Differential Invariant Algebras

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Basic Framework

M — m-dimensional manifold

$\mathrm{J}^n = \mathrm{J}^n(M, p)$ — n^{th} order jet space for p-dimensional submanifolds $N \subset M$

G — transformation group acting on M

$G^{(n)}$ — prolonged action on the submanifold jet space J^n

Differential Invariants

Differential invariant $\qquad I : \mathrm{J}^n \to \mathbb{R}$

$$I(g^{(n)} \cdot (x, u^{(n)})) = I(x, u^{(n)})$$

$\Longrightarrow$ curvature, torsion, ...

Invariant differential operators:

$$\mathcal{D}_1, \ldots, \mathcal{D}_p$$

$\Longrightarrow$ arc length derivative

$\mathcal{I}(G)$ — the algebra of differential invariants

The Basis Theorem

Theorem. The differential invariant algebra $\mathcal{I}(G)$ is generated by a finite number of differential invariants

$$I_1, \ \ldots \ , I_\ell$$

and $p = \dim N$ invariant differential operators

$$\mathcal{D}_1, \ \ldots \ , \mathcal{D}_p$$

meaning that *every* differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives:

$$\mathcal{D}_J I_\kappa = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_n} I_\kappa.$$

$\Longrightarrow$ *Lie, Tresse, Ovsiannikov, Kumpera*

Key Issues

- Minimal basis of generating invariants: $I_1, \ldots, I_\ell$
- Commutation formulae for

  the invariant differential operators:

$$[\mathcal{D}_j, \mathcal{D}_k] = \sum_{i=1}^{p} Y_{j,k}^i \, \mathcal{D}_i$$

$\Longrightarrow$ Non-commutative differential algebra

- Syzygies (functional relations) among

  the differentiated invariants:

$$\Phi(\ \ldots \ \mathcal{D}_J I_\kappa \ \ldots \ ) \equiv 0$$

$\Longrightarrow$ Gauss–Codazzi relations

Applications

- Equivalence and signatures of submanifolds
- Characterization of moduli spaces
- Invariant differential equations:

$$H(\ \ldots\ \mathcal{D}_J I_\kappa\ \ldots\ ) = 0$$

- Group splitting of PDEs and explicit solutions
- Invariant variational problems:

$$\int L(\ \ldots\ \mathcal{D}_J I_\kappa\ \ldots\ )\,\boldsymbol{\omega}$$

The Infinite Jet Bundle

Jet bundles

$$M = \mathrm{J}^0 \longleftarrow \mathrm{J}^1 \longleftarrow \mathrm{J}^2 \longleftarrow \cdots$$

Inverse limit

$$\mathrm{J}^\infty = \lim_{n \to \infty} \mathrm{J}^n$$

Local coordinates

$$z^{(\infty)} = (x, u^{(\infty)}) = (\;\ldots\; x^i \;\ldots\; u^\alpha_J \;\ldots\;)$$

Differential Forms

Coframe — basis for the cotangent space $T^*\mathrm{J}^\infty$:

- Horizontal one-forms

$$dx^1, \ldots, dx^p$$

- Contact (vertical) one-forms

$$\theta_J^\alpha = du_J^\alpha - \sum_{i=1}^p u_{J,i}^\alpha \, dx^i$$

Intrinsic definition of contact form

$$\theta \mid \mathrm{j}_\infty N = 0 \qquad \Longleftrightarrow \qquad \theta = \sum A_J^\alpha \, \theta_J^\alpha$$

The Variational Bicomplex

$\Longrightarrow$ *Vinogradov, Tsujishita, I. Anderson*

Bigrading of the differential forms on J^∞:

$$\Omega^* = \bigoplus_{r,s} \Omega^{r,s}$$

$r = \#$ horizontal forms

$s = \#$ contact forms

The Differentials

Horizontal and vertical differentials: $d = d_H + d_V$

$$d_H : \Omega^{r,s} \longrightarrow \Omega^{r+1,s}$$

$$d_H F = \sum_{i=1}^{p} (D_i F)\, dx^i \qquad \text{— total derivatives}$$

$$d_V : \Omega^{r,s} \longrightarrow \Omega^{r,s+1}$$

$$d_V F = \sum_{\alpha,J} \frac{\partial F}{\partial u_J^\alpha}\, \theta_J^\alpha \qquad \text{— variation}$$

The Simplest Example

$$(x, u) \in M = \mathbb{R}^2$$

x — independent variable

u — dependent variable

Horizontal form

$$dx$$

Contact (vertical) forms

$$\theta = du - u_x \, dx$$

$$\theta_x = du_x - u_{xx} \, dx$$

$$\theta_{xx} = du_{xx} - u_{xxx} \, dx$$

$$\vdots$$

$$\theta = du - u_x\,dx, \qquad \theta_x = du_x - u_{xx}\,dx, \qquad \theta_{xx} = du_{xx} - u_{xxx}\,dx$$

Differential:

$$\begin{aligned} dF &= \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial u}\,du + \frac{\partial F}{\partial u_x}\,du_x + \frac{\partial F}{\partial u_{xx}}\,du_{xx} + \cdots \\ &= (D_x F)\,dx + \frac{\partial F}{\partial u}\theta + \frac{\partial F}{\partial u_x}\theta_x + \frac{\partial F}{\partial u_{xx}}\theta_{xx} + \cdots \\ &= d_H F + d_V F \end{aligned}$$

Total derivative:

$$D_x F = \frac{\partial F}{\partial u}\,u_x + \frac{\partial F}{\partial u_x}\,u_{xx} + \frac{\partial F}{\partial u_{xx}}\,u_{xxx} + \cdots$$

The Variational Bicomplex

$$
\begin{array}{cccccccccccc}
 & \vdots & & \vdots & & & & \vdots & & \vdots & & \vdots \\
 & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle \delta} \\
 & \Omega^{0,3} & \xrightarrow{d_H} & \Omega^{1,3} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,3} & \xrightarrow{d_H} & \Omega^{p,3} & \xrightarrow{\pi} & \mathcal{F}^3 \\
 & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle \delta} \\
 & \Omega^{0,2} & \xrightarrow{d_H} & \Omega^{1,2} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,2} & \xrightarrow{d_H} & \Omega^{p,2} & \xrightarrow{\pi} & \mathcal{F}^2 \\
 & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle \delta} \\
 & \Omega^{0,1} & \xrightarrow{d_H} & \Omega^{1,1} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,1} & \xrightarrow{d_H} & \Omega^{p,1} & \xrightarrow{\pi} & \mathcal{F}^1 \\
 & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & \nearrow{\scriptstyle \delta} & \\
 & \Omega^{0,0} & \xrightarrow{d_H} & \Omega^{1,0} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,0} & \xrightarrow{d_H} & \Omega^{p,0} & & \\
\nearrow & & & & & & & & & & & \\
\mathbb{R} & & & & & & & & & & &
\end{array}
$$

Moving Frames

Definition. An n^{th} order *moving frame* is a G-equivariant map

$$\rho^{(n)} : V^n \subset \mathrm{J}^n \longrightarrow G$$

Equivariance:

$$\rho(g^{(n)} \cdot z^{(n)}) = \begin{cases} g \cdot \rho(z^{(n)}) & \text{left moving frame} \\ \rho(z^{(n)}) \cdot g^{-1} & \text{right moving frame} \end{cases}$$

Note $\qquad \rho_{left}(z^{(n)}) = \rho_{right}(z^{(n)})^{-1}$

Theorem. A moving frame exists in a neighborhood of a point $z^{(n)} \in \mathrm{J}^n$ if and only if G acts freely and regularly near $z^{(n)}$.

Theorem. If G acts locally effectively on subsets, then for $n \gg 0$, the (prolonged) action of G is locally free on an open subset of J^n.

$\Longrightarrow$ *Ovsiannikov, PJO*

The Normalization Construction

1. Write out the explicit formulas for the prolonged group action:

$$w^{(n)}(g, z^{(n)}) = g^{(n)} \cdot z^{(n)}$$

$\Longrightarrow$ *Implicit differentiation*

2. From the components of $w^{(n)}$, choose $r = \dim G$ *normalization equations*:

$$w_1(g, z^{(n)}) = c_1 \qquad \ldots \qquad w_r(g, z^{(n)}) = c_r$$

3. Solve the normalization equations for the group parameters $g = (g_1, \ldots, g_r)$:

$$g = \rho(z^{(n)}) = \rho(x, u^{(n)})$$

The solution is the right moving frame.

4. Substitute the moving frame formulas

$$g = \rho(z^{(n)}) = \rho(x, u^{(n)})$$

for the group parameters into the un-normalized components of $w^{(n)}$ to produce a complete system of functionally independent differential invariants of order $\leq n$:

$$I_k(x, u^{(n)}) = w_k(\rho(z^{(n)}), z^{(n)})), \qquad k = r+1, \ \ldots \ , \dim \mathrm{J}^n$$

Invariantization

The process of replacing group parameters in transformation rules by their moving frame formulae is known as invariantization:

$$\iota: \begin{cases} \text{Functions} & \longrightarrow & \text{Invariants} \\ \text{Forms} & \longrightarrow & \text{Invariant Forms} \\ \begin{array}{c}\text{Differential}\\ \text{Operators}\end{array} & \longrightarrow & \begin{array}{c}\text{Invariant Differential}\\ \text{Operators}\end{array} \\ \vdots & & \vdots \end{cases}$$

- Invariantization defines an exterior algebra morphism.
- Analysis of invariant objects relies on the invariantized bicomplex defined by the moving frame.

The Fundamental Differential Invariants

$$I^{(n)}(x, u^{(n)}) = \iota(x, u^{(n)}) = \rho(x, u^{(n)}) \cdot (x, u^{(n)})$$

$$\boxed{H^i(x, u^{(n)}) = \iota(x^i) \qquad I^\alpha_K(x, u^{(l)}) = \iota(u^\alpha_K)}$$

The constant differential invariants, as dictated by the moving frame normalizations, are known as the *phantom invariants*. The remaining non-constant differential invariants are the *basic invariants* and form a complete system of functionally independent differential invariants for the prolonged group action.

Invariantization of general differential functions:

$$\iota\,[\,F(\;\ldots\;x^i\;\ldots\;u_J^\alpha\;\ldots\;)\,] = F(\;\ldots\;H^i\;\ldots\;I_J^\alpha\;\ldots\;)$$

The Replacement Theorem:

If J is a differential invariant, then $\iota(J) = J$.

$$J(\;\ldots\;x^i\;\ldots\;u_J^\alpha\;\ldots\;) = J(\;\ldots\;H^i\;\ldots\;I_J^\alpha\;\ldots\;)$$

The Double Fibration

$$\mathcal{B}^{(n)} = \mathrm{J}^n \times G$$

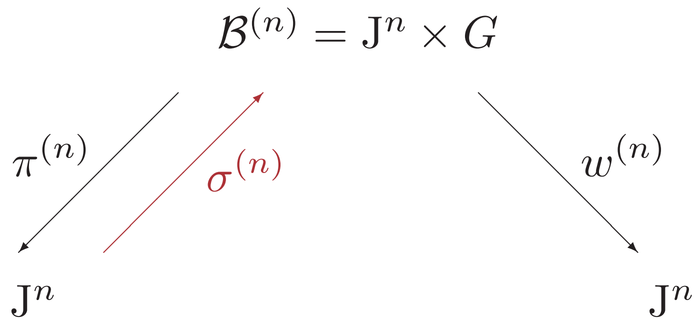
$$\begin{array}{ccc} & \mathcal{B}^{(n)} & \\ {}^{\pi^{(n)}}\swarrow & & \searrow^{w^{(n)}} \\ \mathrm{J}^n & & \mathrm{J}^n \end{array}$$

Source and target maps:

$$\pi^{(n)}(z^{(n)}, g) = z^{(n)}, \qquad w^{(n)}(z^{(n)}, g) = g^{(n)} \cdot z^{(n)}$$

$\Longrightarrow$ groupoid

The Moving Frame Section



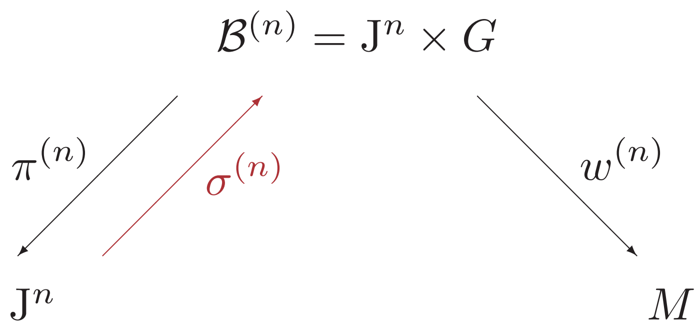
Source and target maps:

$$\pi^{(n)}(z^{(n)}, g) = z^{(n)}, \qquad w^{(n)}(z^{(n)}, g) = g^{(n)} \cdot z^{(n)}$$

Moving frame section:

$$\sigma^{(n)}(z^{(n)}) = (z^{(n)}, \rho^{(n)}(z^{(n)}))$$

Invariantization of Differential Forms



Invariantization of a differential form Ω on J^n:

$$\iota(\Omega) = (\sigma^{(n)})^*[\,\pi_J((w^{(n)})^*\,\Omega)\,]$$

- π_J projects differential forms on $T^*\mathcal{B}^{(n)} = T^*\mathrm{J}^n \times T^*G$ to their *jet components*, i.e., π_J annihilates all the (lifted) Maurer–Cartan forms:

$$\pi_J: \ \mu^\kappa \longmapsto 0, \qquad \kappa = 1, \ldots, r = \dim G.$$

Differential Invariants and Forms

- Fundamental differential invariants

$$H^i(x, u^{(n)}) = \iota(x^i) \qquad I^\alpha_K(x, u^{(l)}) = \iota(u^\alpha_K)$$

- Invariant horizontal forms

$$\varpi^i = \iota(dx^i)$$

- Invariant contact forms

$$\vartheta^\alpha_J = \iota(\theta^\alpha_J)$$

The Invariant "Quasi–Tricomplex"

Differential forms

$$\Omega^* = \bigoplus_{r,s} \widehat{\Omega}^{r,s}$$

Differential

$$d = d_{\mathcal{H}} + d_{\mathcal{V}} + d_{\mathcal{W}}$$

$$d_{\mathcal{H}} : \quad \widehat{\Omega}^{r,s} \longrightarrow \widehat{\Omega}^{r+1,s}$$

$$d_{\mathcal{V}} : \quad \widehat{\Omega}^{r,s} \longrightarrow \widehat{\Omega}^{r,s+1}$$

$$d_{\mathcal{W}} : \quad \widehat{\Omega}^{r,s} \longrightarrow \widehat{\Omega}^{r-1,s+2}$$

$$\iota(\Omega) = (\sigma^{(n)})^* [\, \pi_J((w^{(n)})^* \, \Omega) \,].$$

$\Longrightarrow$ Due to π_J, in general differentiation and invariantization do not commute:

$$d\iota(\Omega) \neq \iota(d\Omega)$$

The Universal Recurrence Formula

$$d\,\iota(\Omega) = \iota(d\Omega) + \sum_{\kappa=1}^{r} \gamma^\kappa \wedge \, \iota[\mathbf{v}_\kappa(\Omega)]$$

$\mathbf{v}_1, \ldots, \mathbf{v}_r$ — basis for (prolonged) infinitesimal generators

$\gamma^\kappa = (\sigma^{(n)})^*(\mu^\kappa)$ — invariantized dual Maurer–Cartan forms

The Universal Recurrence Formula

$$d\,\iota(\Omega) = \iota(d\Omega) + \sum_{\kappa=1}^{r} \gamma^{\kappa} \wedge \iota[\mathbf{v}_{\kappa}(\Omega)]$$

- The invariantized Maurer–Cartan forms γ^{κ} are uniquely determined by the recurrence formulae for the Maurer–Cartan forms!

- All identities, commutation formulae, etc., in the invariant variational bicomplex are obtained as consequences of this universal formula — by specializing Ω to the various basic functions and differential forms!

Euclidean Curves

Prolonged group action (implicit differentiation):

$$y = w^*(x) = x\cos\phi - u\sin\phi + a$$

$$v = w^*(u) = x\sin\phi + u\cos\phi + b$$

$$v_y = w^*(u_x) = \frac{\sin\phi + u_x\cos\phi}{\cos\phi - u_x\sin\phi}$$

$$v_{yy} = w^*(u_{xx}) = \frac{u_{xx}}{(\cos\phi - u_x\sin\phi)^3}$$

$$v_{yyy} = w^*(u_{xxx}) = \frac{(\cos\phi\ - u_x\sin\phi\ )u_{xxx} - 3u_{xx}^2\sin\phi}{(\cos\phi\ - u_x\sin\phi\ )^5}$$

Normalization:

$$y = w^*(x) = x\cos\phi - u\sin\phi + a = 0$$

$$v = w^*(u) = x\cos\phi + u\sin\phi + b = 0$$

$$v_y = w^*(u_x) = \frac{\sin\phi + u_x\cos\phi}{\cos\phi - u_x\sin\phi} = 0$$

$$v_{yy} = w^*(u_{xx}) = \frac{u_{xx}}{(\cos\phi - u_x\sin\phi)^3}$$

Right moving frame $\qquad \rho : \mathrm{J}^1 \longrightarrow \mathrm{SE}(2)$

$$\phi = -\tan^{-1} u_x \qquad a = -\frac{x + u u_x}{\sqrt{1+u_x^2}} \qquad b = \frac{x u_x - u}{\sqrt{1+u_x^2}}$$

Fundamental normalized differential invariants

$$\left.\begin{aligned} \iota(x) &= H = 0 \\ \iota(u) &= I_0 = 0 \\ \iota(u_x) &= I_1 = 0 \end{aligned}\right\} \quad \text{phantom diff. invs.}$$

$$\left.\begin{aligned} \iota(u_{xx}) &= I_2 = \kappa \\ \iota(u_{xxx}) &= I_3 = \kappa_s \\ \iota(u_{xxxx}) &= I_4 = \kappa_{ss} + 3\,\kappa^3 \\ &\;\;\vdots \end{aligned}\right\} \quad \text{basic diff. invs.}$$

$$dy = w^*(dx) = (\cos\phi - u_x \sin\phi)\, dx - (\sin\phi)\,\theta + da - v\, d\phi$$

$$\pi_J(dy) = (\cos\phi - u_x \sin\phi)\, dx - (\sin\phi)\,\theta$$

$$\phi = -\tan^{-1} u_x$$

Invariant horizontal one-form:

$$\iota(dx) = \sigma^*(\pi_J(dy)) = \varpi \quad = \quad \omega \quad + \quad \eta$$

$$= \sqrt{1+u_x^2}\, dx \quad + \quad \frac{u_x}{\sqrt{1+u_x^2}}\,\theta$$

Dual invariant differential operator:

$$\mathcal{D} = \iota(D_x) = \sigma^*(D_y) = \sigma^*\left(\frac{1}{\cos\phi - u_x \sin\phi}\, D_x\right) = \frac{1}{\sqrt{1+u_x^2}}\, D_x = D_s$$

$$\theta = du - u_x\, dx$$

$$\pi_J(w^*(\theta)) = \pi_J(dv - v_y\, dy) = (\cos\phi - v_y \sin\phi)\,\theta = \frac{\theta}{\cos\phi - u_x \sin\phi}$$

$$\pi_J(w^*(\theta_x)) = \pi_J(dv_y - v_{yy}\, dy) = \frac{\theta_x}{(\cos\phi - u_x \sin\phi)^2} + \frac{(u_{xx}\sin\phi)\,\theta}{(\cos\phi - u_x \sin\phi)^3}$$

$$\phi = -\tan^{-1} u_x$$

Invariant contact forms:

$$\iota(\theta) = \vartheta = \frac{\theta}{\sqrt{1+u_x^2}} \qquad \iota(\theta_x) = \vartheta_1 = \frac{(1+u_x^2)\,\theta_x - u_x u_{xx}\theta}{(1+u_x^2)^2}$$

Prolonged infinitesimal generators

$$\mathbf{v}_1 = \partial_x \qquad \mathbf{v}_2 = \partial_u$$

$$\mathbf{v}_3 = -\,u\,\partial_x + x\,\partial_u + (1 + u_x^2)\,\partial_{u_x} + 3\,u_x u_{xx}\,\partial_{u_{xx}} + \cdots$$

Recurrence formula

$$d\iota(F) = \iota(dF) + \iota(\mathbf{v}_1(F))\,\gamma^1 + \iota(\mathbf{v}_2(F))\,\gamma^2 + \iota(\mathbf{v}_3(F))\,\gamma^3$$

Use phantom invariants

$$\begin{aligned} 0 = dH &= \iota(dx) + \sum\, \iota(\mathbf{v}_\kappa(x))\, \gamma^\kappa = \varpi + \gamma^1, \\ 0 = dI_0 &= \iota(du) + \sum\, \iota(\mathbf{v}_\kappa(u))\, \gamma^\kappa \\ &= \iota(u_x\, dx + \theta) + \sum\, \iota(\mathbf{v}_\kappa(u))\, \gamma^\kappa = \vartheta + \gamma^2, \\ 0 = dI_1 &= \iota(du_x) + \sum\, \iota(\mathbf{v}_\kappa(u_x))\, \gamma^\kappa \\ &= \iota(u_{xx}\, dx + \theta_x) + \sum\, \iota(\mathbf{v}_\kappa(u))\, \gamma^\kappa = \kappa\, \varpi + \vartheta_1 + \gamma^3, \end{aligned}$$

to solve for

$$\gamma^1 = -\varpi, \qquad \gamma^2 = -\,\vartheta, \qquad \gamma^3 = -\kappa\, \varpi - \vartheta_1.$$

$$\gamma^1 = -\varpi, \qquad \gamma^2 = -\vartheta, \qquad \gamma^3 = -\kappa\,\varpi - \vartheta_1.$$

Recurrence formulae

$$\begin{aligned}
\kappa_s\varpi + d_{\mathcal{V}}\,\kappa &= d(I_2) = \iota(du_{xx}) + \iota(\mathbf{v}_3(u_{xx}))\,\gamma^3 \\
&= \iota(u_{xxx}\,dx + \theta_{xx}) - \iota(3u_x u_{xx}))\,\kappa\,\varpi \\
&= I_3\,\varpi + \vartheta_2 \\
(\mathcal{D}_s I_3)\,\varpi + d_{\mathcal{V}}\,I_3 &= d_{\mathcal{H}}\,(I_3) = \iota(d_H u_{xxx}) + \iota(\mathbf{v}_3(u_{xxx})\,\gamma^3 \\
&= \iota(u_{xxxx}\,dx + \theta_{xxx}) - \iota(4u_x u_{xxx} + 3u_{xx}^2)\,\kappa\,\varpi \\
&= (I_4 - 3\,\kappa^3)\,\varpi + \vartheta_3
\end{aligned}$$

$$d\varpi = d\iota(dx) = \iota(d^2x) + \sum_{\kappa} \gamma^{\kappa} \wedge \iota(\mathbf{v}_{\kappa}(dx))$$

$$= \gamma^3 \wedge \iota(-\,du) = -\,\kappa\,\vartheta \wedge \varpi + \vartheta_1 \wedge \vartheta,$$

and so

$$d_{\mathcal{H}}\,\varpi = 0, \qquad d_{\mathcal{V}}\,\varpi = -\,\kappa\,\vartheta \wedge \varpi, \qquad d_{\mathcal{W}}\,\varpi = \vartheta_1 \wedge \vartheta.$$

$$d\vartheta = d\iota(\theta) = \iota(d\theta) + \sum_{\kappa} \gamma^{\kappa} \wedge \iota(\mathbf{v}_{\kappa}(\theta))$$

$$= \iota(-\,\theta_x \wedge dx) + \gamma^3 \wedge \iota(u_x\theta) = -\,\vartheta_1 \wedge \varpi,$$

and so

$$d_{\mathcal{H}}\,\vartheta = -\,\vartheta_1 \wedge \varpi, \qquad d_{\mathcal{V}}\,\vartheta = d_{\mathcal{W}}\,\vartheta = 0,$$

etc.

Key Recurrence Formulae

$$\kappa = I_2 \qquad I_2 = \kappa$$

$$\kappa_s = I_3 \qquad I_3 = \kappa_s$$

$$\kappa_{ss} = I_4 - 3I_2^3 \qquad I_4 = \kappa_{ss} + 3\kappa^3$$

$$\kappa_{sss} = I_5 - 19I_2^2 I_3 \qquad I_5 = \kappa_{sss} + 19\kappa^2\kappa_s$$

$$d_{\mathcal{V}}\,\kappa = (\mathcal{D}^2 + \kappa^2)\,\vartheta,$$

$$d_{\mathcal{V}}\,\kappa_s = (\mathcal{D}^3 + \kappa^2\mathcal{D} + 3\,\kappa\,\kappa_s)\vartheta,$$

$$d_{\mathcal{V}}\,\kappa_{ss} = (\mathcal{D}^4 + \kappa^2\mathcal{D}^2 + 5\,\kappa\,\kappa_s\mathcal{D} + 4\,\kappa\,\kappa_{ss} + 3\,\kappa_s^2)\vartheta.$$

$$\mathcal{D}\vartheta = \vartheta_1 \qquad \mathcal{D}\vartheta_1 = \vartheta_2 - \kappa^2\,\vartheta \qquad \mathcal{D}\vartheta_2 = \vartheta_3 - 3\kappa^2\,\vartheta_1 - \kappa\kappa_s\,\vartheta$$

$$d_{\mathcal{V}}\,\varpi = -\,\kappa\,\vartheta \wedge \varpi$$

Invariantized infinitesimal generator coefficients:

$$\eta^i_\kappa(H, I) = \iota[\,\xi^i_\kappa(x, u))\,],$$

$$\psi^\alpha_\kappa(H, I) = \iota[\,\varphi^\alpha_\kappa(x, u)\,], \qquad\qquad \kappa = 1, \ldots, r$$

$$\psi^\alpha_{J,\kappa}(H, I^{(k)}) = \iota[\,\varphi^\alpha_{J,\kappa}(x, u^{(k)}))\,],$$

Assemble into vectors:

$$\boldsymbol{\eta}^j(H, I) = (\,\eta^i_r(H, I), \ \ldots\ , \eta^i_r(H, I)\,)$$

$$\boldsymbol{\psi}^\alpha_J(H, I^{(k)}) = (\,\psi^\alpha_{J,1}(H, I^{(k)}), \ \ldots\ , \psi^\alpha_{J,r}(H, I^{(k)})\,)$$

$\Longrightarrow$ Rows of invariantized Lie matrix.

Recurrence Formulae for Differential Invariants

$$\mathcal{D}_i H^j = \delta_i^j + \boldsymbol{\eta}^j(H,I)\,\mathbf{R}_i \qquad \mathcal{D}_i I_J^\alpha = I_{J,i}^\alpha + \boldsymbol{\psi}_J^\alpha(H,I^{(k)})\,\mathbf{R}_i$$

Maurer–Cartan invariants:

$$\mathbf{R}_i = (R_i^1,\ldots,R_i^r)^T, \qquad i = 1,\ldots,p$$

Pulled-back Maurer–Cartan forms:

$$\gamma^\kappa = (\rho^{(n)})^*(\mu^\kappa) = \sum_{i=1}^{p} R_i^\kappa\,\omega^i + \cdots$$

The *phantom recurrence relations*

$$0 = \mathcal{D}_i H^l = \delta_i^l + \boldsymbol{\eta}^l\,\mathbf{R}_i \qquad 0 = \mathcal{D}_i I_K^\beta = I_{K,i}^\beta + \boldsymbol{\psi}_K^\beta\,\mathbf{R}_i$$

can be uniquely solved for the Maurer–Cartan invariants.

Indices

$$\mathcal{T} = \{ \ \ldots \ i \ \ldots \ (J;\alpha) \, \}$$

— jet variable indices $(\ \ldots \ x^i \ \ldots \ u^\alpha_J \ \ldots \ )$

$\mathcal{P} \subset \mathcal{T}$ — phantom indices: $\quad |\, \mathcal{P} \,| = r = \dim G.$

Coordinate cross-section:

$$x^i = c^i, \qquad u^\beta_K = c^\beta_K, \qquad \text{for all} \qquad i, (K;\beta) \in \mathcal{P},$$

Minimal cross-section:

$$|\, \mathcal{P}^{(k)} \,| = r_k = \text{max orbit dim of } G^{(k)}$$

$\mathcal{B} = \mathcal{T} \setminus \mathcal{P}$ — basic indices

$\mathcal{E} = \mathcal{B}^{(0)} \cup \{ \, (J, i; \alpha) \in \mathcal{B} \,|\, (J;\alpha) \in \mathcal{P} \, \}$ — edge indices.

Minimal Order Moving Frames

Definition. A cross-section $K^n \subset \mathrm{J}^n$, and, hence its induced moving frame $\rho^{(n)}\colon \mathrm{J}^n \longrightarrow G$, is said to be of *minimal order* if, for each $0 \le k \le n$, its projection $K^k = \pi_k^n(K^n) \subset \mathrm{J}^k$ forms a cross-section to the orbits of $G^{(k)}$ on J^k.

Theorem. Given a moving frame of order n, the normalized differential invariants corresponding to indices in $\mathcal{B}^{(n)} \cup \mathcal{E}$ form a generating system.

Theorem. The edge differential invariants arising from a minimal order moving frame form a generating system of differential invariants.

Euclidean Space Curves

$M = \mathbb{R}^3$ — coordinates: $z = (x, u, v)$

$C = \{\, u = u(x),\ v = v(x)\,\}$ — curve

$G = \mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$

Infinitesimal generators:

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_u, \qquad \mathbf{v}_3 = \partial_v,$$

$$\mathbf{v}_4 = v\partial_u - u\partial_v, \qquad \mathbf{v}_5 = -u\partial_x + x\partial_u, \qquad \mathbf{v}_6 = -v\partial_x + x\partial_v.$$

The classical moving frame cross-section:

$$x = 0, \quad u = 0, \quad v = 0, \quad u_x = 0, \quad v_x = 0, \quad v_{xx} = 0,$$

Phantom indices:

$$\mathcal{P} = \{ 1, (0;1), (0;2), (1;1), (1;2), (2;2) \},$$

Basic indices

$$\mathcal{B} = \{ (k\,;1), (l\,;2) \quad \text{for all} \quad k \geq 2, \; l \geq 3 \}$$

Edge indices:

$$\mathcal{E} = \{ (2;1), (3;2) \} \quad \longleftrightarrow \quad u_{xx}, v_{xxx}$$

Phantom invariants

$$H = \iota(x) = 0, \qquad I_0 = \iota(u) = 0, \qquad J_0 = \iota(v) = 0,$$

$$I_1 = \iota(u_x) = 0, \qquad J_1 = \iota(v_x) = 0, \qquad J_2 = \iota(v_{xx}) = 0,$$

Edge invariants:

$$I_2 = \iota(u_{xx}) = \kappa, \qquad J_3 = \iota(v_{xxx}) = \kappa\,\tau$$

Non-edge basic invariants:

$$I_3 = \iota(u_{xxx}) = \kappa_s,$$

$$I_4 = \iota(u_{xxxx}) = \kappa_{ss} + 3\,\kappa^3 - \kappa\,\tau^2,$$

$$J_4 = \iota(v_{xxxx}) = 2\,\kappa_s\tau + \kappa\,\tau_s,$$

$$\vdots$$

Recurrence formulae:

$$\mathcal{D}H = 1 + R_1,$$

$$\mathcal{D}I_0 = I_1 + R_2 = R_2, \qquad \mathcal{D}J_0 = J_1 + R_3 = R_3,$$

$$\mathcal{D}I_1 = I_2 + R_5, \qquad \mathcal{D}J_1 = J_2 + R_6 = R_6,$$

$$\mathcal{D}I_2 = I_3, \qquad \mathcal{D}J_2 = J_3 - I_2 R_4,$$

$$\mathcal{D}I_3 = I_4 + J_3 R_4 + 3\, I_2^2 R_5, \qquad \mathcal{D}J_3 = J_4 - I_3 R_4,$$

$$\mathcal{D}I_4 = I_5 + J_4 R_4 + 10\, I_2 I_3 R_5 + 4\, I_2 J_3 R_6, \quad \mathcal{D}J_4 = J_5 - I_4 R_4 + 6\, I_2 J_3 R_5,$$

Maurer–Cartan invariants:

$$R_1 = -1, \quad R_2 = 0, \quad R_3 = 0, \quad R_4 = J_3 / I_2, \quad R_5 = -I_2, \quad R_6 = 0.$$

Normalizations:

$$0 = \mathcal{D}H = 1 + R_1,$$

$$0 = \mathcal{D}I_0 = I_1 + R_2 = R_2, \qquad 0 = \mathcal{D}J_0 = J_1 + R_3 = R_3,$$

$$0 = \mathcal{D}I_1 = I_2 + R_5, \qquad 0 = \mathcal{D}J_1 = J_2 + R_6 = R_6,$$

$$\mathcal{D}I_2 = I_3, \qquad 0 = \mathcal{D}J_2 = J_3 - I_2 R_4,$$

$$\mathcal{D}I_3 = I_4 + J_3 R_4 + 3\, I_2^2 R_5, \qquad \mathcal{D}J_3 = J_4 - I_3 R_4,$$

$$\mathcal{D}I_4 = I_5 + J_4 R_4 + 10\, I_2 I_3 R_5 + 4\, I_2 J_3 R_6, \quad \mathcal{D}J_4 = J_5 - I_4 R_4 + 6\, I_2 J_3 R_5,$$

Solve for Maurer–Cartan invariants:

$$R_1 = -1, \quad R_2 = 0, \quad R_3 = 0, \quad R_4 = J_3 / I_2, \quad R_5 = -I_2, \quad R_6 = 0.$$

Recurrence relations

$$\mathcal{D}I_2 = I_3, \qquad \mathcal{D}I_3 = I_4 - 3\,I_2^3 + J_3^2/I_2, \qquad \mathcal{D}J_3 = J_4 - I_3J_3/I_2,$$

$$\mathcal{D}I_4 = I_5 - 10\,I_2^2I_3 + J_3J_4/I_2, \qquad \mathcal{D}J_4 = J_5 - 6\,I_2^2J_3 + J_3I_4/I_2,$$

Non-traditional cross-section:

$$x = 0, \quad u = 0, \quad v = 0, \quad v_x = 0, \quad v_{xx} = 0, \quad v_{xxx} = 1,$$

Phantom indices:

$$\mathcal{P} = \{\, 1, (0;1), (0;2), (1;2), (2;2), (3;2) \,\};$$

Basic indices:

$$\mathcal{B} = \{\, (k\;;1), (l\;;2) \;\; \text{for all} \;\; k \geq 1, \;\; l \geq 4 \,\};$$

Edge indices:

$$\mathcal{E} = \{\, (1;1), (4;2) \,\} \quad \longleftrightarrow \quad u_x, v_{xxxx}.$$

Phantom invariants:

$$\widetilde{H} = \tilde{\iota}\,(x) = 0, \qquad \tilde{I}_0 = \tilde{\iota}\,(u) = 0, \qquad \tilde{J}_0 = \tilde{\iota}\,(v) = 0,$$

$$\tilde{J}_1 = \tilde{\iota}\,(v_x) = 0, \qquad \tilde{J}_2 = \tilde{\iota}\,(v_{xx}) = 0, \qquad \tilde{J}_3 = \tilde{\iota}\,(v_{xxx}) = 1,$$

Basic differential invariants:

$$\tilde{I}_1 = \tilde{\iota}\,(u_x), \qquad \tilde{I}_2 = \tilde{\iota}\,(u_{xx}), \qquad \tilde{I}_3 = \tilde{\iota}\,(u_{xxx}), \qquad \tilde{I}_4 = \tilde{\iota}\,(u_{xxxx}),$$

$$\tilde{J}_4 = \tilde{\iota}\,(v_{xxxx}), \qquad \ldots,$$

Phantom recurrence formulae:

$$0 = \widetilde{\mathcal{D}} H = 1 + \widetilde{R}_1,$$

$$0 = \widetilde{\mathcal{D}} \widetilde{I}_0 = \widetilde{I}_1 + \widetilde{R}_2,$$

$$0 = \widetilde{\mathcal{D}} \widetilde{J}_0 = \widetilde{J}_1 + \widetilde{R}_3 = \widetilde{R}_3,$$

$$0 = \widetilde{\mathcal{D}} \widetilde{J}_1 = \widetilde{J}_2 - \widetilde{I}_1 \widetilde{R}_4 + \widetilde{R}_6 = - \widetilde{I}_1 \widetilde{R}_4 + \widetilde{R}_6,$$

$$0 = \widetilde{\mathcal{D}} \widetilde{J}_2 = \widetilde{J}_3 - \widetilde{I}_2 \widetilde{R}_4 = 1 - \widetilde{I}_2 \widetilde{R}_4,$$

$$0 = \widetilde{\mathcal{D}} \widetilde{J}_3 = \widetilde{J}_4 - \widetilde{I}_3 \widetilde{R}_4 + 3 \widetilde{I}_1 \widetilde{R}_5.$$

Maurer–Cartan invariants:

$$\widetilde{R}_1 = -1, \qquad \widetilde{R}_2 = -\widetilde{I}_1, \qquad \widetilde{R}_3 = 0,$$

$$\widetilde{R}_4 = \frac{1}{\widetilde{I}_2}, \qquad \widetilde{R}_5 = \frac{1}{3\,\widetilde{I}_1}\left(\frac{\widetilde{I}_3}{\widetilde{I}_2} - \widetilde{J}_4\right), \qquad \widetilde{R}_6 = \frac{\widetilde{I}_1}{\widetilde{I}_2}\,.$$

Fundamental recurrence formulae:

$$\widetilde{\mathcal{D}}\,\widetilde{I}_1 = \widetilde{I}_2 + \frac{1+\widetilde{I}_1^2}{3\,\widetilde{I}_1}\left(\frac{\widetilde{I}_3}{\widetilde{I}_2} - \widetilde{J}_4\right), \qquad \widetilde{\mathcal{D}}\,\widetilde{I}_2 = 2\,\widetilde{I}_3 - \widetilde{I}_2\,\widetilde{J}_4,$$

$$\widetilde{\mathcal{D}}\,\widetilde{I}_3 = \widetilde{I}_4 + \frac{4\,\widetilde{I}_1\,\widetilde{I}_3 + 3\,\widetilde{I}_2^2}{3\,\widetilde{I}_1}\left(\frac{\widetilde{I}_3}{\widetilde{I}_2} - \widetilde{J}_4\right) + \frac{1+\widetilde{I}_1^2}{\widetilde{I}_2},$$

The edge invariants $\widetilde{I}_1$ and $\widetilde{J}_4$ do not generate!

$$\widetilde{I}_3 = \tfrac{1}{2}\widetilde{\mathcal{D}}\,\widetilde{I}_2 + \tfrac{1}{2}\,\widetilde{I}_2\,\widetilde{J}_4, \qquad \widetilde{I}_4 = \widetilde{\mathcal{D}}\,\widetilde{I}_3 - \frac{4\,\widetilde{I}_1\,\widetilde{I}_3 + 3\,\widetilde{I}_2^2}{3\,\widetilde{I}_1}\left(\frac{\widetilde{I}_3}{\widetilde{I}_2} - \widetilde{J}_4\right) + \frac{1+\widetilde{I}_1^2}{\widetilde{I}_2},$$

But

$$\widetilde{\mathcal{D}}\,\widetilde{I}_1 = \widetilde{I}_2 + \frac{1+\widetilde{I}_1^2}{6\,\widetilde{I}_1}\left(\frac{\widetilde{\mathcal{D}}\,\widetilde{I}_2}{\widetilde{I}_2} - \widetilde{J}_4\right)$$

Replacement Rule:

$$\kappa = \widetilde{\iota}\,\left(\frac{\sqrt{(u_x v_{xx} - u_{xx} v_x)^2 + u_{xx}^2 + v_{xx}^2}}{(1+u_x^2+v_x^2)^{3/2}}\right) = \frac{|\,\widetilde{I}_2\,|}{(1+\widetilde{I}_1^2)^{3/2}}\,,$$

$$\tau = \widetilde{\iota}\,\left(\frac{u_{xx} v_{xxx} - u_{xxx} v_{xx}}{(u_x v_{xx} - u_{xx} v_x)^2 + u_{xx}^2 + v_{xx}^2}\right) = \frac{\widetilde{I}_2}{\widetilde{I}_2^2} = \frac{1}{\widetilde{I}_2}\,.$$

$$\widetilde{I}_1 = \sqrt{(\kappa\,\tau)^{-2/3} - 1}\,, \qquad \widetilde{I}_2 = \frac{1}{\tau}\,.$$

★ Only valid for curves with $\kappa\,\tau > 1$.

$$\widetilde{\mathcal{D}} = (\kappa\,\tau)^{-1/3}\mathcal{D} = (\kappa\,\tau)^{-1/3}\,\frac{d}{ds}\,.$$

$$\widetilde{J}_4 = \frac{2\,\kappa_s\tau + \kappa\,\tau_s}{(\kappa\,\tau)^{4/3}} \;+\; 6\,\frac{\kappa^{2/3}}{\tau^{1/3}}\sqrt{(\kappa\,\tau)^{-2/3} - 1}\,,$$

$$\widetilde{I}_3 = \frac{\kappa_s}{(\kappa\,\tau)^{4/3}} \;+\; 3\,\frac{\kappa^{2/3}}{\tau^{4/3}}\sqrt{(\kappa\,\tau)^{-2/3} - 1}\,.$$

Equi-affine Surfaces

$M = \mathbb{R}^3 \qquad G = \mathrm{SA}(3) \qquad \dim G = 11.$

$$g \cdot z = A z + b, \qquad \det A = 1, \qquad z = \begin{pmatrix} x \\ y \\ u \end{pmatrix} \in \mathbb{R}^3.$$

Surfaces $S \subset M = \mathbb{R}^3$:

$$u = f(x, y)$$

Hyperbolic case

$$u_{xx}u_{yy} - u_{xy}^2 < 0$$

Cross-section:

$$x = y = u = u_x = u_y = u_{xy} = 0, \qquad u_{xx} = 1, \qquad u_{yy} = -1,$$

$$u_{xyy} = u_{xxx}, \qquad u_{xxy} = u_{yyy} = 0.$$

Power series normal form:

$$u(x,y) = \tfrac{1}{2}(x^2 - y^2) + \tfrac{1}{6}\,c\,(x^3 + 3\,xy^2) + \cdots$$

$\Longrightarrow$ *Nonsingular*: $c \neq 0$.

Invariantization — differential invariants: $\quad I_{jk} = \iota(u_{jk})$

Phantom differential invariants:

$$\iota(x) = \iota(y) = \iota(u) = \iota(u_x) = \iota(u_y) = \iota(u_{xy}) = \iota(u_{xxy}) = \iota(u_{yyy}) = 0,$$

$$\iota(u_{xx}) = 1, \qquad \iota(u_{yy}) = -1, \qquad \iota(u_{xxx}) - \iota(u_{xyy}) = 0.$$

Pick invariant:

$$P = \iota(u_{xxx}) = \iota(u_{xyy}).$$

Basic differential invariants of order 4:

$$Q_0 = \iota(u_{xxxx}), \quad Q_1 = \iota(u_{xxxy}), \quad Q_2 = \iota(u_{xxyy}),$$

$$Q_3 = \iota(u_{xyyy}), \quad Q_4 = \iota(u_{yyyy}),$$

Invariant differential operators:

$$\mathcal{D}_1 = \iota(D_x), \qquad \mathcal{D}_2 = \iota(D_y).$$

$\Longrightarrow$ Since the moving frame has order 3, one can generate all higher order differential invariants from the basic differential invariants of order ≤ 4.

Thus, to prove the Theorem, it suffices to generate $Q_0, \ldots, Q_4$ from P by invariant differentiation.

Infinitesimal generators:

$$\mathbf{v}_1 = x\,\partial_x - u\,\partial_u, \quad \mathbf{v}_2 = y\,\partial_y - u\,\partial_u,$$

$$\mathbf{v}_3 = y\,\partial_x, \quad \mathbf{v}_4 = u\,\partial_x, \quad \mathbf{v}_5 = x\,\partial_y,$$

$$\mathbf{v}_6 = u\,\partial_y, \quad \mathbf{v}_7 = x\,\partial_u, \quad \mathbf{v}_8 = y\,\partial_u,$$

$$\mathbf{w}_1 = \partial_x, \quad \mathbf{w}_2 = \partial_y, \quad \mathbf{w}_3 = \partial_u,$$

Recurrence formulae

$$\mathcal{D}_1 I_{jk} = I_{j+1,k} + \sum_{\kappa=1}^{8} \varphi_\kappa^{jk}(0,0,I^{(j+k)}) R_1^\kappa,$$

$$\mathcal{D}_2 I_{jk} = I_{j,k+1} + \sum_{\kappa=1}^{8} \varphi_\kappa^{jk}(0,0,I^{(j+k)}) R_2^\kappa, \qquad j+k \geq 1.$$

$\varphi_\kappa^{jk}(x,y,u^{(j+k)})$ — Prolonged infinitesimal generator coefficients

$\varphi_\kappa^{jk}(0,0,I^{(j+k)})$ — invariantized coefficients

R_i^κ — Maurer–Cartan invariants

Phantom recurrence formulae:

$$0 = \mathcal{D}_1 I_{10} = 1 + R_1^7, \qquad 0 = \mathcal{D}_2 I_{10} = R_2^7,$$

$$0 = \mathcal{D}_1 I_{01} = R_1^8, \qquad 0 = \mathcal{D}_2 I_{01} = -1 + R_2^8,$$

$$0 = \mathcal{D}_1 I_{20} = I_{30} - 3R_1^1 - R_1^2, \qquad 0 = \mathcal{D}_2 I_{20} = -3R_2^1 - R_2^2,$$

$$0 = \mathcal{D}_1 I_{11} = -R_1^3 + R_1^5, \qquad 0 = \mathcal{D}_2 I_{11} = I_{30} - R_2^3 + R_2^5,$$

$$0 = \mathcal{D}_1 I_{02} = I_{12} + R_1^1 + 3R_1^2, \qquad 0 = \mathcal{D}_2 I_{02} = R_2^1 + 3R_2^2,$$

$$0 = \mathcal{D}_1 I_{21} = I_{31} - I_{30}R_1^3 - 2I_{30}R_1^5 + R_1^6,$$

$$0 = \mathcal{D}_2 I_{21} = I_{22} - I_{30}R_2^3 - 2I_{30}R_2^5 + R_2^6,$$

$$0 = \mathcal{D}_1 I_{03} = I_{13} - 3I_{30}R_2^3 - 3R_2^6, \qquad 0 = \mathcal{D}_2 I_{03} = I_{04} - 3I_{30}R_2^3 - 3R_2^6.$$

Maurer–Cartan invariants:

$$R_1 = \left(\tfrac{1}{2}\,P, -\tfrac{1}{2}\,P, \frac{3\,Q_1 + Q_3}{12\,P}\,, \tfrac{1}{4}\,Q_0 - \tfrac{1}{4}\,Q_2 - \tfrac{1}{2}\,P^2, \frac{3\,Q_1 + Q_3}{12\,P}\,, -\tfrac{1}{4}\,Q_1 + \tfrac{1}{4}\,Q_3, -1, 0 \right)$$

$$R_2 = \left(0, 0, \frac{3\,Q_2 + Q_4}{12\,P} + \tfrac{1}{2}\,P, \tfrac{1}{4}\,Q_1 - \tfrac{1}{4}\,Q_3, \frac{3\,Q_2 + Q_4}{12\,P} - \tfrac{1}{2}\,P, -\tfrac{1}{4}\,Q_2 + \tfrac{1}{4}\,Q_4 - \tfrac{1}{2}\,P^2, 0, 1 \right)$$

Fourth order invariants:

$$P_1 = \mathcal{D}_1 P = \tfrac{1}{4}\,Q_0 + \tfrac{3}{4}\,Q_2, \qquad P_2 = \mathcal{D}_2 P = \tfrac{1}{4}\,Q_1 + \tfrac{3}{4}\,Q_3.$$

Commutator:

$$\mathcal{D}_3 = [\,\mathcal{D}_1, \mathcal{D}_2\,] = \mathcal{D}_1\,\mathcal{D}_2 - \mathcal{D}_2\,\mathcal{D}_1 = Y_1\,\mathcal{D}_1 + Y_2\,\mathcal{D}_2,$$

where

$$Y_1 = R_2^1 - R_1^3 = -\,\frac{3\,Q_1 + Q_3}{12\,P}\,, \qquad Y_2 = R_2^5 - R_1^2 = \frac{3\,Q_2 + Q_4}{12\,P}\,.$$

Another fourth order invariant:

$$P_3 = \mathcal{D}_3 P = \mathcal{D}_1 \mathcal{D}_2 P - \mathcal{D}_2 \mathcal{D}_1 P = Y_1 P_1 + Y_2 P_2. \qquad (*)$$

Nondegeneracy condition: If

$$\det \begin{pmatrix} P_1 & P_2 \\ \mathcal{D}_1 P_j & \mathcal{D}_2 P_j \end{pmatrix} \neq 0 \qquad \text{for} \qquad j = 1, 2, \text{ or } 3,$$

we can solve $(*)$ and

$$\mathcal{D}_3 P_j = Y_1 \mathcal{D}_1 P_j + Y_2 \mathcal{D}_2 P_j$$

for the fourth order commutator invariants:

$$Y_1 = -\frac{3\,Q_1 + Q_3}{12\,P}, \qquad Y_2 = \frac{3\,Q_2 + Q_4}{12\,P}.$$

So far, we have constructed four combinations of the fourth order differential invariants

$$S_1 = Q_0 + 3\,Q_2, \qquad S_2 = Q_1 + 3\,Q_3,$$

$$S_3 = 3\,Q_1 + Q_3, \qquad S_4 = 3\,Q_2 + Q_4.$$

as rational functions of the invariant derivatives of the Pick invariant. To obtain the final fourth order differential invariant:

$$12\,P(\,\mathcal{D}_1 S_4 - \mathcal{D}_2 S_3\,) = 48\,P^2 Q_0 - 30\,P^2 S_1 + 18\,P^2 S_4 - 3\,S_2 S_3 - S_3^2 + 3\,S_1 S_4 + S_4^2.$$

⋆ ⋆ ⋆ This completes the proof ⋆ ⋆ ⋆

General Problem:

Find minimal generating sets of differential invariants.