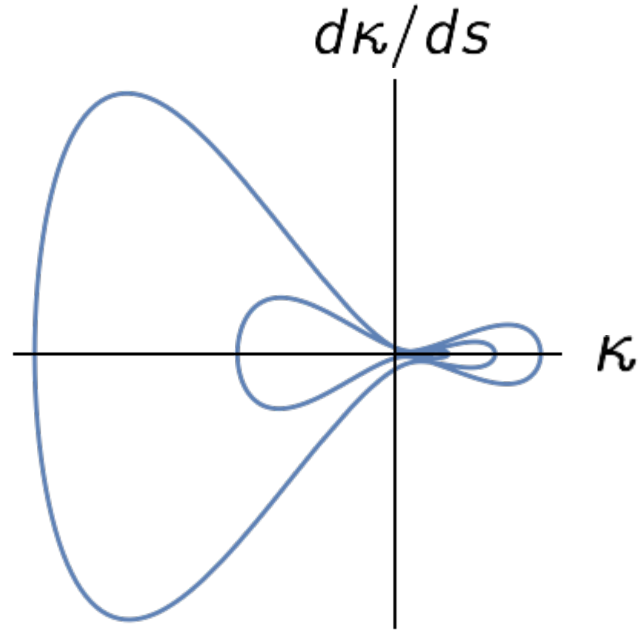
= rate of change
(derivative)
of curvature

The Invariant Signature

The invariant signature of a planar curve is the set traced out by curvature and the rate of change of curvature (its arclength derivative).



original curve



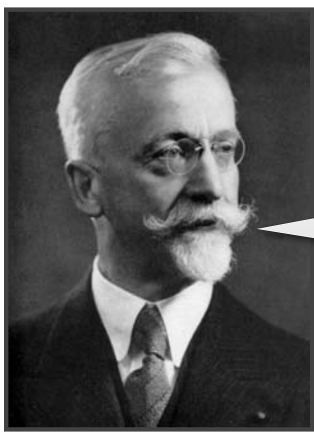
invariant signature

The invariant signature

Theorem

Two regular curves are related by a group transformation if and only if they have the same invariant signatures.

Proof idea



Theorem (Élie Cartan 1908)

Shapes are related if and only if they have the same relationships among their **differential invariants**.

(Calabi, Haker, Olver, Shakiban, Tannenbaum 1998)

Moving Frames

The mathematical theory is all based on the new equivariant method of moving frames, which provides a systematic and algorithmic calculus for constructing complete systems of differential invariants, joint invariants, joint differential invariants, invariant differential operators, invariant differential forms, invariant variational problems, invariant conservation laws, invariant numerical algorithms, invariant signatures, etc., etc.

Acta Applicandae Mathematicae **55:** 127–208, 1999.

Moving Coframes: II. Regularization and Theoretical Foundations

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(Received: 16 November 1998)

Abstract. The primary goal of this paper is to provide a rigorous theoretical justification of Cartan's method of moving frames for arbitrary finite-dimensional Lie group actions on manifolds. The general theorems are based a new regularized version of the moving frame algorithm, which is of both theoretical and practical use. Applications include a new approach to the construction and classification of differential invariants and invariant differential operators on jet bundles, as well as equivalence, symmetry, and rigidity theorems for submanifolds under general transformation groups. The method also leads to complete classifications of generating systems of differential invariants, explicit commutation formulae for the associated invariant differential operators, and a general classification theorem for syzygies of the higher order differentiated differential invariants. A variety of illustrative examples demonstrate how the method can be directly applied to practical problems arising in geometry, invariant theory, and differential equations.

Mathematics Subject Classifications (1991): 53A55, 58D19, 58H05, 68U10.

Key words: moving frame, Lie group, jet bundle, prolongation, differential invariant, equivalence, symmetry, rigidity, syzygy.

1. Introduction

This paper is the second in a series devoted to the analysis and applications of the method of moving frames and its generalizations. In the first paper [9], we introduced the method of moving coframes, which can be used to practically compute moving frames and differential invariants, and is applicable to finite-dimensional Lie transformation groups as well as infinite-dimensional pseudo-group actions. In this paper, we introduce a second method, called regularization, that not only provides, in a simple manner, the theoretical justification for the method of moving frames in the case of finite-dimensional Lie group actions, but also gives an alternative, practical approach to their construction. The regularized method successfully bypasses many of the complications inherent in traditional approaches by completely avoiding the usual process of normalization during the general computation. In this way, the issues of branching and regularity do not arise. Once a moving

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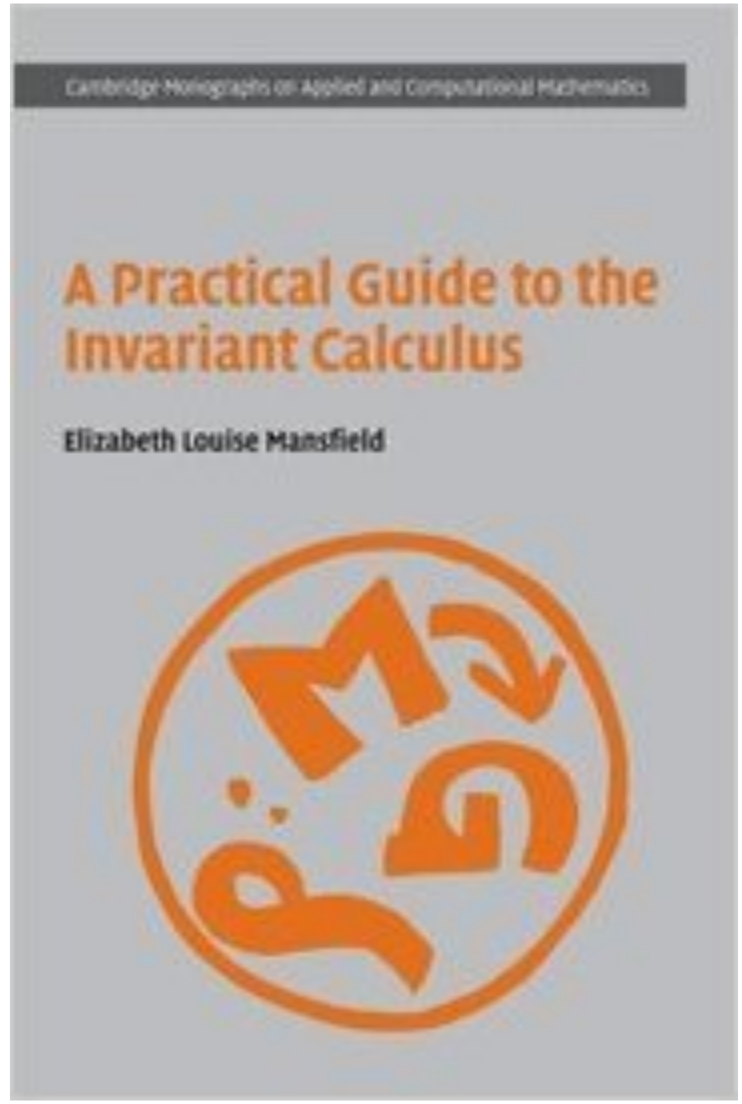
** Supported in part by NSF Grant DMS 95-00931.



3D Differential Invariant Signatures

Euclidean space curves: $\quad C \subset \mathbb{R}^3$

$$\Sigma = \{\, (\,\kappa\,,\,\kappa_s\,,\,\tau\,)\,\} \;\subset\; \mathbb{R}^3$$

- κ — curvature, τ — torsion

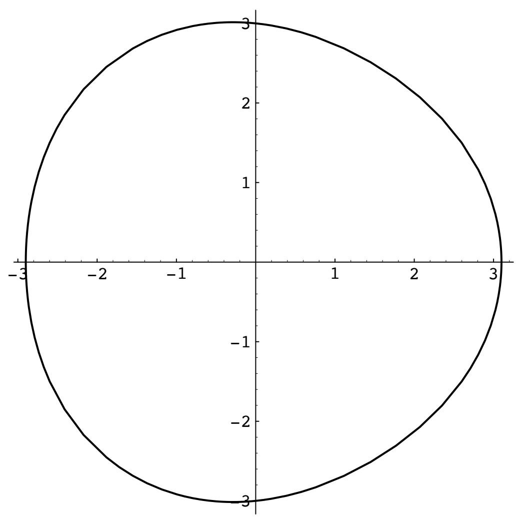
Euclidean surfaces: $\quad S \subset \mathbb{R}^3$ (generic)

$$\Sigma = \left\{\, \Bigl(\, H\,,\,K\,,\,H_{,1}\,,\,H_{,2}\,,\,K_{,1}\,,\,K_{,2} \Bigr)\,\right\} \;\subset\; \mathbb{R}^6$$

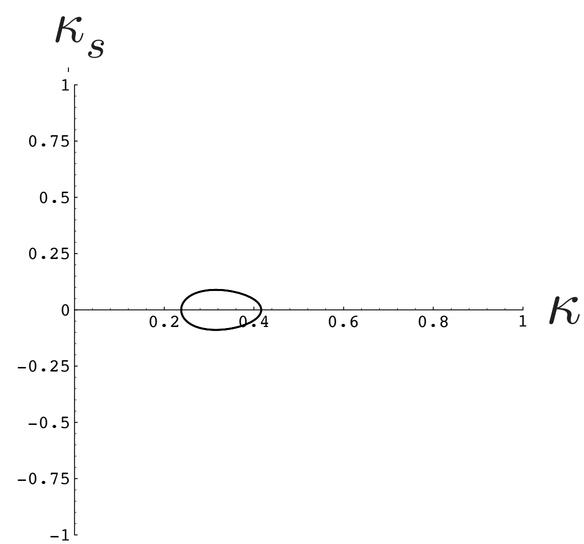
or $\quad \widehat{\Sigma} = \left\{\, \Bigl(\, H\,,\,H_{,1}\,,\,H_{,2}\,,\,H_{,11} \Bigr)\,\right\} \;\subset\; \mathbb{R}^4$

- H — mean curvature, K — Gauss curvature

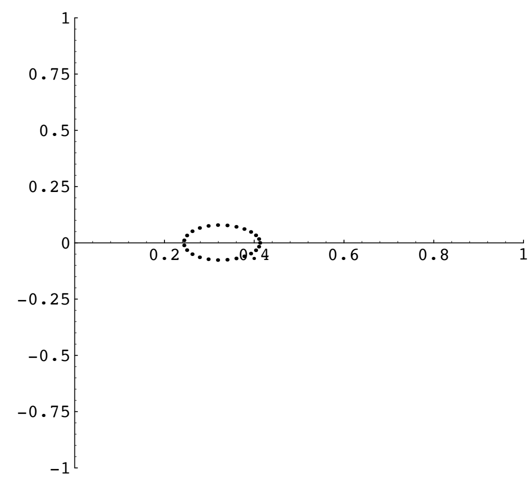
The polar curve $r = 3 + \frac{1}{10} \cos 3\theta$



The Original Curve

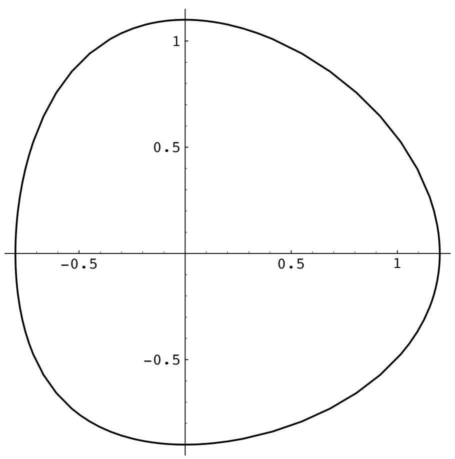


Euclidean Signature

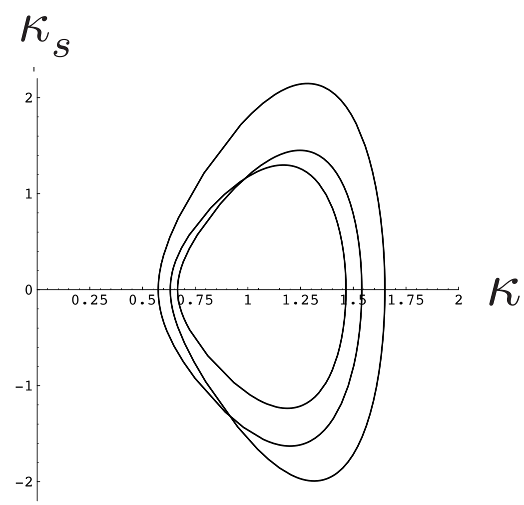


Numerical Signature

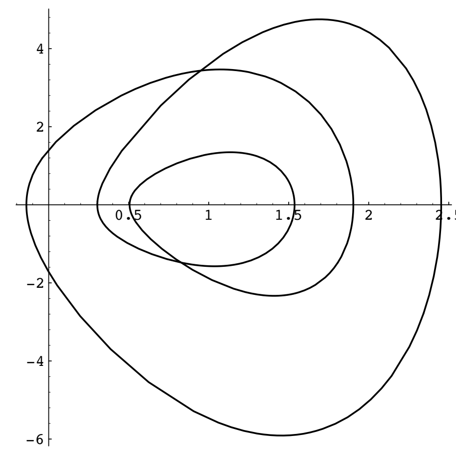
The Curve $x = \cos t + \frac{1}{5}\cos^2 t, \;\; y = \sin t + \frac{1}{10}\sin^2 t$



The Original Curve

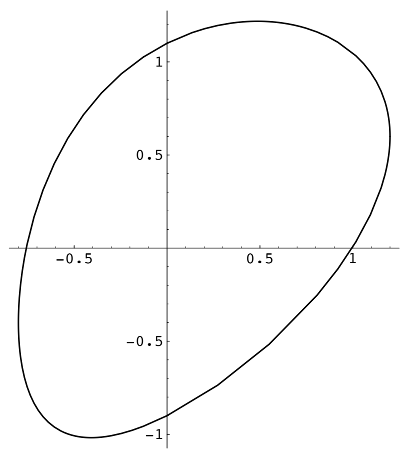


Euclidean Signature

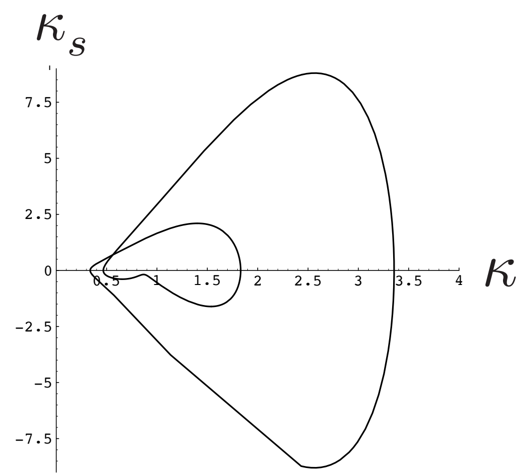


Equi-affine Signature

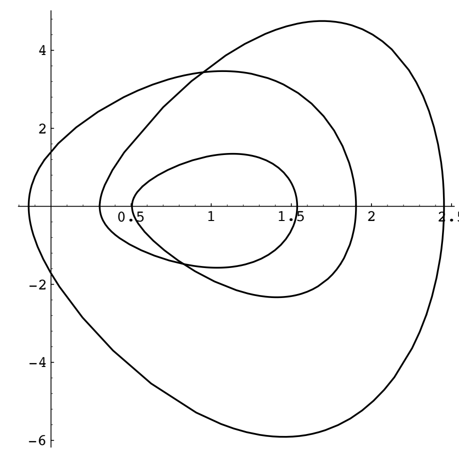
The Curve $x = \cos t + \frac{1}{5}\cos^2 t, \quad y = \frac{1}{2}\,x + \sin t + \frac{1}{10}\sin^2 t$



The Original Curve



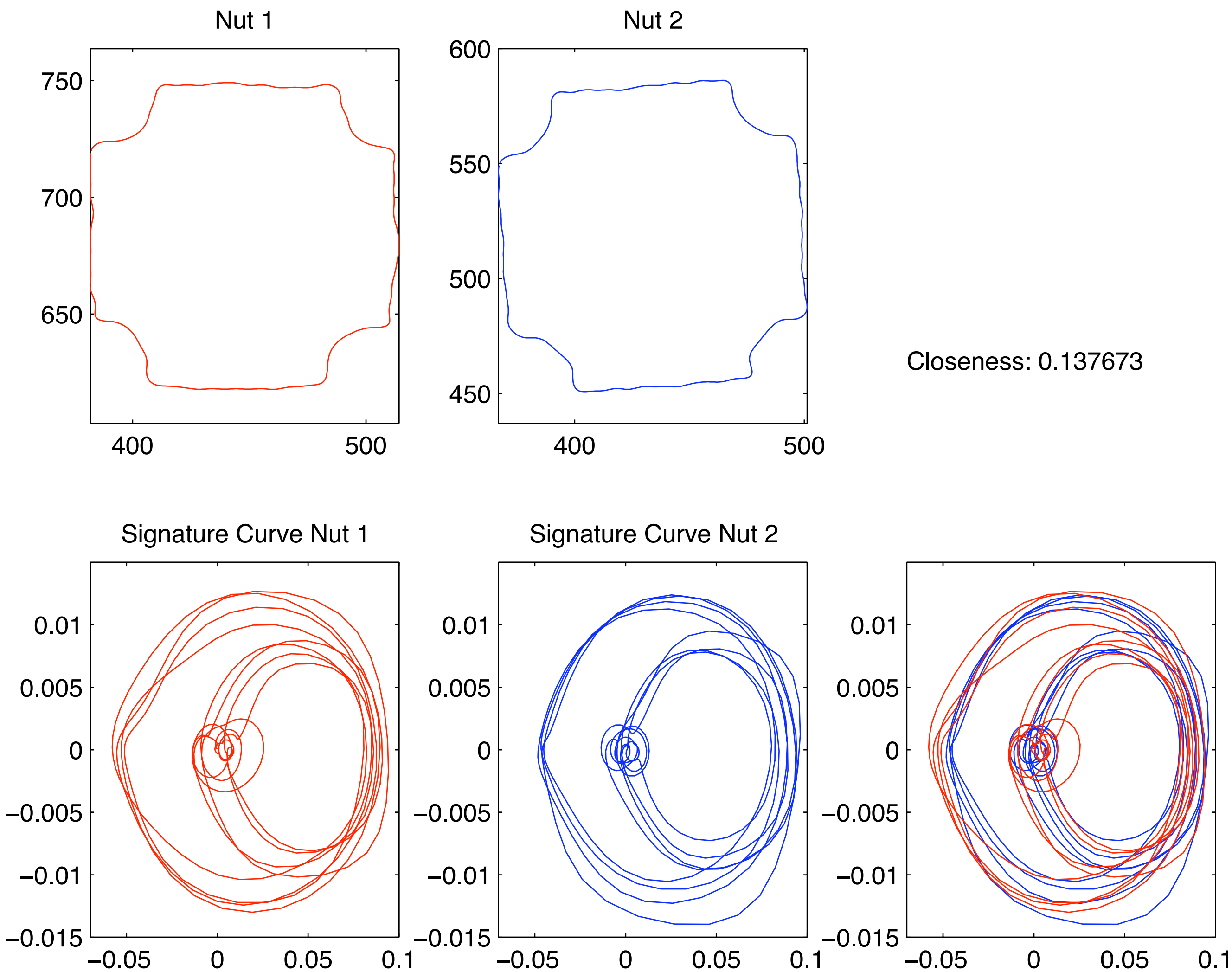
Euclidean Signature

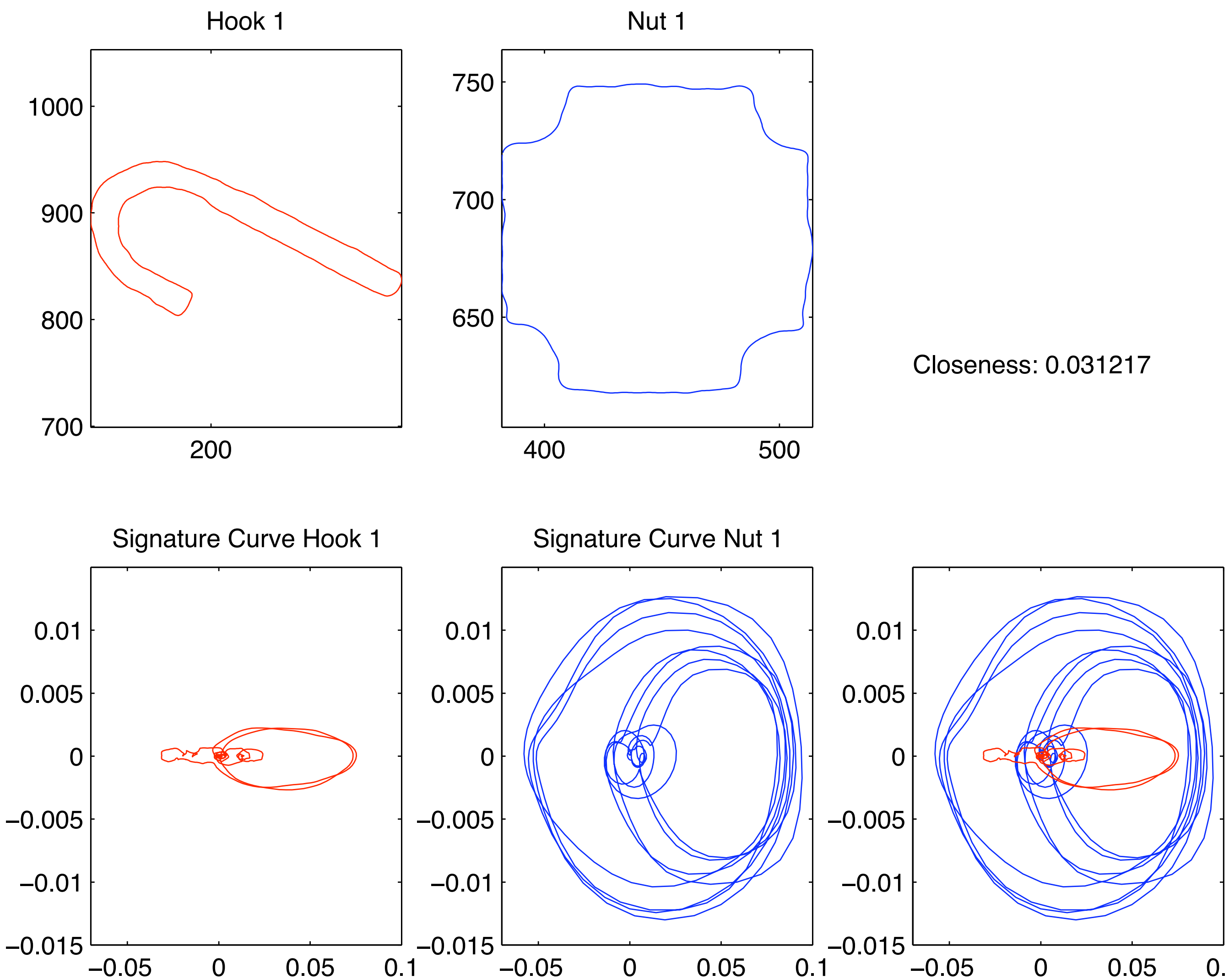


Equi-affine Signature

Object Recognition

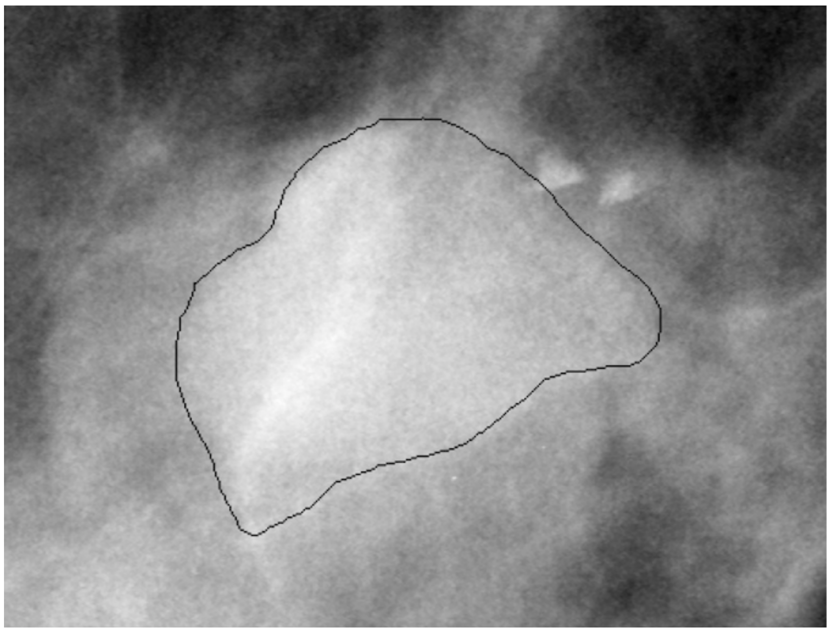
$\Longrightarrow$ Steve Haker

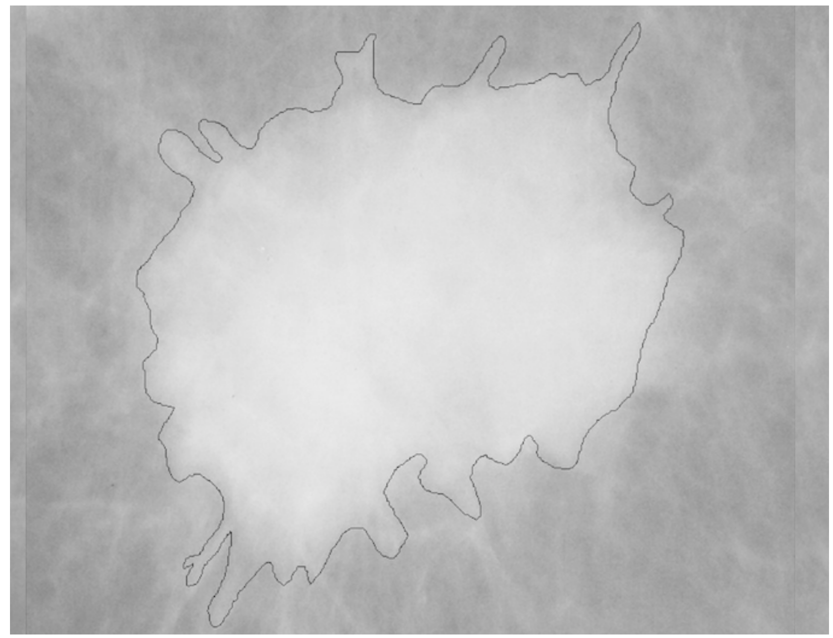

Nut 1
Nut 2
Closeness: 0.137673
Signature Curve Nut 1
Signature Curve Nut 2
750
700
650
400
500
600
550
500
450
400
500
0.01
0.005
0
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1


Hook 1
Nut 1
Closeness: 0.031217
Signature Curve Hook 1
Signature Curve Nut 1
1000
900
800
700
200
750
700
650
400
500
0.01
0.005
0
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1

Diagnosing breast tumors

Anna Grim, Cheri Shakiban



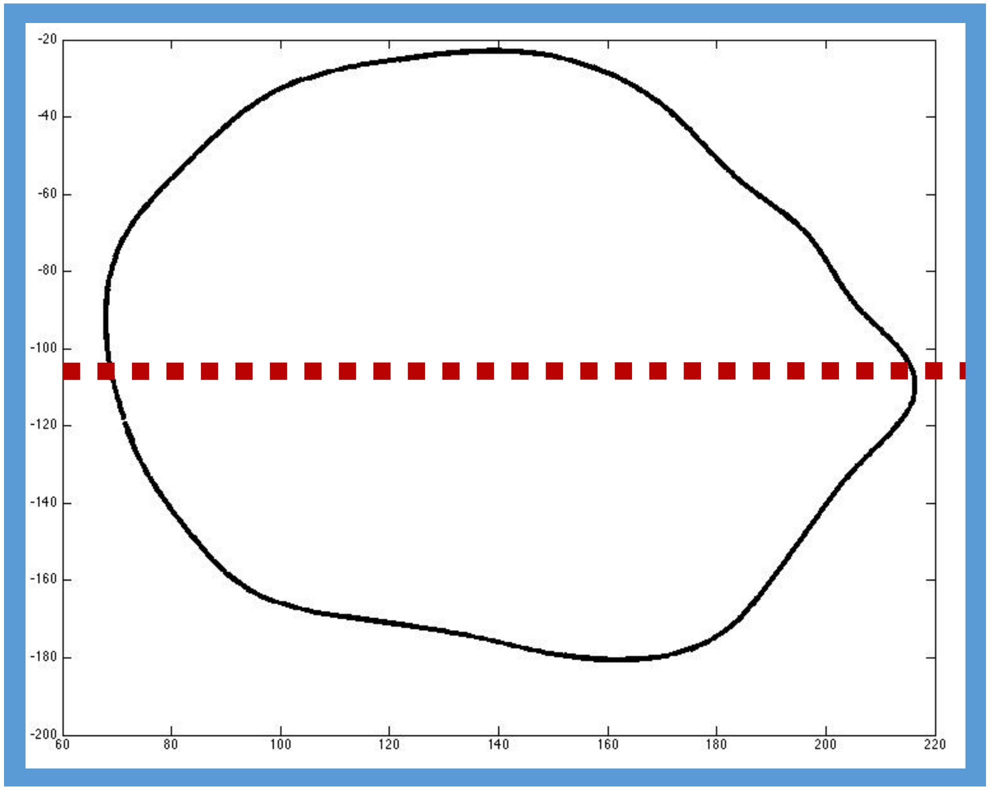


Benign — cyst

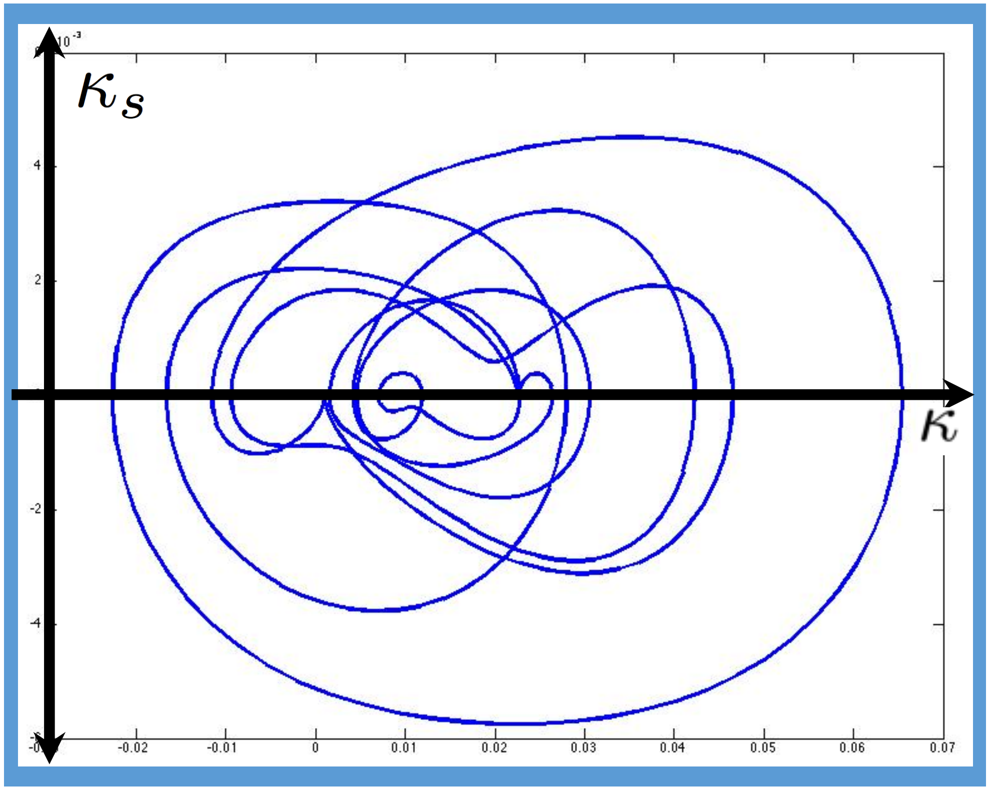
Malignant — cancerous

A BENIGN TUMOR

Contour

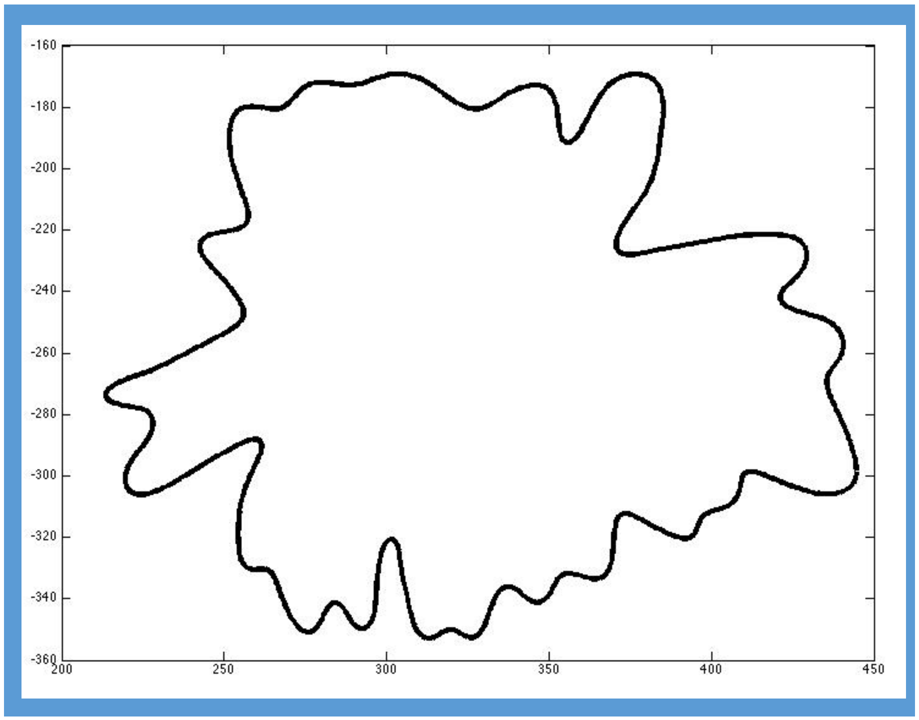


Signature Curve

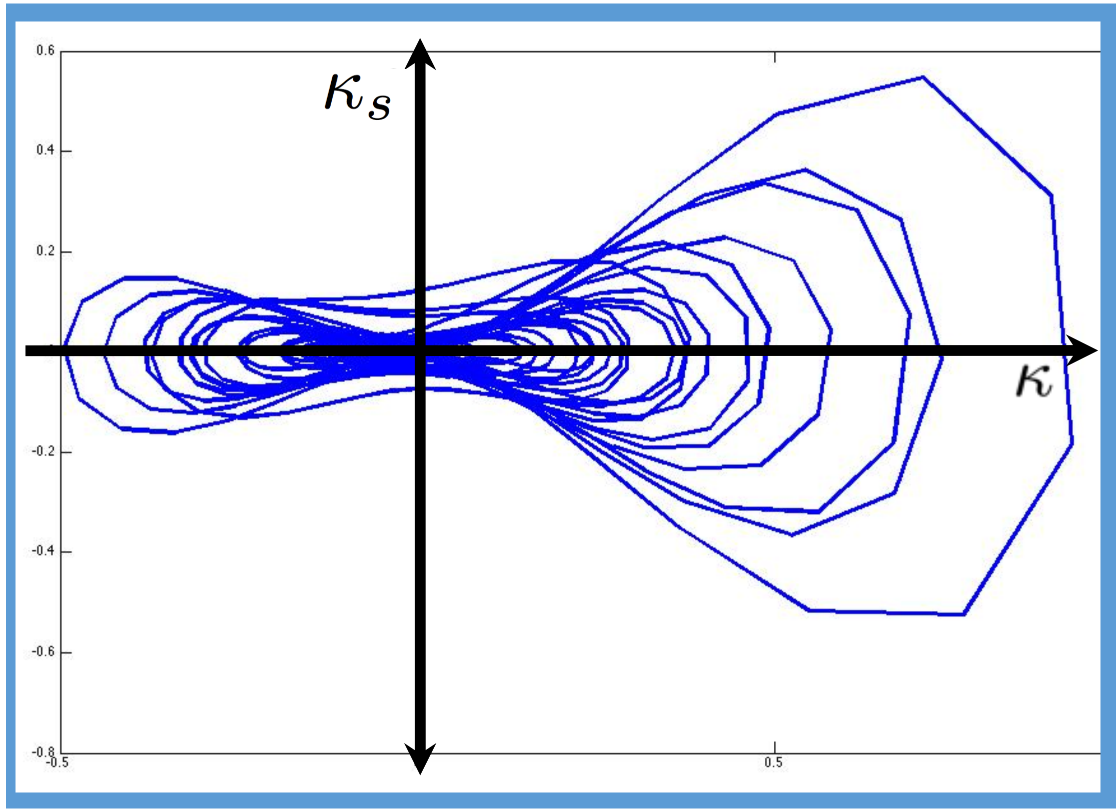


A MALIGNANT TUMOR

Contour

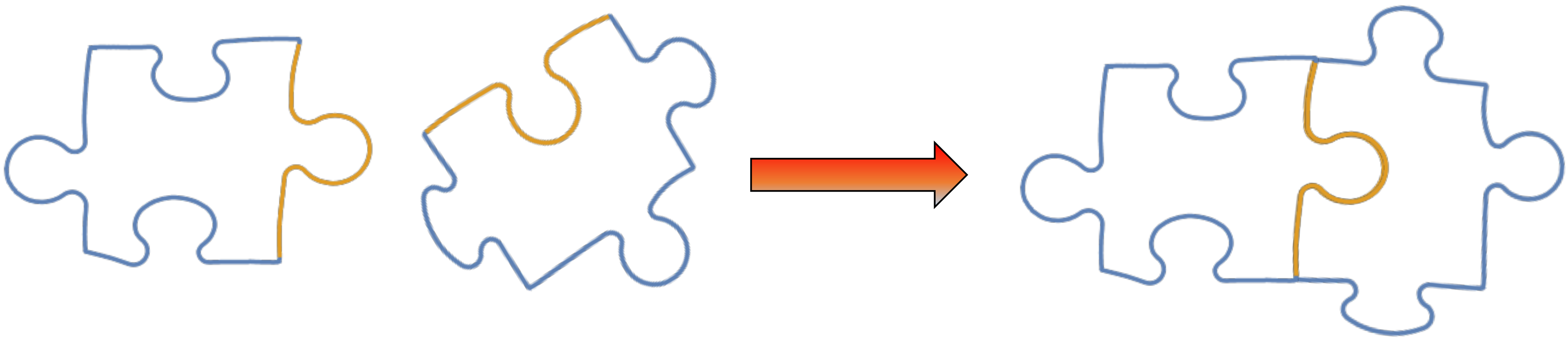


Signature Curve



Applications to Jigsaw Puzzles and Broken Objects

Automatic puzzle reassembly



Step 0. Digitally photograph and smooth the puzzle pieces.

Step 1. Numerically compute invariant signatures of (parts of) pieces.

Step 2. Compare signatures to find potential fits.

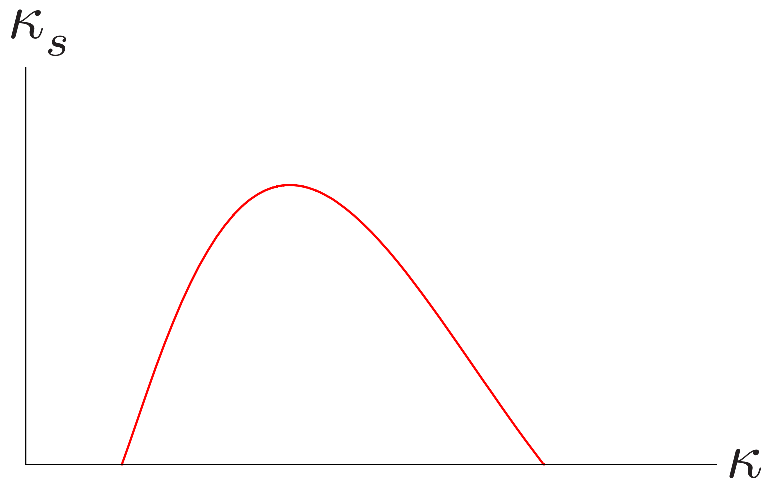
Step 3. Put them together, if they fit, as closely as possible.

Repeat steps 1–3 until puzzle is assembled….

Localization of Signatures

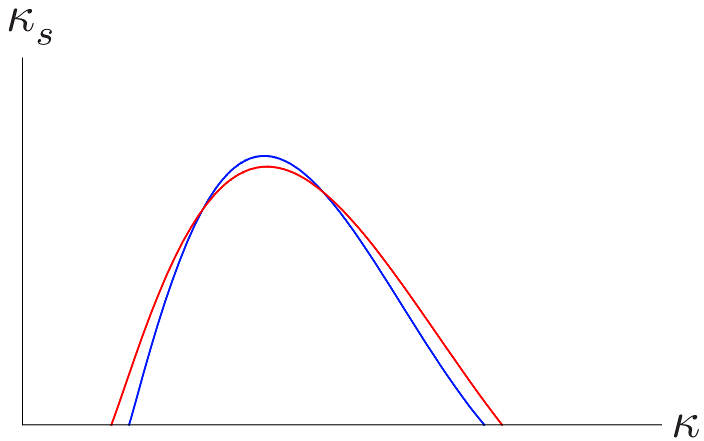
Bivertex arc: $\kappa_s \neq 0$ everywhere
except $\kappa_s = 0$ at the two endpoints

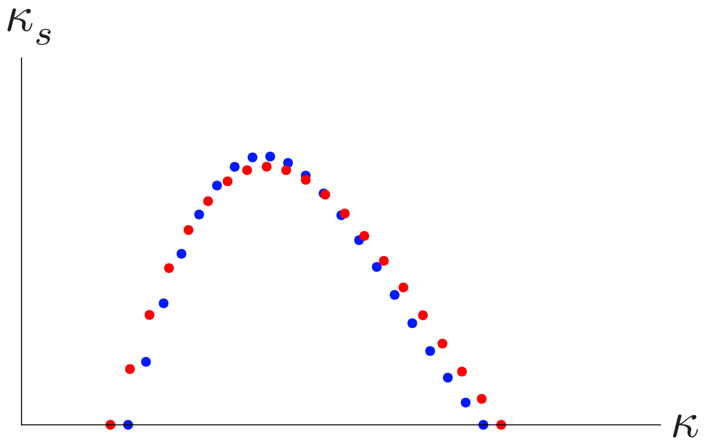
The signature Σ of a bivertex arc is a single arc that starts and ends on the κ–axis.



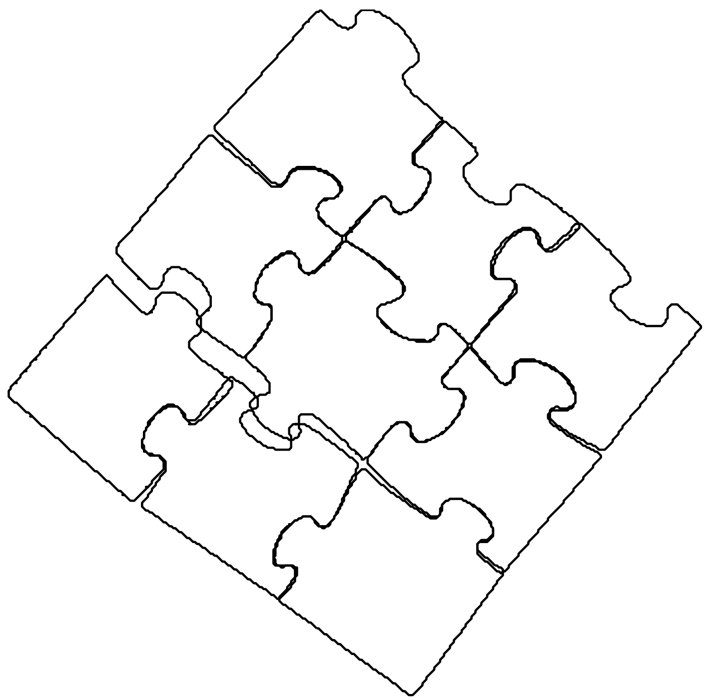
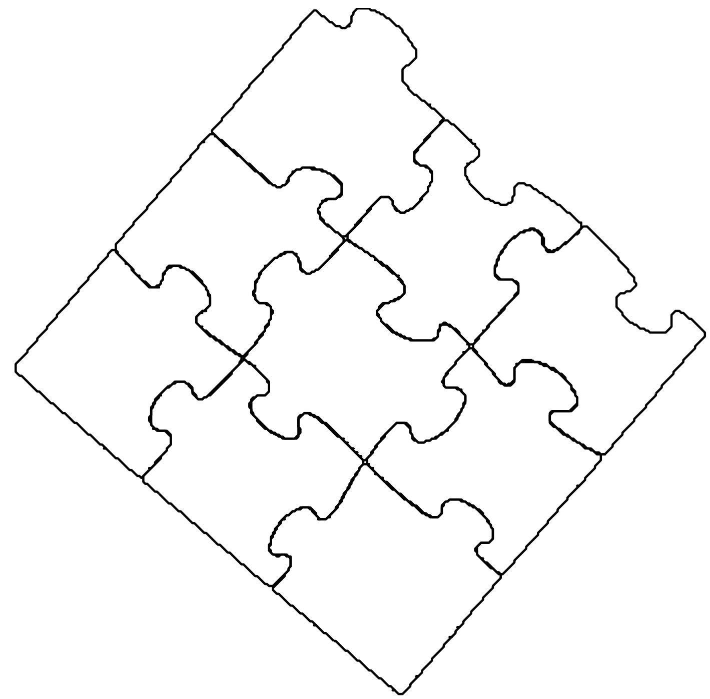
Gravitational/Electrostatic Attraction

- ⋆ Treat the two (signature) curves as masses or as oppositely charged wires. The higher their mutual attraction, the closer they are together.
- ⋆ In practice, we are dealing with discrete data (pixels) and so treat the curves and signatures as point masses/charges.





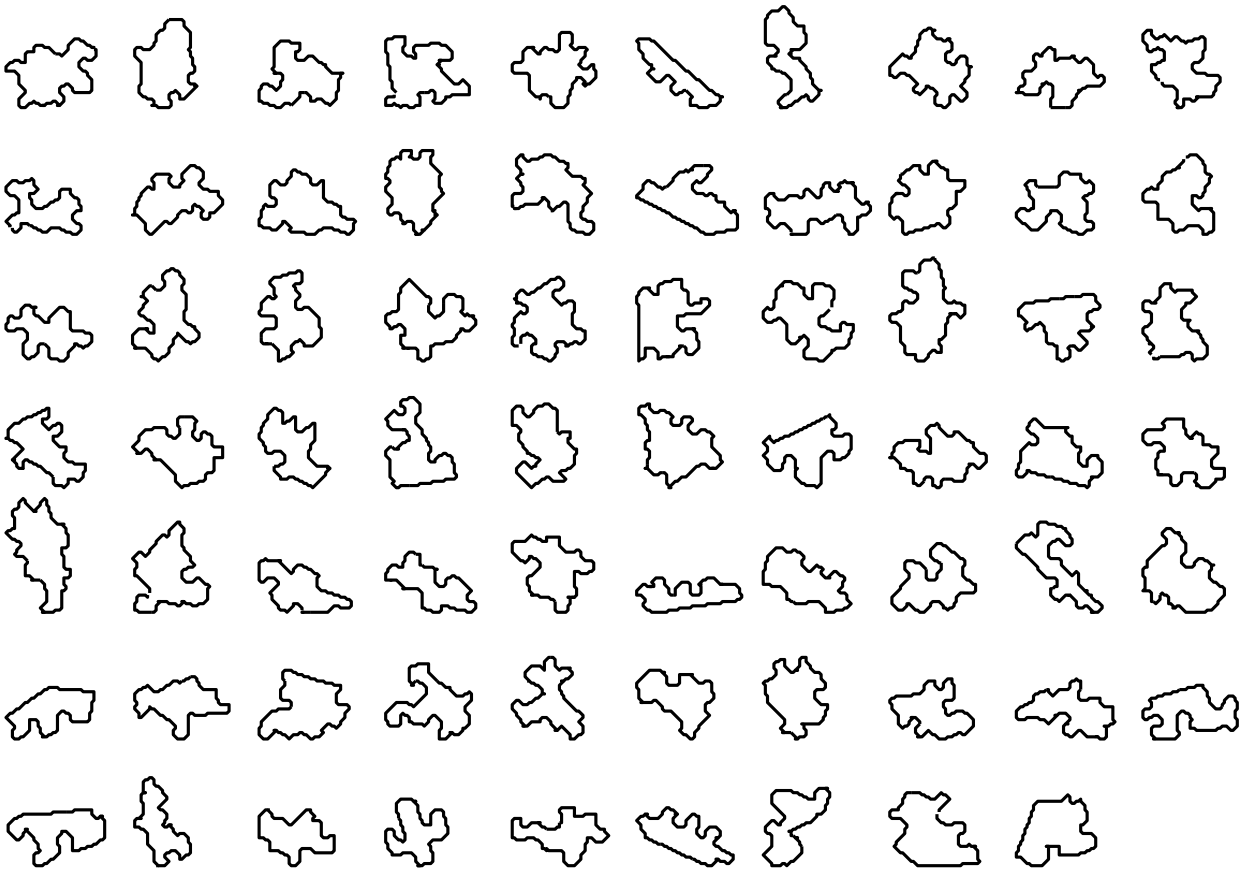
Piece Locking

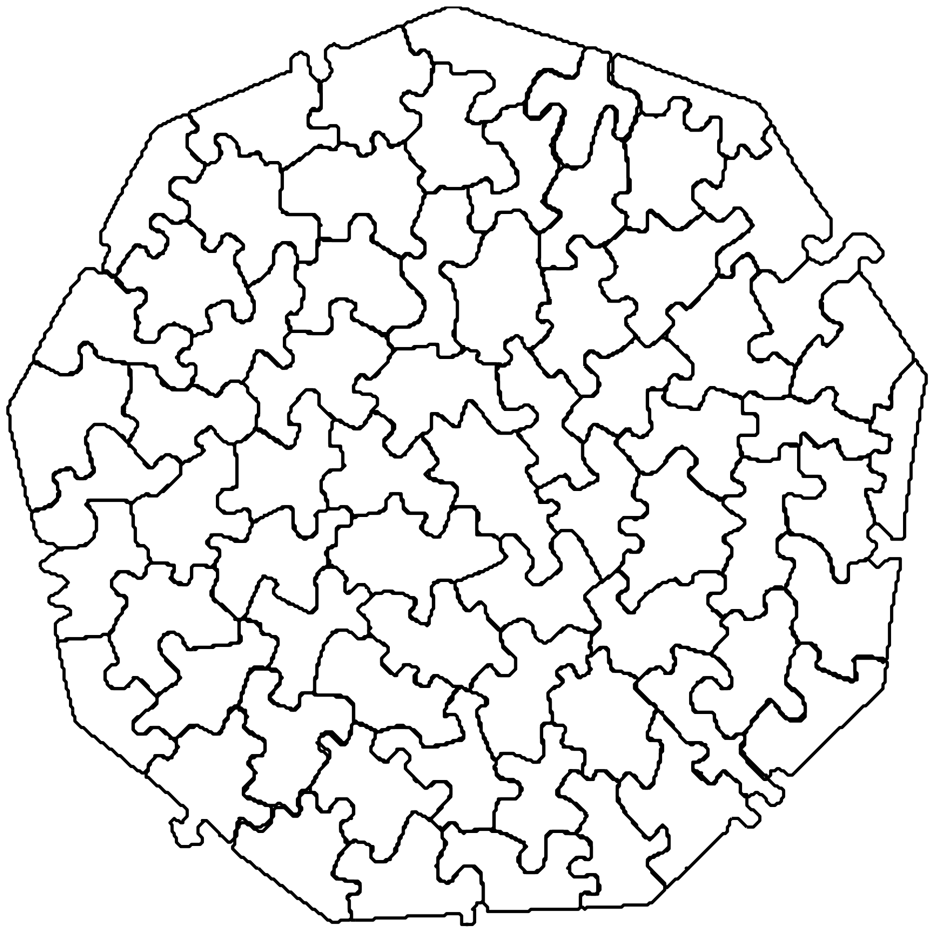
⋆⋆ Minimize force and torque based on gravitational attraction of the two matching edges.


the most unique
puzzle ever
the BAFFLER™
by CHRIS YATES
The Nonagon
67 pieces

The Baffler Nonagon



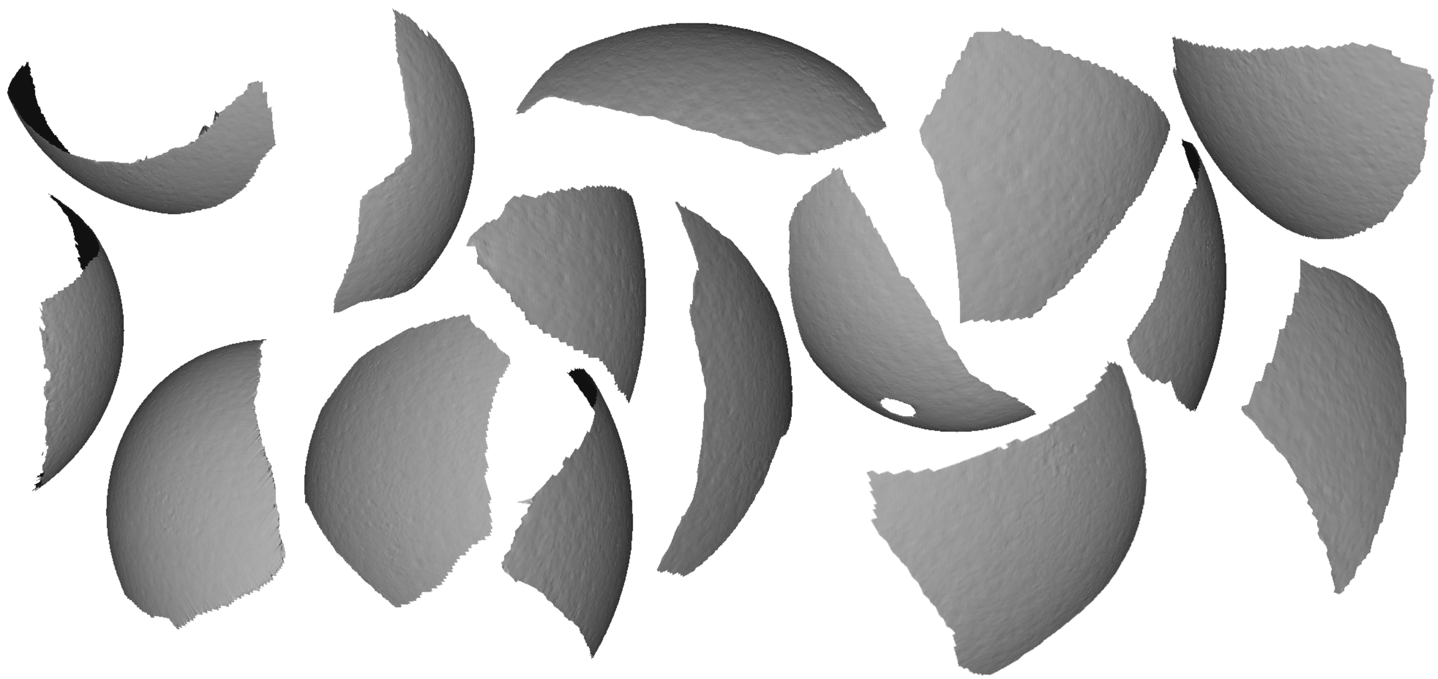
The Baffler Nonagon — Solved



Putting Humpty Dumpty Together Again

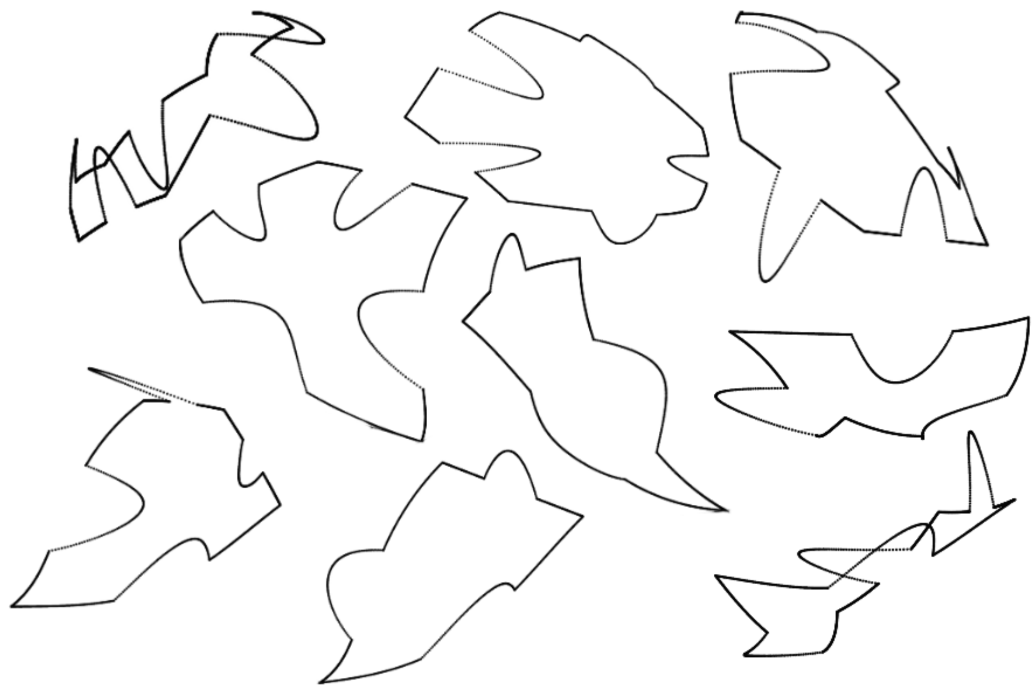
Anna Grim, Ryan Slechta, Tim O'Connor, Rob Thompson, Cheri Shakiban, Peter Olver

A broken ostrich egg

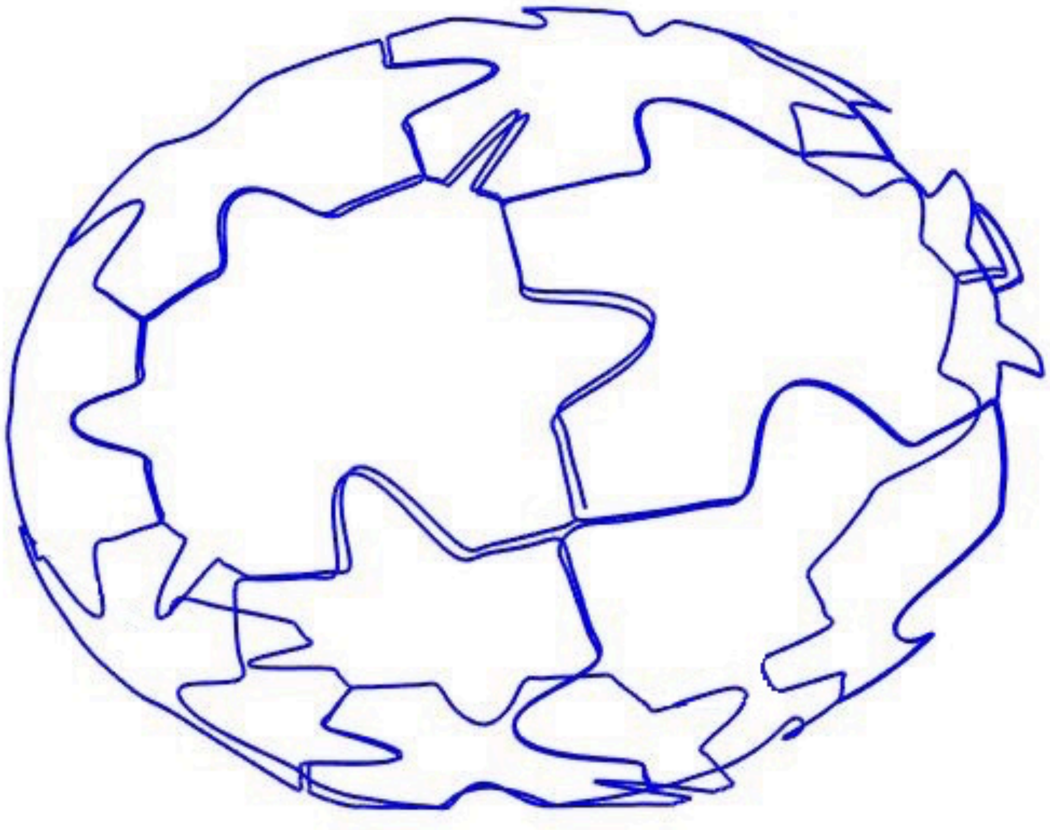


(Scanned by M. Bern, Xerox PARC)

A synthetic 3d jigsaw puzzle

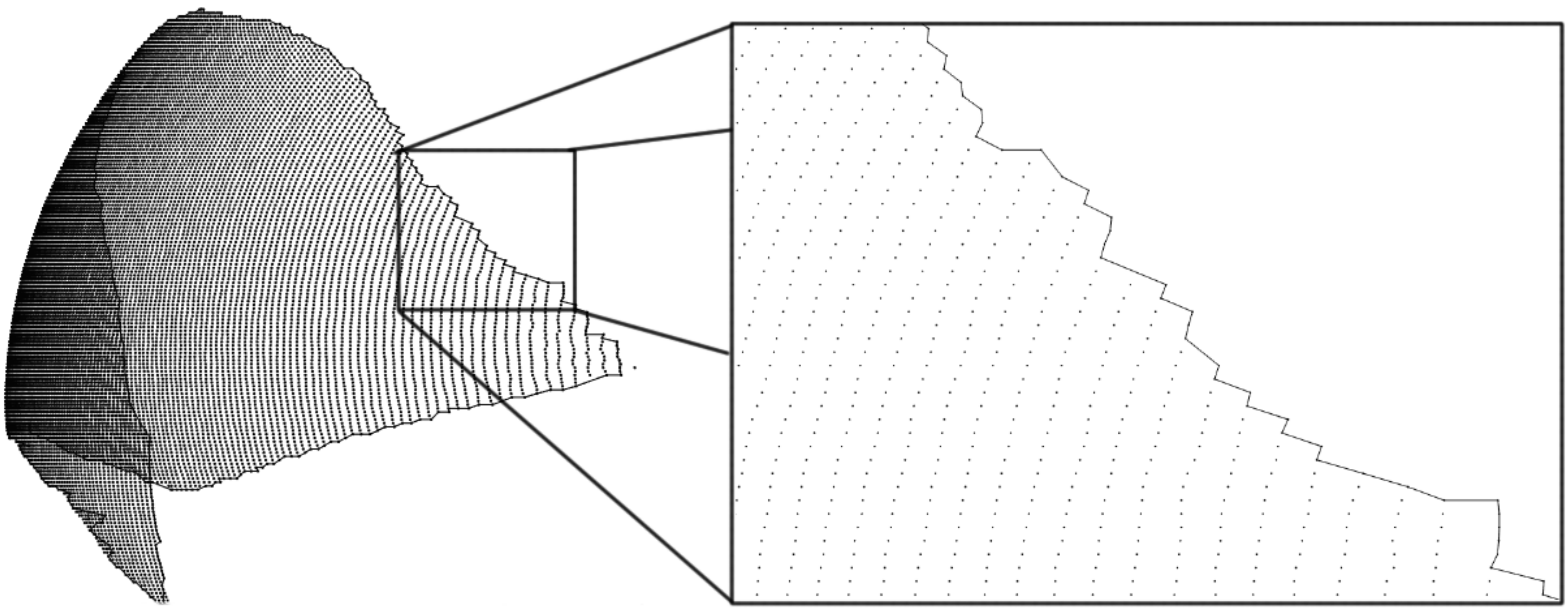


Assembly of synthetic spherical puzzle

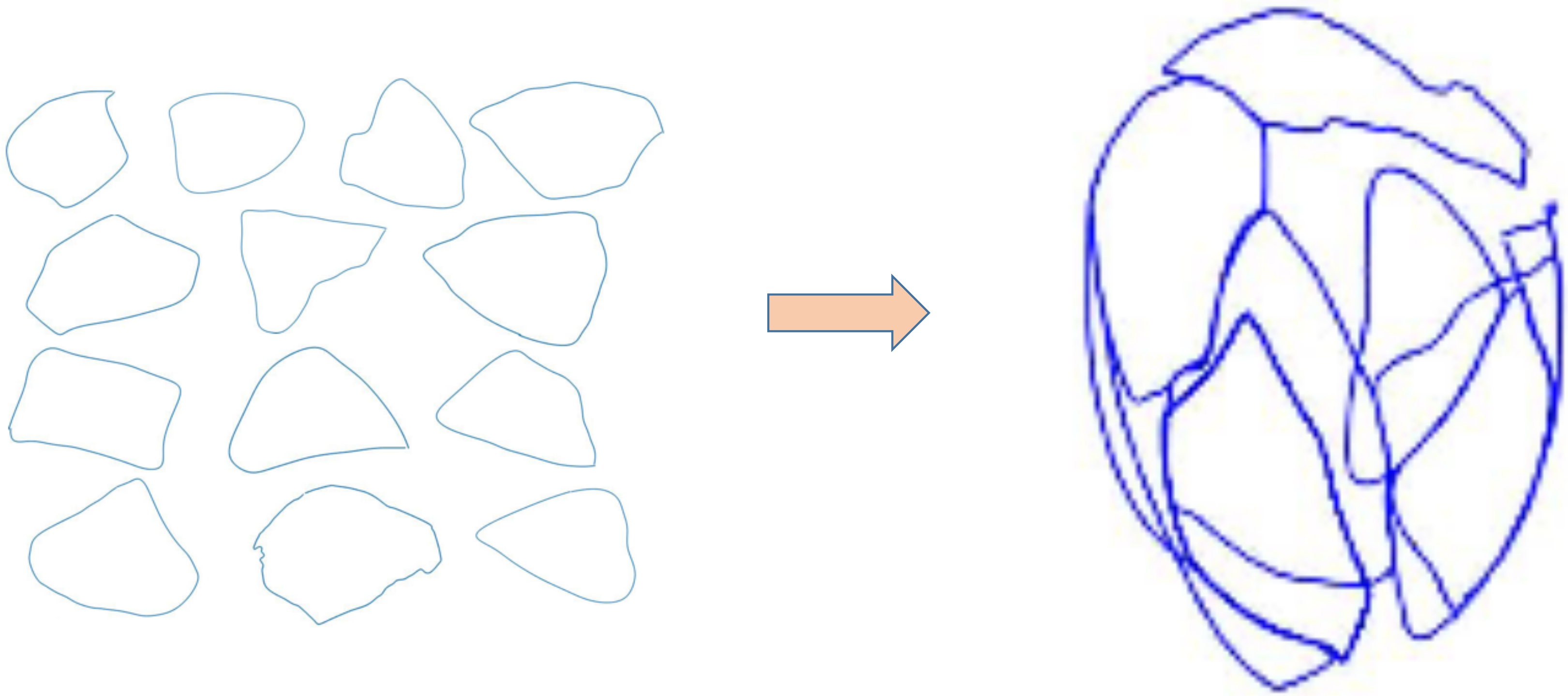


- Uses curvature and torsion invariants

An egg piece

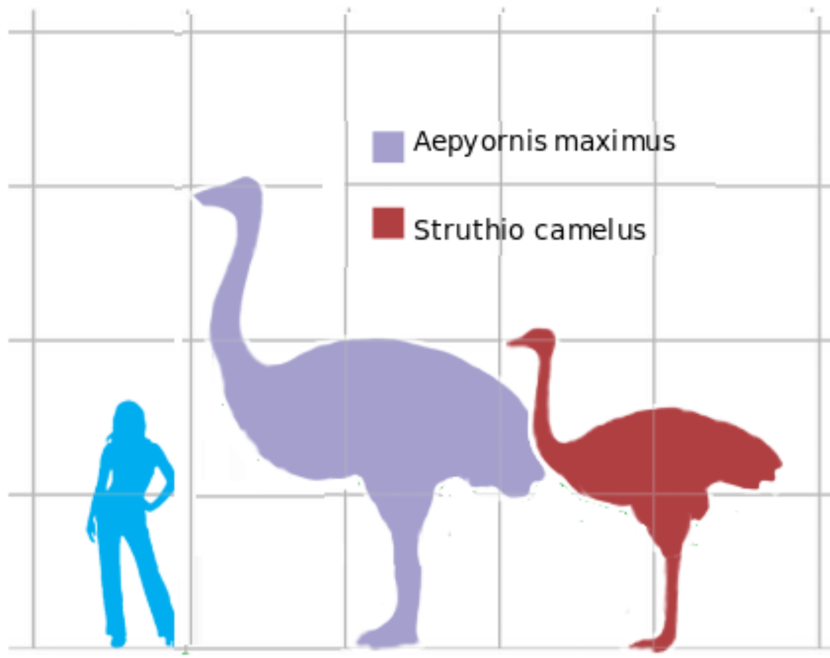


All the king's horses and men



The elephant bird business plan

The elephant bird of Madagascar



(Image from wikipedia.org)

- more than 3 meters tall
- extinct by the 1700's
- one egg could make about 160 omelets

Elephant bird egg shells

(Extract from “Zoo Quest to Madagascar”, BBC 1961)

The elephant bird of Madagascar

(Image from Tennant's Auctioneers)

- pictured egg is 70% complete
- complete egg recently sold for $100,000

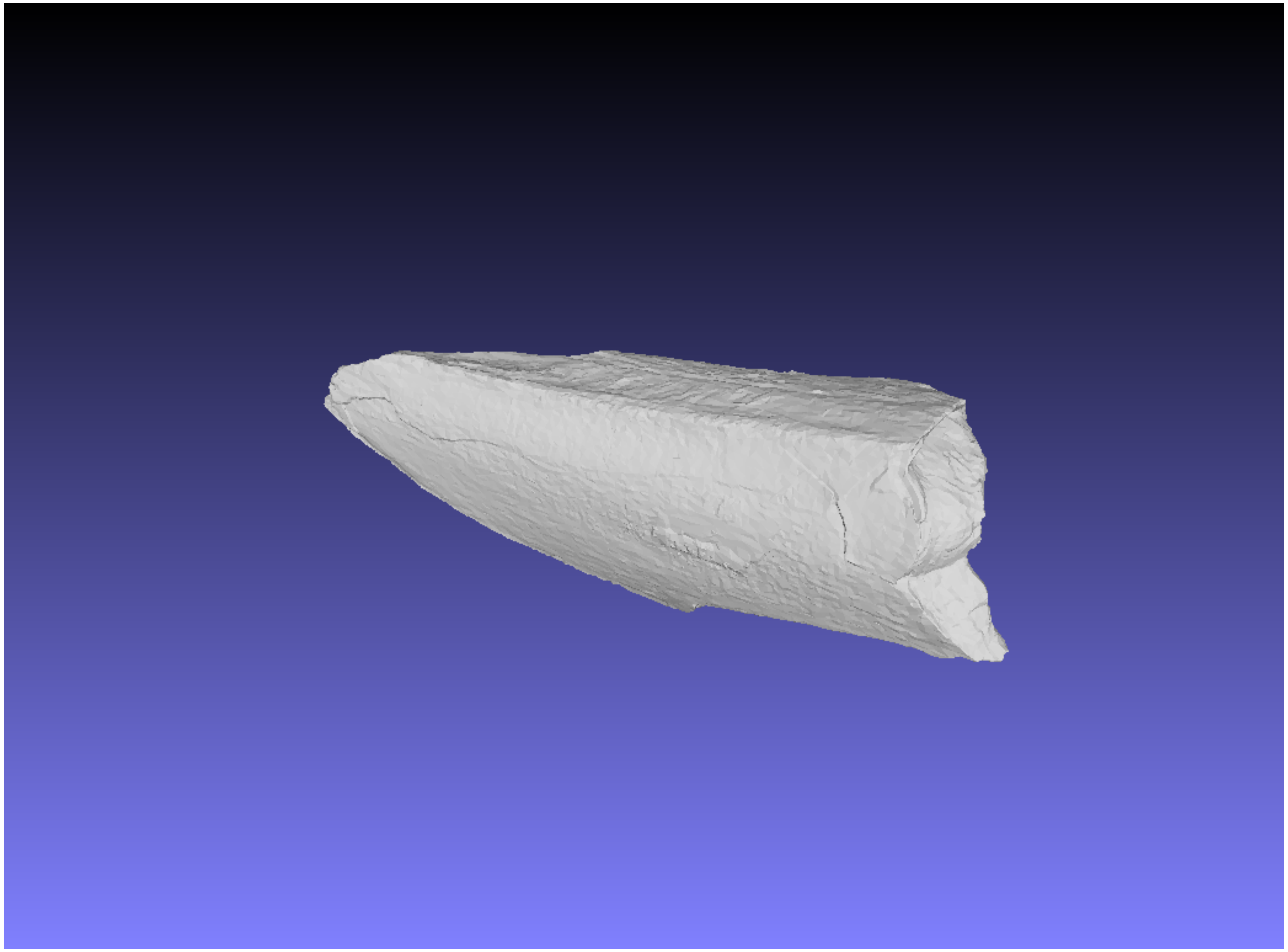
Puzzles in archaeology

Puzzles in archaeology

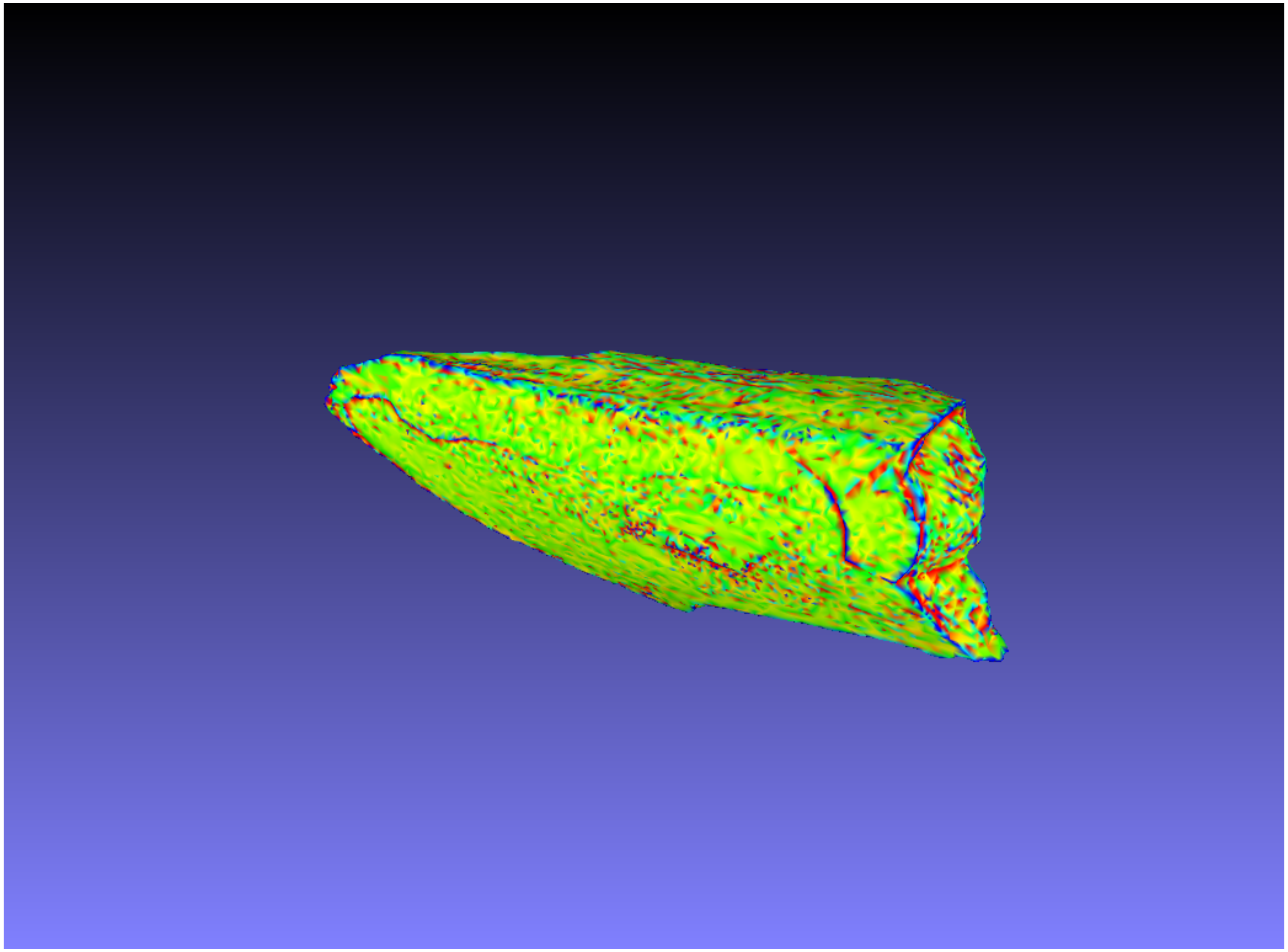
Puzzles in surgery

Puzzles in anthropology

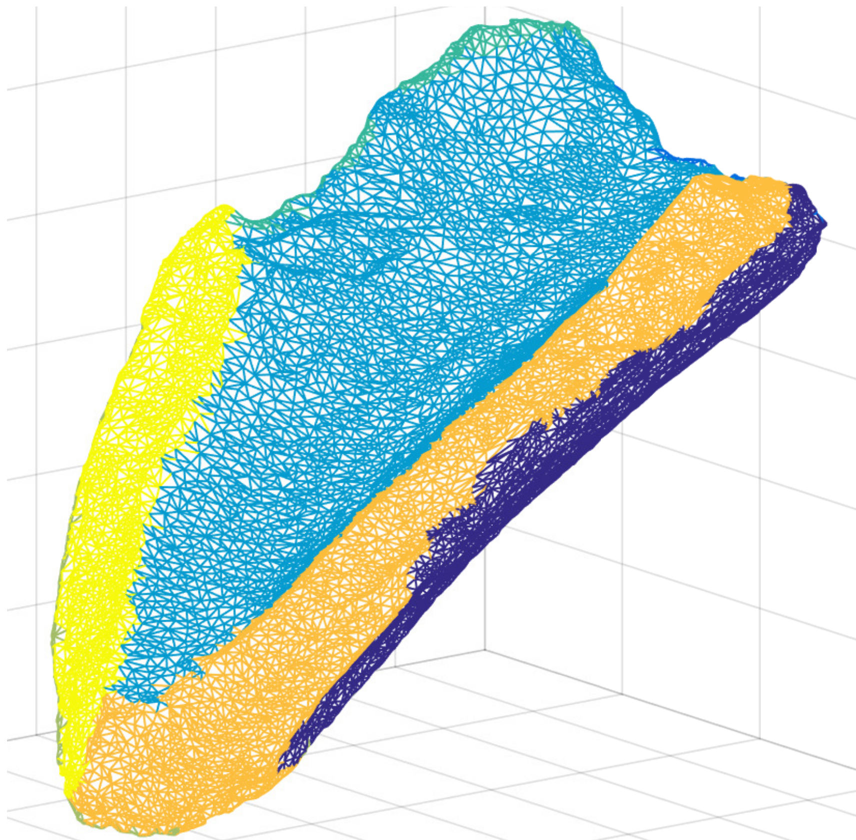
Bone fragment



Mean curvature



Segmentation



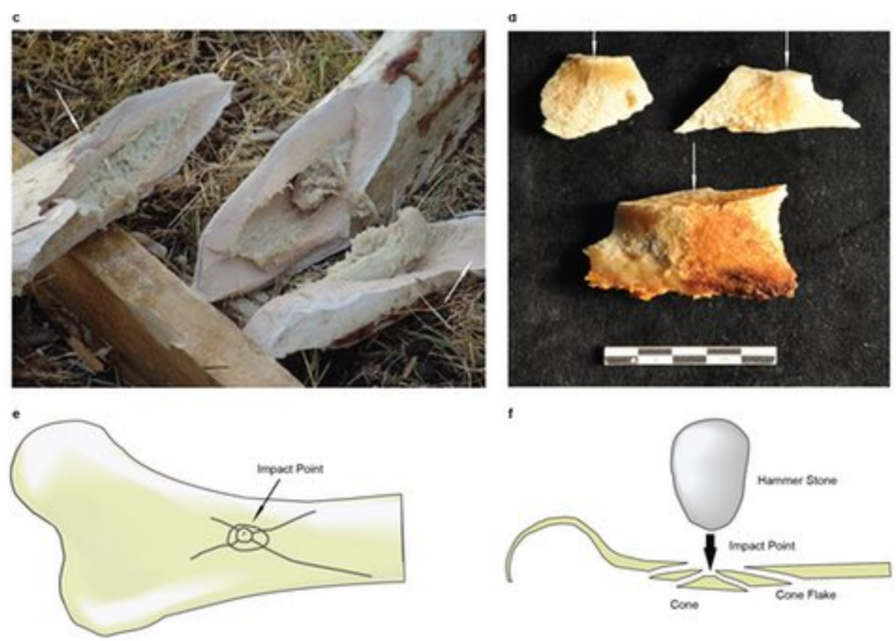
theguardian

Could history of humans in North America be rewritten by broken bones?

Smashed mastodon bones show humans arrived over 100,000 years earlier than previously thought say researchers, although other experts are sceptical

Ian Sample Science editor

Wednesday 26 April 2017 13.00 EDT



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Ph.D. students (past and present): Mimi Boutin, Steve Haker, David Richter, Jessica Senou, Rob Thompson, Katrina Yezzi-Woody

Collaborators: Eugene Calabi, Jeff Calder, Cheri Shakiban, Allen Tannenbaum