Algebras of Differential Invariants

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Examples of Differential Invariants

Euclidean Group on $\mathbb{R}^3$

$$G = \mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$$

$\Longrightarrow$ group of rigid motions

$$z \longmapsto R\,z + b \qquad R \in \mathrm{SO}(3)$$

- Induced action on curves and surfaces.

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- κ — curvature: order $= 2$
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Thus, κ and τ *generate* the differential invariants of space curves under the Euclidean group.

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Thus, H, K *generate* the differential invariant algebra of (generic) Euclidean surfaces.

Equi-affine Group on $\mathbb{R}^3$

$$G = \mathrm{SA}(3) = \mathrm{SL}(3) \ltimes \mathbb{R}^3 \qquad \text{— volume preserving}$$

$$z \longmapsto A\,z + b, \qquad \det A = 1$$

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Curves in $\mathbb{R}^3$:

- κ — equi-affine curvature: order = 4
- τ — equi-affine torsion: order = 5
- $\kappa_s, \tau_s, \kappa_{ss}, \ldots$ — diff. w.r.t. equi-affine arc length

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Surfaces in $\mathbb{R}^3$:

- P — Pick invariant: order $= 3$
- $Q_0, Q_1, \ldots, Q_4$ — fourth order invariants
- $\mathcal{D}_1 P, \mathcal{D}_2 P, \mathcal{D}_1 Q_\nu, \ldots$ diff. w.r.t. the equi-affine frame

General Problems

Determine the structure of the algebra of differential invariants.

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Find a minimal system of generating differential invariants.

Curves

Theorem. Let G be an ordinary* Lie group acting on the m-dimensional manifold M. Then, locally, there exist $m-1$ generating differential invariants $\kappa_1, \ldots, \kappa_{m-1}$. Every other differential invariant can be written as a function of the generating differential invariants and their derivatives with respect to the G-invariant arc length element ds.

$\star$ ordinary $=$ transitive $+$ no pseudo-stabilization.

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$\Longrightarrow \quad m = 3 \quad$ — $\quad$ curvature κ & torsion τ

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In particular:

$$Q_\nu = \Phi_\nu(P, \mathcal{D}_1 P, \mathcal{D}_2 P, \ \ldots\ )$$

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In particular:

$$K = \Phi(H, \mathcal{D}_1 H, \mathcal{D}_2 H, \ \ldots\ )$$

Euclidean Proof

Commutation relation:

$$[\mathcal{D}_1, \mathcal{D}_2] = \mathcal{D}_1 \mathcal{D}_2 - \mathcal{D}_2 \mathcal{D}_1 = Y_2 \mathcal{D}_1 - Y_1 \mathcal{D}_2,$$

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Commutator invariants:

$$Y_1 = \frac{\mathcal{D}_1\kappa_2}{\kappa_1 - \kappa_2} \qquad Y_2 = \frac{\mathcal{D}_2\kappa_1}{\kappa_2 - \kappa_1}$$

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$\Longrightarrow$ Gauss' Theorema Egregium

(*Guggenheimer*)

The Commutator Trick

$$\boxed{K = -(\mathcal{D}_1 + Y_1)\,Y_1 - (\mathcal{D}_2 + Y_2)\,Y_2}$$

To determine the commutator invariants:

$$\begin{aligned} \mathcal{D}_1\mathcal{D}_2 H - \mathcal{D}_2\mathcal{D}_1 H &= Y_2\,\mathcal{D}_1 H - Y_1\,\mathcal{D}_2 H \\ \mathcal{D}_1\mathcal{D}_2\mathcal{D}_J H - \mathcal{D}_2\mathcal{D}_1\mathcal{D}_J H &= Y_2\,\mathcal{D}_1\mathcal{D}_J H - Y_1\,\mathcal{D}_2\mathcal{D}_J H \end{aligned} \qquad (*)$$

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Non-degeneracy condition:

$$\det \begin{pmatrix} \mathcal{D}_1 H & \mathcal{D}_2 H \\ \mathcal{D}_1\mathcal{D}_J H & \mathcal{D}_2\mathcal{D}_J H \end{pmatrix} \neq 0,$$

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Solve $(*)$ for Y_1, Y_2 in terms of derivatives of H, producing a universal formula

$$\boxed{K = \Psi(H, \mathcal{D}_1 H, \mathcal{D}_2 H, \ \ldots\ , \mathcal{D}_1\mathcal{D}_2^3 H, \mathcal{D}_2^4 H)}$$

for the Gauss curvature as a rational function of the mean curvature and its invariant derivatives up to order 4!

Definition. A surface $S \subset \mathbb{R}^3$ is mean curvature degenerate if, near any non-umbilic point $p_0 \in S$, there exist scalar functions $F_1(t), F_2(t)$ such that

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- surfaces with symmetry: rotation, helical;
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Theorem. If a surface is mean curvature non-degenerate then the algebra of Euclidean differential invariants is generated entirely by the mean curvature and its successive invariant derivatives.

Further Results

For suitably non-degenerate surfaces $S \subset \mathbb{R}^3$:

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Theorem. $G = \mathrm{PSL}(4)$

The algebra of projective differential invariants is generated by a single fourth order differential invariant.

$\Longrightarrow$ (*with E. Hubert*)

Example. $G\colon\ (x,y,u) \longmapsto (\,x+a, y+b, u+P(x,y)\,)$

$$a, b \in \mathbb{R}, \qquad P \text{ is an arbitrary polynomial of degree} \leq n$$

Differential invariants:

$$u_{i,j} = \frac{\partial^{i+j} u}{\partial x^i \partial y^j} \qquad i+j \geq n+1$$

Invariant differential operators:

$$\mathcal{D}_1 = D_x, \qquad \mathcal{D}_2 = D_y.$$

Minimal generating set:

$$u_{i,j}, \qquad i+j = n+1$$

♠ For submanifolds of dimension $p \geq 2$, the number of generating differential invariants can be arbitrarily large.

Equivariant Moving Frames

M — m-dimensional manifold

$\mathrm{J}^n = \mathrm{J}^n(M, p)$ — n^{th} order jet space for p-dimensional submanifolds $S \subset M$

$z^{(n)} = (x, u^{(n)}) = (\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ )$

— local coordinates on J^n viewing $S = \{\, u = f(x)\,\}$

G — transformation group acting on M

$G^{(n)}$ — prolonged action on the submanifold jet space J^n

Differential Invariants

Differential invariant $\quad I : U \subset \mathrm{J}^n \to \mathbb{R}$

$$I(g^{(n)} \cdot z^{(n)}) = I(z^{(n)}) \qquad \text{for all} \qquad z^{(n)} \in U = \operatorname{dom} I$$

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$\mathcal{I}(G)$ — the algebra (sheaf) of differential invariants

The Basis Theorem

Theorem. The differential invariant algebra $\mathcal{I}(G)$ is locally generated by a finite number of differential invariants

$$I_1, \ \ldots \ , I_\ell$$

and $p = \dim S$ invariant differential operators

$$\mathcal{D}_1, \ \ldots \ , \mathcal{D}_p$$

meaning that *every* differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives:

$$\mathcal{D}_J I_\kappa = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_n} I_\kappa.$$

$\Longrightarrow$ Lie groups: *Lie, Ovsiannikov*

$\Longrightarrow$ Lie pseudo-groups: *Tresse, Kumpera, Pohjanpelto–PJO, Krugkilov–Lychagin*

Key Issues

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  the invariant differential operators:

$$[\mathcal{D}_j, \mathcal{D}_k] = \sum_{i=1}^{p} Y^i_{jk} \, \mathcal{D}_i$$

$\Longrightarrow$ Non-commutative differential algebra

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$\Longrightarrow$ Non-commutative differential algebra

- Syzygies (functional relations) among

  the differentiated invariants:

$$\Phi(\ \ldots \ \mathcal{D}_J I_\kappa \ \ldots \ ) \equiv 0$$

$\Longrightarrow$ Codazzi relations

Applications

- Equivalence and signatures of submanifolds

  $\Longrightarrow$ image processing

- Characterization of moduli spaces
- Invariant differential equations:

$$H(\; \ldots \; \mathcal{D}_J I_\kappa \; \ldots \;) = 0$$

- Group splitting/foliation of PDEs

  — explicit solutions & Bäcklund transformations

- Invariant variational problems:

$$\int L(\; \ldots \; \mathcal{D}_J I_\kappa \; \ldots \;) \, \boldsymbol{\omega}$$

- conservation laws and characteristic classes

Equivariant Moving Frames

Definition. An n^{th} order *moving frame* is a G-equivariant map

$$\rho^{(n)} : V^n \subset \mathrm{J}^n \longrightarrow G$$

- *Élie Cartan*
- *Guggenheimer, Griffiths, Green, Jensen*
- *Fels, Kogan, Pohjanpelto, PJO*

Equivariance:

$$\rho(g^{(n)} \cdot z^{(n)}) = \begin{cases} g \cdot \rho(z^{(n)}) & \text{left moving frame} \\ \rho(z^{(n)}) \cdot g^{-1} & \text{right moving frame} \end{cases}$$

Note: $\rho_{left}(z^{(n)}) = \rho_{right}(z^{(n)})^{-1}$

Theorem. A moving frame exists in a neighborhood of a jet $z^{(n)} \in \mathrm{J}^n$ if and only if G acts freely and regularly near $z^{(n)}$.

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Theorem. If G acts locally effectively on all open subsets $U \subset M$, then for $n \gg 0$, the (prolonged) action of G is locally free on an open subset of J^n.

$\Longrightarrow$ *Ovsiannikov, PJO, S. Adams*

- free — the only group element $g \in G$ which fixes *one* point $z^{(n)} \in \mathrm{J}^n$ is the identity:

  $$g^{(n)} \cdot z^{(n)} = z^{(n)} \text{ if and only if } g = e.$$

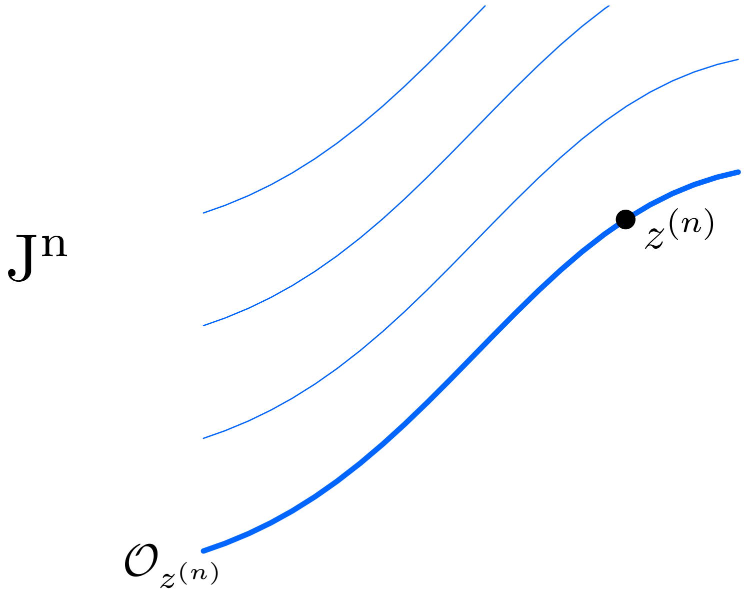
- locally free — the orbits have the same dimension as G.

- regular — all orbits have the same dimension and intersect sufficiently small coordinate charts only once ($\not\approx$ irrational flow on the torus)

- effective — the only group element $g \in G$ which fixes *every* point $z \in U \subset M$ is the identity:

  $$g \cdot z = z \text{ for all } z \in U \text{ if and only if } g = e.$$

Geometric Construction



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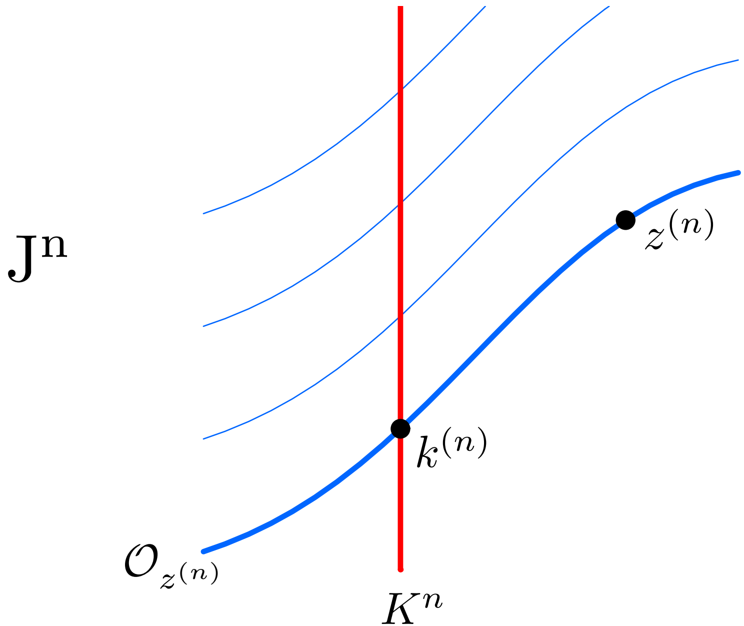

Normalization = choice of cross-section to the group orbits

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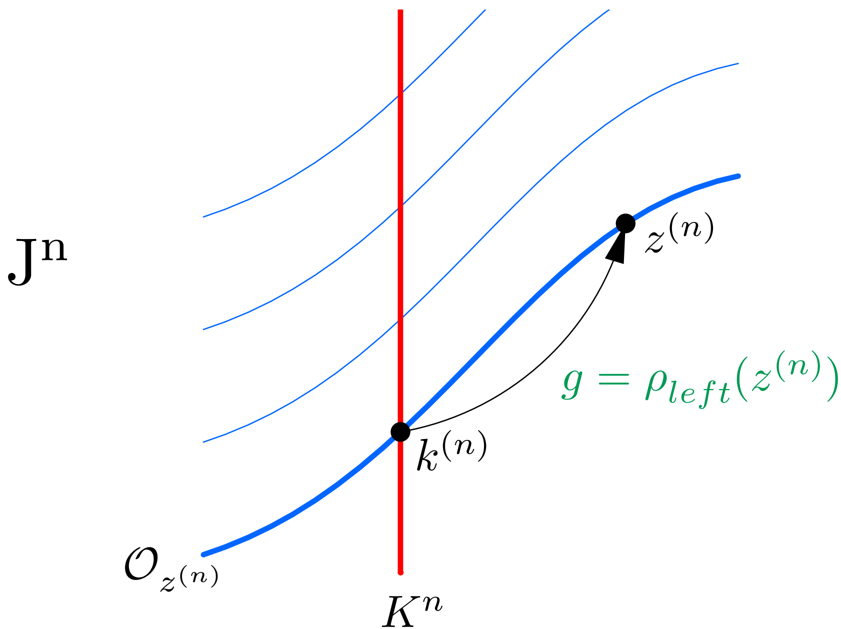

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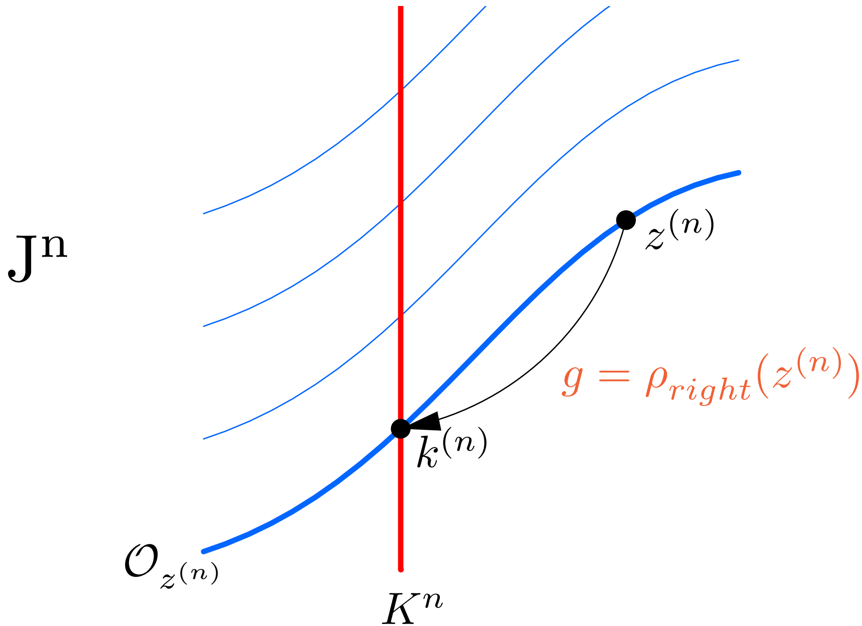

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Algebraic Construction

1. Write out the explicit formulas for the prolonged group action:

$$w^{(n)}(g, z^{(n)}) = g^{(n)} \cdot z^{(n)}$$

$$\Longrightarrow \quad \textit{Implicit differentiation}$$

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2. From the components of $w^{(n)}$, choose $r = \dim G$ *normalization equations* to define the cross-section:

$$w_1(g, z^{(n)}) = c_1 \qquad \ldots \qquad w_r(g, z^{(n)}) = c_r$$

3. Solve the normalization equations for the group parameters $g = (g_1, \ldots, g_r)$:

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4. Substitute the moving frame formulas

$$g = \rho(z^{(n)}) = \rho(x, u^{(n)})$$

for the group parameters into the un-normalized components of $w^{(n)}$ to produce a complete system of functionally independent differential invariants of order $\leq n$:

$$I_k(x, u^{(n)}) = w_k(\rho(z^{(n)}), z^{(n)})), \qquad k = r+1, \ \ldots \ , \dim \mathrm{J}^n$$

Invariantization

The process of replacing group parameters in transformed objects by their moving frame formulae:

$$
\iota : \left\{
\begin{array}{ccc}
\text{Functions} & \longrightarrow & \text{Invariants} \\[2ex]
\text{Forms} & \longrightarrow & \text{Invariant Forms} \\[2ex]
\begin{array}{c}\text{Differential} \\ \text{Operators}\end{array} & \longrightarrow & \begin{array}{c}\text{Invariant Differential} \\ \text{Operators}\end{array} \\[2ex]
\vdots & & \vdots
\end{array}
\right.
$$

- Invariantization defines an (exterior) algebra morphism.
- Invariantization does not affect invariants: $\iota(I) = I$

The Fundamental Differential Invariants

Invariantized jet coordinate functions:

$$H^i(x, u^{(n)}) = \iota(x^i) \qquad I^\alpha_K(x, u^{(l)}) = \iota(u^\alpha_K)$$

- The constant differential invariants, as dictated by the moving frame normalizations, are known as the phantom invariants.
- The remaining non-constant differential invariants are the basic invariants and form a complete system of functionally independent differential invariants for the prolonged group action.

Invariantization of general differential functions:

$$\iota\,[\,F(\;\ldots\;x^i\;\ldots\;u_J^\alpha\;\ldots\;)\,] = F(\;\ldots\;H^i\;\ldots\;I_J^\alpha\;\ldots\;)$$

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The Replacement Theorem:

If J is a differential invariant, then $\iota(J) = J$.

$$J(\,\ldots\;x^i\;\ldots\;u^\alpha_J\;\ldots\,) = J(\,\ldots\;H^i\;\ldots\;I^\alpha_J\;\ldots\,)$$

Invariantization of general differential functions:

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Key fact: Invariantization and differentiation do not commute:

$$\iota(D_iF) \;\neq\; \mathcal{D}_i\iota(F)$$

Infinitesimal Generators

Infinitesimal generators of action of G on M:

$$\mathbf{v}_\kappa = \sum_{i=1}^{p} \xi_\kappa^i(x,u)\,\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi_\kappa^\alpha(x,u)\,\frac{\partial}{\partial u^\alpha} \qquad \kappa = 1,\ldots,r$$

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Prolonged infinitesimal generators on J^n:

$$\mathbf{v}_\kappa^{(n)} = \mathbf{v}_\kappa + \sum_{\alpha=1}^{q} \sum_{j=\#J=1}^{n} \varphi_{J,\kappa}^\alpha(x,u^{(j)}) \frac{\partial}{\partial u_J^\alpha}$$

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Prolonged infinitesimal generators on J^n:

$$\mathbf{v}^{(n)}_\kappa = \mathbf{v}_\kappa + \sum_{\alpha=1}^{q} \sum_{j=\#J=1}^{n} \varphi^\alpha_{J,\kappa}(x,u^{(j)})\;\frac{\partial}{\partial u^\alpha_J}$$

Prolongation formula:

$$\varphi^\alpha_{J,\kappa} = D_K\left(\varphi^\alpha_\kappa - \sum_{i=1}^{p} u^\alpha_i\,\xi^i_\kappa\right) + \sum_{i=1}^{p} u^\alpha_{J,i}\,\xi^i_\kappa$$

$D_1,\ldots,D_p$ — total derivatives

Recurrence Formulae

$$\mathcal{D}_j\, \iota(F) = \iota(D_j F) \; + \; \sum_{\kappa=1}^{r} R_j^\kappa \, \iota(\mathbf{v}_\kappa^{(n)}(F))$$

$\omega^i = \iota(dx^i)$ — invariant coframe

$\mathcal{D}_i = \iota(D_{x^i})$ — dual invariant differential operators

R_j^κ — Maurer–Cartan invariants

$\mathbf{v}_1, \ \ldots \ \mathbf{v}_r \in \mathfrak{g}$ — infinitesimal generators

$\mu^1, \ \ldots \ \mu^r \in \mathfrak{g}^*$ — dual Maurer–Cartan forms

The Maurer–Cartan Invariants

Invariantized Maurer–Cartan forms:

$$\gamma^\kappa = \rho^*(\mu^\kappa) \equiv \sum_{j=1}^{p} R_j^\kappa\, \omega^j$$

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Remark: When $G \subset \mathrm{GL}(N)$, the Maurer–Cartan invariants R_j^κ are the entries of the Frenet matrices

$$\mathcal{D}_i \, \rho(x, u^{(n)}) \cdot \rho(x, u^{(n)})^{-1}$$

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Theorem. (*E. Hubert*) The Maurer–Cartan invariants and, in the intransitive case, the order zero invariants generate the differential invariant algebra $\mathcal{I}(G)$.

Recurrence Formulae

$$\mathcal{D}_j\, \iota(F) = \iota(D_j F) \;+\; \sum_{\kappa=1}^{r} R_j^\kappa\, \iota(\mathbf{v}_\kappa^{(n)}(F))$$

♠ If $\iota(F) = c$ is a phantom differential invariant, then the left hand side of the recurrence formula is zero. The collection of all such phantom recurrence formulae form a linear algebraic system of equations that can be uniquely solved for the Maurer–Cartan invariants R_j^κ!

♡ Once the Maurer–Cartan invariants are replaced by their explicit formulae, the induced recurrence relations completely determine the structure of the differential invariant algebra $\mathcal{I}(G)$!

The Universal Recurrence Formula

Let Ω be any differential form on J^n.

$$d\,\iota(\Omega) = \iota(d\Omega) + \sum_{\kappa=1}^{r} \gamma^\kappa \wedge \iota[\mathbf{v}_\kappa(\Omega)]$$

$\Longrightarrow$ *The invariant variational bicomplex*

Commutator invariants:

$$d\omega^i = d[\iota(dx^i)] = \iota(d^2 x^i) + \sum_{\kappa=1}^{r} \gamma^\kappa \wedge \iota[\mathbf{v}_\kappa(dx^i)]$$

$$= - \sum_{j<k} Y^i_{jk}\, \omega^j \wedge \omega^k + \cdots$$

$$[\, \mathcal{D}_j, \mathcal{D}_k \,] = \sum_{i=1}^{p} Y^i_{jk}\, \mathcal{D}_i$$

The Differential Invariant Algebra

Thus, remarkably, the structure of $\mathcal{I}(G)$ can be determined without knowing the explicit formulae for either the moving frame, or the differential invariants, or the invariant differential operators!

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The only required ingredients are the specification of the cross-section, and the standard formulae for the prolonged infinitesimal generators.

The Differential Invariant Algebra

Thus, remarkably, the structure of $\mathcal{I}(G)$ can be determined without knowing the explicit formulae for either the moving frame, or the differential invariants, or the invariant differential operators!

The only required ingredients are the specification of the cross-section, and the standard formulae for the prolonged infinitesimal generators.

Theorem. If G acts transitively on M, or if the infinitesimal generator coefficients depend rationally in the coordinates, then all recurrence formulae are rational in the basic differential invariants and so $\mathcal{I}(G)$ is a rational, non-commutative differential algebra.

Euclidean Surfaces

Euclidean group $\mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$ acts on surfaces $S \subset \mathbb{R}^3$.

For simplicity, we assume the surface is (locally) the graph of a function

$$z = u(x, y)$$

Infinitesimal generators:

$$\mathbf{v}_1 = -y\,\partial_x + x\,\partial_y, \qquad \mathbf{v}_2 = -u\,\partial_x + x\,\partial_u, \qquad \mathbf{v}_3 = -u\,\partial_y + y\,\partial_u,$$

$$\mathbf{w}_1 = \partial_x, \qquad \mathbf{w}_2 = \partial_y, \qquad \mathbf{w}_3 = \partial_u.$$

- The translations $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ will be ignored, as they play no role in the higher order recurrence formulae.

Cross-section (Darboux frame):

$$x = y = u = u_x = u_y = u_{xy} = 0.$$

Phantom differential invariants:

$$\iota(x) = \iota(y) = \iota(u) = \iota(u_x) = \iota(u_y) = \iota(u_{xy}) = 0$$

Principal curvatures

$$\kappa_1 = \iota(u_{xx}), \qquad \kappa_2 = \iota(u_{yy})$$

Mean curvature and Gauss curvature:

$$H = \tfrac{1}{2}(\kappa_1 + \kappa_2), \qquad K = \kappa_1 \kappa_2$$

Higher order differential invariants — invariantized jet coordinates:

$$I_{jk} = \iota(u_{jk}) \qquad \text{where} \qquad u_{jk} = \frac{\partial^{j+k} u}{\partial x^j \partial y^k}$$

★ ★ Nondegeneracy condition: non-umbilic point $\kappa_1 \neq \kappa_2$.

Algebra of Euclidean Differential Invariants

Principal curvatures:

$$\kappa_1 = \iota(u_{xx}), \qquad \kappa_2 = \iota(u_{yy})$$

Mean curvature and Gauss curvature:

$$H = \tfrac{1}{2}(\kappa_1 + \kappa_2), \qquad K = \kappa_1 \kappa_2$$

Invariant differentiation operators:

$$\mathcal{D}_1 = \iota(D_x), \qquad \mathcal{D}_2 = \iota(D_y)$$

$\Longrightarrow$ Differentiation with respect to the diagonalizing Darboux frame.

Algebra of Euclidean Differential Invariants

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Invariant differentiation operators:

$$\mathcal{D}_1 = \iota(D_x), \qquad \mathcal{D}_2 = \iota(D_y)$$

$\Longrightarrow$ Differentiation with respect to the diagonalizing Darboux frame.

The recurrence formulae enable one to express the higher order differential invariants in terms of the principal curvatures, or, equivalently, the mean and Gauss curvatures, and their invariant derivatives:

$$I_{jk} = \iota(u_{jk}) = \widetilde{\Phi}_{jk}(\kappa_1, \kappa_2, \mathcal{D}_1\kappa_1, \mathcal{D}_2\kappa_1, \mathcal{D}_1\kappa_2, \mathcal{D}_2\kappa_2, \mathcal{D}_1^2\kappa_1, \ \ldots\ )$$

$$= \Phi_{jk}(H, K, \mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1 K, \mathcal{D}_2 K, \mathcal{D}_1^2 H, \ \ldots\ )$$

Recurrence Formulae

$$\iota(D_i u_{jk}) = \mathcal{D}_i\, \iota(u_{jk}) - \sum_{\kappa=1}^{3} R_i^\kappa\, \iota[\,\varphi_\kappa^{jk}(x, y, u^{(j+k)})\,], \qquad j + k \geq 1$$

$I_{jk} = \iota(u_{jk})$ — normalized differential invariants

R_i^κ — Maurer–Cartan invariants

Recurrence Formulae

$$\iota(D_i u_{jk}) = \mathcal{D}_i\, \iota(u_{jk}) - \sum_{\kappa=1}^{3} R_i^\kappa\, \iota[\,\varphi_\kappa^{jk}(x,y,u^{(j+k)})\,], \qquad j+k \geq 1$$

$I_{jk} = \iota(u_{jk})$ — normalized differential invariants

R_i^κ — Maurer–Cartan invariants

$\varphi_\kappa^{jk}(0,0,I^{(j+k)}) = \iota[\,\varphi_\kappa^{jk}(x,y,u^{(j+k)})\,]$

— invariantized prolonged infinitesimal generator coefficients.

$$I_{j+1,k} = \mathcal{D}_1 I_{jk} - \sum_{\kappa=1}^{3} \varphi_\kappa^{jk}(0,0,I^{(j+k)})\, R_1^\kappa$$

$$I_{j,k+1} = \mathcal{D}_1 I_{jk} - \sum_{\kappa=1}^{3} \varphi_\kappa^{jk}(0,0,I^{(j+k)})\, R_2^\kappa$$

Prolonged infinitesimal generators:

$$
\begin{aligned}
\operatorname{pr} \mathbf{v}_1 &= -y\partial_x + x\partial_y - u_y\partial_{u_x} + u_x\partial_{u_y} \\
&\quad - 2u_{xy}\partial_{u_{xx}} + (u_{xx} - u_{yy})\partial_{u_{xy}} - 2u_{xy}\partial_{u_{yy}} + \cdots , \\
\operatorname{pr} \mathbf{v}_2 &= -u\partial_x + x\partial_u + (1+u_x^2)\partial_{u_x} + u_x u_y\partial_{u_y} \\
&\quad + 3u_x u_{xx}\partial_{u_{xx}} + (u_y u_{xx} + 2u_x u_{xy})\partial_{u_{xy}} + (2u_y u_{xy} + u_x u_{yy})\partial_{u_{yy}} + \cdots , \\
\operatorname{pr} \mathbf{v}_3 &= -u\partial_y + y\partial_u + u_x u_y\partial_{u_x} + (1+u_y^2)\partial_{u_y} \\
&\quad + (u_y u_{xx} + 2u_x u_{xy})\partial_{u_{xx}} + (2u_y u_{xy} + u_x u_{yy})\partial_{u_{xy}} + 3u_y u_{yy}\partial_{u_{yy}} + \cdots .
\end{aligned}
$$

Prolonged infinitesimal generators:

$$\begin{aligned}
\operatorname{pr} \mathbf{v}_1 &= -y\partial_x + x\partial_y - u_y\partial_{u_x} + u_x\partial_{u_y} \\
&\qquad - 2u_{xy}\partial_{u_{xx}} + (u_{xx} - u_{yy})\partial_{u_{xy}} - 2u_{xy}\partial_{u_{yy}} + \cdots , \\
\operatorname{pr} \mathbf{v}_2 &= -u\partial_x + x\partial_u + (1 + u_x^2)\partial_{u_x} + u_x u_y \partial_{u_y} \\
&\qquad + 3u_x u_{xx}\partial_{u_{xx}} + (u_y u_{xx} + 2u_x u_{xy})\partial_{u_{xy}} + (2u_y u_{xy} + u_x u_{yy})\partial_{u_{yy}} + \cdots , \\
\operatorname{pr} \mathbf{v}_3 &= -u\partial_y + y\partial_u + u_x u_y\partial_{u_x} + (1 + u_y^2)\partial_{u_y} \\
&\qquad + (u_y u_{xx} + 2u_x u_{xy})\partial_{u_{xx}} + (2u_y u_{xy} + u_x u_{yy})\partial_{u_{xy}} + 3u_y u_{yy}\partial_{u_{yy}} + \cdots .
\end{aligned}$$

$$I_{jk} = \iota(u_{jk})$$

Phantom differential invariants:

$$I_{00} = I_{10} = I_{01} = I_{11} = 0$$

Principal curvatures:

$$I_{20} = \kappa_1 \qquad I_{02} = \kappa_2$$

Phantom recurrence formulae:

$$
\begin{aligned}
\kappa_1 = I_{20} &= \mathcal{D}_1 I_{10} - R_1^2 = -R_1^2, \\
0 = I_{11} &= \mathcal{D}_1 I_{01} - R_1^3 = -R_1^3, \\
I_{21} &= \mathcal{D}_1 I_{11} - (\kappa_1 - \kappa_2)\, R_1^1 = -(\kappa_1 - \kappa_2)\, R_1^1, \\
0 = I_{11} &= \mathcal{D}_2 I_{10} - R_2^2 = -R_2^2, \\
\kappa_2 = I_{02} &= \mathcal{D}_2 I_{01} - R_2^3 = -R_2^3, \\
I_{12} &= \mathcal{D}_2 I_{11} - (\kappa_1 - \kappa_2)\, R_2^1 = -(\kappa_1 - \kappa_2)\, R_2^1.
\end{aligned}
$$

Phantom recurrence formulae:

$$\kappa_1 = I_{20} = \mathcal{D}_1 I_{10} - R_1^2 = -R_1^2,$$

$$0 = I_{11} = \mathcal{D}_1 I_{01} - R_1^3 = -R_1^3,$$

$$I_{21} = \mathcal{D}_1 I_{11} - (\kappa_1 - \kappa_2)\, R_1^1 = -(\kappa_1 - \kappa_2)\, R_1^1,$$

$$0 = I_{11} = \mathcal{D}_2 I_{10} - R_2^2 = -R_2^2,$$

$$\kappa_2 = I_{02} = \mathcal{D}_2 I_{01} - R_2^3 = -R_2^3,$$

$$I_{12} = \mathcal{D}_2 I_{11} - (\kappa_1 - \kappa_2)\, R_2^1 = -(\kappa_1 - \kappa_2)\, R_2^1.$$

Maurer–Cartan invariants:

$$R_1^1 = -Y_1, \qquad R_1^2 = -\kappa_1, \qquad R_1^3 = 0,$$

$$R_1^2 = -Y_2, \qquad R_2^2 = 0, \qquad R_3^2 = -\kappa_2.$$

Commutator invariants:

$$Y_1 = \frac{I_{21}}{\kappa_1 - \kappa_2} = \frac{\mathcal{D}_1 \kappa_2}{\kappa_1 - \kappa_2} \qquad\qquad Y_2 = \frac{I_{12}}{\kappa_1 - \kappa_2} = \frac{\mathcal{D}_2 \kappa_1}{\kappa_2 - \kappa_1}$$

Phantom recurrence formulae:

$$\kappa_1 = I_{20} = \mathcal{D}_1 I_{10} - R_1^2 = -R_1^2,$$

$$0 = I_{11} = \mathcal{D}_1 I_{01} - R_1^3 = -R_1^3,$$

$$I_{21} = \mathcal{D}_1 I_{11} - (\kappa_1 - \kappa_2)\, R_1^1 = -(\kappa_1 - \kappa_2)\, R_1^1,$$

$$0 = I_{11} = \mathcal{D}_2 I_{10} - R_2^2 = -R_2^2,$$

$$\kappa_2 = I_{02} = \mathcal{D}_2 I_{01} - R_2^3 = -R_2^3,$$

$$I_{12} = \mathcal{D}_2 I_{11} - (\kappa_1 - \kappa_2)\, R_2^1 = -(\kappa_1 - \kappa_2)\, R_2^1.$$

Maurer–Cartan invariants:

$$R_1^1 = -Y_1, \qquad R_1^2 = -\kappa_1, \qquad R_1^3 = 0,$$

$$R_1^2 = -Y_2, \qquad R_2^2 = 0, \qquad R_3^2 = -\kappa_2.$$

Commutator invariants:

$$Y_1 = \frac{I_{21}}{\kappa_1 - \kappa_2} = \frac{\mathcal{D}_1 \kappa_2}{\kappa_1 - \kappa_2} \qquad\qquad Y_2 = \frac{I_{12}}{\kappa_1 - \kappa_2} = \frac{\mathcal{D}_2 \kappa_1}{\kappa_2 - \kappa_1}$$

$$\boxed{[\,\mathcal{D}_1, \mathcal{D}_2\,] = \mathcal{D}_1\, \mathcal{D}_2 - \mathcal{D}_2\, \mathcal{D}_1 = Y_2\, \mathcal{D}_1 - Y_1\, \mathcal{D}_2,}$$

Third order recurrence relations:

$$I_{30} = \mathcal{D}_1 \kappa_1 = \kappa_{1,1}, \quad I_{21} = \mathcal{D}_2 \kappa_1 = \kappa_{1,2}, \quad I_{12} = \mathcal{D}_1 \kappa_2 = \kappa_{2,1}, \quad I_{03} = \mathcal{D}_2 \kappa_2 = \kappa_{2,2},$$

Third order recurrence relations:

$$I_{30} = \mathcal{D}_1\kappa_1 = \kappa_{1,1}, \quad I_{21} = \mathcal{D}_2\kappa_1 = \kappa_{1,2}, \quad I_{12} = \mathcal{D}_1\kappa_2 = \kappa_{2,1}, \quad I_{03} = \mathcal{D}_2\kappa_2 = \kappa_{2,2},$$

Fourth order recurrence relations:

$$\begin{aligned}
I_{40} &= \kappa_{1,11} - \frac{3\,\kappa_{1,2}^2}{\kappa_1 - \kappa_2} + 3\,\kappa_1^3, \\
I_{31} &= \kappa_{1,12} - \frac{3\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2} &&= \kappa_{1,21} + \frac{\kappa_{1,1}\kappa_{1,2} - 2\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2}, \\
I_{22} &= \kappa_{1,22} + \frac{\kappa_{1,1}\kappa_{2,1} - 2\,\kappa_{2,1}^2}{\kappa_1 - \kappa_2} + \kappa_1\kappa_2^2 &&= \kappa_{2,11} - \frac{\kappa_{1,2}\kappa_{2,2} - 2\,\kappa_{1,2}^2}{\kappa_1 - \kappa_2} + \kappa_1^2\kappa_2, \\
I_{13} &= \kappa_{2,21} + \frac{3\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2} &&= \kappa_{2,12} - \frac{\kappa_{2,1}\kappa_{2,2} - 2\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2}, \\
I_{04} &= \kappa_{2,22} + \frac{3\,\kappa_{2,1}^2}{\kappa_1 - \kappa_2} + 3\,\kappa_2^3.
\end{aligned}$$

★ The two expressions for I_{31} and I_{13} follow from the commutator formula.

Fourth order recurrence relations

$$I_{40} = \kappa_{1,11} - \frac{3\,\kappa_{1,2}^2}{\kappa_1 - \kappa_2} + 3\,\kappa_1^3,$$

$$I_{31} = \kappa_{1,12} - \frac{3\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2} = \kappa_{1,21} + \frac{\kappa_{1,1}\kappa_{1,2} - 2\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2},$$

$$I_{22} = \kappa_{1,22} + \frac{\kappa_{1,1}\kappa_{2,1} - 2\,\kappa_{2,1}^2}{\kappa_1 - \kappa_2} + \kappa_1\kappa_2^2 = \kappa_{2,11} - \frac{\kappa_{1,2}\kappa_{2,2} - 2\,\kappa_{1,2}^2}{\kappa_1 - \kappa_2} + \kappa_1^2\kappa_2,$$

$$I_{13} = \kappa_{2,21} + \frac{3\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2} = \kappa_{2,12} - \frac{\kappa_{2,1}\kappa_{2,2} - 2\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2},$$

$$I_{04} = \kappa_{2,22} + \frac{3\,\kappa_{2,1}^2}{\kappa_1 - \kappa_2} + 3\,\kappa_2^3.$$

★ ★ The two expressions for I_{22} imply the Codazzi syzygy

$$\kappa_{1,22} - \kappa_{2,11} + \frac{\kappa_{1,1}\,\kappa_{2,1} + \kappa_{1,2}\,\kappa_{2,2} - 2\,\kappa_{2,1}^2 - 2\,\kappa_{1,2}^2}{\kappa_1 - \kappa_2} - \kappa_1\,\kappa_2\,(\kappa_1 - \kappa_2) = 0,$$

which can be written compactly as

$$K = \kappa_1\kappa_2 = -\,(\mathcal{D}_1 + Y_1)\,Y_1 - (\mathcal{D}_2 + Y_2)\,Y_2.$$

$\Longrightarrow$ Gauss' Theorema Egregium

Generating Differential Invariants

♡ From the general structure of the recurrence relations, one proves that the Euclidean differential invariant algebra $\mathcal{I}_{\mathrm{SE}(3)}$ is generated by the principal curvatures κ_1, κ_2 or, equivalently, the mean and Gauss curvatures, H, K, through the process of invariant differentiation:

$$I = \Phi(H, K, \mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1 K, \mathcal{D}_2 K, \mathcal{D}_1^2 H, \ \ldots \ )$$

♢ Remarkably, for suitably generic surfaces, the Gauss curvature can be written as a universal rational function of the mean curvature and its invariant derivatives of order ≤ 4:

$$K = \Psi(H, \mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1^2 H, \ \ldots \ , \mathcal{D}_2^4 H)$$

and hence $\mathcal{I}_{\mathrm{SE}(3)}$ is generated by mean curvature alone!

♠ To prove this, given

$$K = \kappa_1 \kappa_2 = -(\mathcal{D}_1 + Y_1)\, Y_1 - (\mathcal{D}_2 + Y_2)\, Y_2$$

it suffices to write the commutator invariants Y_1, Y_2 in terms of H.

Equi-affine Surfaces

$$M = \mathbb{R}^3 \qquad G = \mathrm{SA}(3) = \mathrm{SL}(3) \ltimes \mathbb{R}^3 \qquad \dim G = 11.$$

$$g \cdot z = A\,z + b, \qquad \det A = 1, \qquad z = \begin{pmatrix} x \\ y \\ u \end{pmatrix} \in \mathbb{R}^3.$$

Surfaces $S \subset M = \mathbb{R}^3$:

$$u = f(x, y)$$

Hyperbolic case

$$u_{xx}u_{yy} - u_{xy}^2 < 0$$

Cross-section:

$$x = y = u = u_x = u_y = u_{xy} = 0, \qquad u_{xx} = 1, \qquad u_{yy} = -1,$$

$$u_{xyy} = u_{xxx}, \qquad u_{xxy} = u_{yyy} = 0.$$

Power series normal form:

$$u(x,y) = \tfrac{1}{2}(x^2 - y^2) + \tfrac{1}{6}c\,(x^3 + 3\,xy^2) + \cdots$$

$$\Longrightarrow \quad \textit{Nonsingular}: c \neq 0.$$

Invariantization — differential invariants: $\qquad I_{jk} = \iota(u_{jk})$

Phantom differential invariants:

$$\iota(x) = \iota(y) = \iota(u) = \iota(u_x) = \iota(u_y) = \iota(u_{xy}) = \iota(u_{xxy}) = \iota(u_{yyy}) = 0,$$

$$\iota(u_{xx}) = 1, \qquad \iota(u_{yy}) = -1, \qquad \iota(u_{xxx}) - \iota(u_{xyy}) = 0.$$

Pick invariant:

$$P = \iota(u_{xxx}) = \iota(u_{xyy}).$$

Basic differential invariants of order 4:

$$Q_0 = \iota(u_{xxxx}), \quad Q_1 = \iota(u_{xxxy}), \quad Q_2 = \iota(u_{xxyy}),$$

$$Q_3 = \iota(u_{xyyy}), \quad Q_4 = \iota(u_{yyyy}),$$

Invariant differential operators:

$$\mathcal{D}_1 = \iota(D_x), \qquad \mathcal{D}_2 = \iota(D_y).$$

- Since the moving frame has order 3, one can generate all higher order differential invariants from the basic differential invariants of order ≤ 4.

- This is a consequence of a general theorem, that follows directly from the recurrence formulae.

- Thus, to prove that the Pick invariant generates $\mathcal{I}(G)$, it suffices to generate $Q_0, \ldots, Q_4$ from P by invariant differentiation.

Infinitesimal generators:

$$\mathbf{v}_1 = x\,\partial_x - u\,\partial_u, \qquad \mathbf{v}_2 = y\,\partial_y - u\,\partial_u,$$

$$\mathbf{v}_3 = y\,\partial_x, \qquad \mathbf{v}_4 = u\,\partial_x, \qquad \mathbf{v}_5 = x\,\partial_y,$$

$$\mathbf{v}_6 = u\,\partial_y, \quad \mathbf{v}_7 = x\,\partial_u, \quad \mathbf{v}_8 = y\,\partial_u,$$

$$\mathbf{w}_1 = \partial_x, \qquad \mathbf{w}_2 = \partial_y, \qquad \mathbf{w}_3 = \partial_u,$$

- The translations will be ignored, as they play no role in the higher order recurrence formulae.

Recurrence formulae

$$\mathcal{D}_i \iota(u_{jk}) = \iota(D_i u_{jk}) + \sum_{\kappa=1}^{8} \varphi_\kappa^{jk}(x,y,u^{(j+k)}) R_i^\kappa, \qquad j+k \geq 1$$

$$\mathcal{D}_1 I_{jk} = I_{j+1,k} + \sum_{\kappa=1}^{8} \varphi_\kappa^{jk}(0,0,I^{(j+k)}) R_1^\kappa$$

$$\mathcal{D}_2 I_{jk} = I_{j,k+1} + \sum_{\kappa=1}^{8} \varphi_\kappa^{jk}(0,0,I^{(j+k)}) R_2^\kappa$$

$\varphi_\kappa^{jk}(0,0,I^{(j+k)}) = \iota[\,\varphi_\kappa^{jk}(x,y,u^{(j+k)})\,]$ — invariantized prolonged infinitesimal generator coefficients

R_i^κ — Maurer–Cartan invariants

Phantom recurrence formulae:

$$0 = \mathcal{D}_1 I_{10} = 1 + R_1^7, \qquad 0 = \mathcal{D}_2 I_{10} = R_2^7,$$

$$0 = \mathcal{D}_1 I_{01} = R_1^8, \qquad 0 = \mathcal{D}_2 I_{01} = -1 + R_2^8,$$

$$0 = \mathcal{D}_1 I_{20} = I_{30} - 3R_1^1 - R_1^2, \qquad 0 = \mathcal{D}_2 I_{20} = -3R_2^1 - R_2^2,$$

$$0 = \mathcal{D}_1 I_{11} = -R_1^3 + R_1^5, \qquad 0 = \mathcal{D}_2 I_{11} = I_{30} - R_2^3 + R_2^5,$$

$$0 = \mathcal{D}_1 I_{02} = I_{12} + R_1^1 + 3R_1^2, \qquad 0 = \mathcal{D}_2 I_{02} = R_2^1 + 3R_2^2,$$

$$0 = \mathcal{D}_1 I_{21} = I_{31} - I_{30}R_1^3 - 2I_{30}R_1^5 + R_1^6,$$

$$0 = \mathcal{D}_2 I_{21} = I_{22} - I_{30}R_2^3 - 2I_{30}R_2^5 + R_2^6,$$

$$0 = \mathcal{D}_1 I_{03} = I_{13} - 3I_{30}R_2^3 - 3R_2^6, \qquad 0 = \mathcal{D}_2 I_{03} = I_{04} - 3I_{30}R_2^3 - 3R_2^6.$$

Maurer–Cartan invariants:

$$R_1 = \left(\tfrac{1}{2}P, -\tfrac{1}{2}P, \frac{3Q_1+Q_3}{12P}, \tfrac{1}{4}Q_0 - \tfrac{1}{4}Q_2 - \tfrac{1}{2}P^2, \frac{3Q_1+Q_3}{12P}, -\tfrac{1}{4}Q_1 + \tfrac{1}{4}Q_3, -1, 0 \right)$$

$$R_2 = \left(0, 0, \frac{3Q_2+Q_4}{12P} + \tfrac{1}{2}P, \tfrac{1}{4}Q_1 - \tfrac{1}{4}Q_3, \frac{3Q_2+Q_4}{12P} - \tfrac{1}{2}P, -\tfrac{1}{4}Q_2 + \tfrac{1}{4}Q_4 - \tfrac{1}{2}P^2, 0, 1 \right)$$

Fourth order invariants:

$$P_1 = \mathcal{D}_1 P = \tfrac{1}{4}Q_0 + \tfrac{3}{4}Q_2, \qquad P_2 = \mathcal{D}_2 P = \tfrac{1}{4}Q_1 + \tfrac{3}{4}Q_3.$$

Commutator:

$$\mathcal{D}_3 = [\,\mathcal{D}_1, \mathcal{D}_2\,] = \mathcal{D}_1\mathcal{D}_2 - \mathcal{D}_2\mathcal{D}_1 = Y_1\mathcal{D}_1 + Y_2\mathcal{D}_2,$$

Commutator invariants:

$$Y_1 = R_2^1 - R_1^3 = -\frac{3Q_1+Q_3}{12P}, \qquad Y_2 = R_2^5 - R_1^2 = \frac{3Q_2+Q_4}{12P}.$$

Another fourth order invariant:

$$P_3 = \mathcal{D}_3 P = \mathcal{D}_1 \mathcal{D}_2 P - \mathcal{D}_2 \mathcal{D}_1 P = Y_1 P_1 + Y_2 P_2. \qquad (*)$$

Nondegeneracy condition: If

$$\det \begin{pmatrix} P_1 & P_2 \\ \mathcal{D}_1 P_j & \mathcal{D}_2 P_j \end{pmatrix} \neq 0 \qquad \text{for} \qquad j = 1, 2, \text{ or } 3,$$

we can solve $(*)$ and

$$\mathcal{D}_3 P_j = Y_1 \mathcal{D}_1 P_j + Y_2 \mathcal{D}_2 P_j$$

for the fourth order commutator invariants:

$$Y_1 = -\frac{3\,Q_1 + Q_3}{12\,P}, \qquad Y_2 = \frac{3\,Q_2 + Q_4}{12\,P}.$$

So far, we have constructed four combinations of the fourth order differential invariants

$$S_1 = Q_0 + 3\,Q_2, \qquad S_2 = Q_1 + 3\,Q_3,$$

$$S_3 = 3\,Q_1 + Q_3, \qquad S_4 = 3\,Q_2 + Q_4.$$

as rational functions of the invariant derivatives of the Pick invariant. To obtain the final fourth order differential invariant:

$$12\,P(\mathcal{D}_1 S_4 - \mathcal{D}_2 S_3) = 48\,P^2 Q_0 - 30\,P^2 S_1 + 18\,P^2 S_4 - 3\,S_2 S_3 - S_3^2 + 3\,S_1 S_4 + S_4^2.$$

★ ★ ★ This completes the proof ★ ★ ★

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★ Structure theory for differential invariant algebras?

In particular, minimal generating sets require a syzygy bound:

$$\boxed{K = \Psi(H, \ \ldots \ , \mathcal{D}^{(n)} H) \qquad n \leq N \ ???}$$

THANK YOU!