Invariant Signatures and Histograms for Object Recognition and Symmetry Detection

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AIM, May, 2016

Plane Geometries/Groups

Euclidean geometry:

SE(2) — rigid motions (rotations and translations)

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

E(2) — plus reflections?

Equi-affine geometry:

SA(2) — area-preserving affine transformations:

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} \qquad \alpha\,\delta - \beta\,\gamma = 1$$

Projective geometry:

PSL(3) — projective transformations:

$$\bar{x} = \frac{\alpha\,x + \beta\,y + \gamma}{\rho\,x + \sigma\,y + \tau} \qquad \bar{y} = \frac{\lambda\,x + \mu\,y + \nu}{\rho\,x + \sigma\,y + \tau}$$

The Basic Equivalence Problem

G — transformation group acting on M

Equivalence:

Determine when two subsets

$$N \quad \text{and} \quad \overline{N} \subset M$$

are congruent:

$$\overline{N} = g \cdot N \qquad \text{for} \qquad g \in G$$

Symmetry:

Find all symmetries,

i.e., self-equivalences or *self-congruences*:

$$N = g \cdot N$$

Tennis, Anyone?

Duck = Rabbit?

Limitations of Projective Geometry

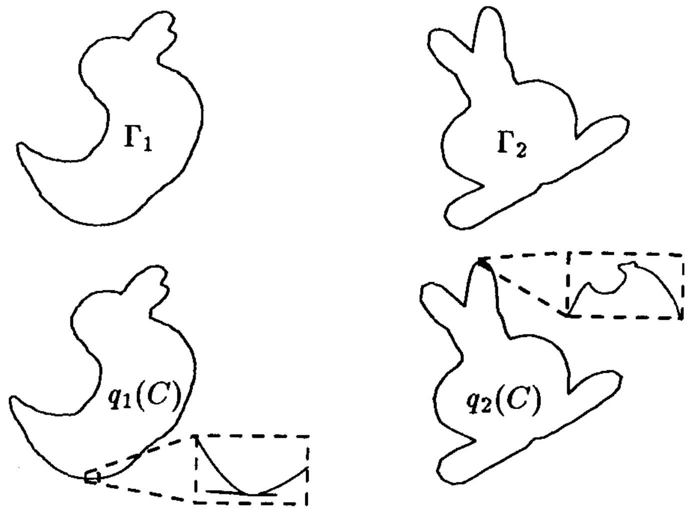
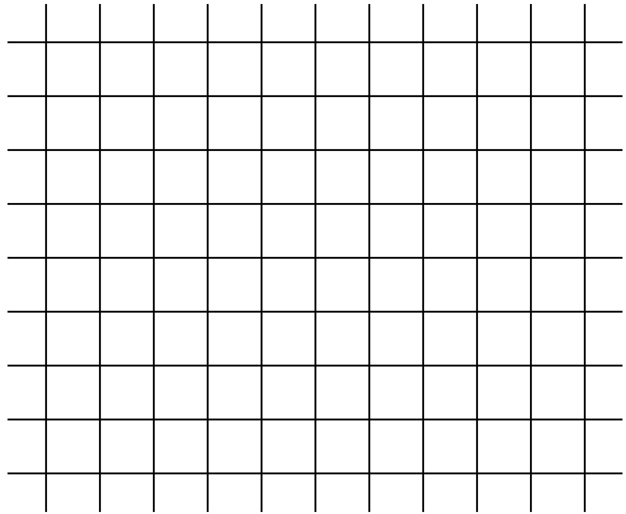


Fig. 3. The upper two curves are not projectively equivalent, but the lower two curves are. The lower curves are constructed by introducing small ripples along the convex hull, these are illustrated in the magnified pictures.

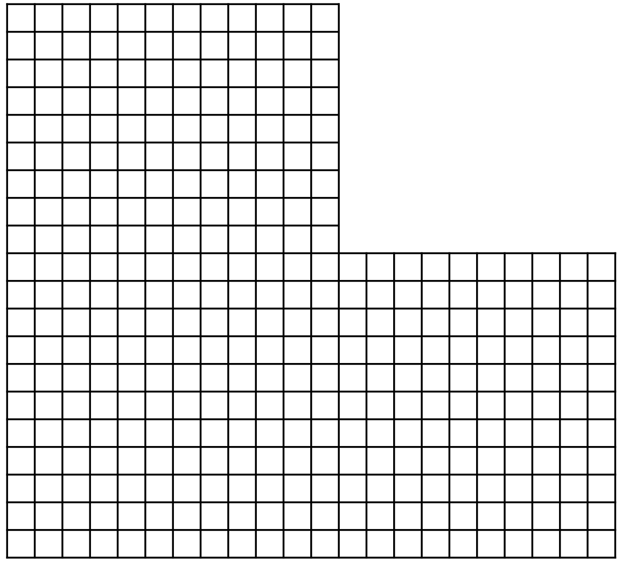
$\Longrightarrow$ K. Åström

Thatcher Illusion

Local Symmetry and Equivalence



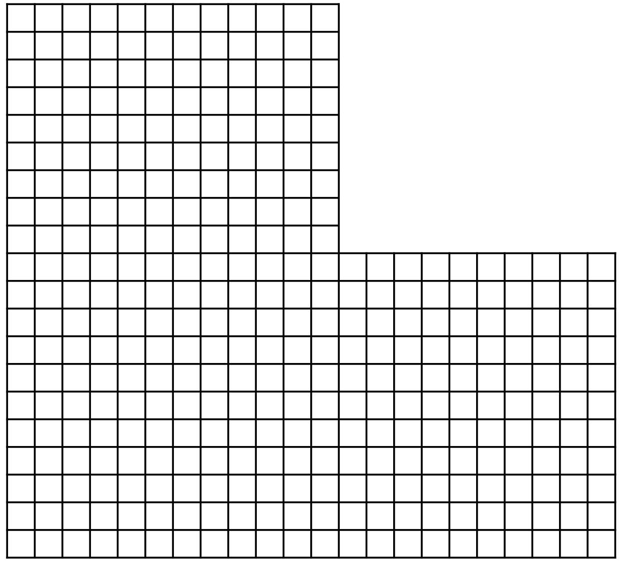
Local Symmetry and Equivalence



$\Longrightarrow$ Alan Weinstein

♠ A groupoid is a small category such that every morphism has an inverse.

Local Symmetry and Equivalence



$\Longrightarrow$ Alan Weinstein

♠ Groupoids are the appropriate structure for local symmetry and equivalence problems ...

Invariants

The solution to an equivalence problem rests on understanding its invariants.

≈ Invariants describe the moduli space of objects under group transformations.

⋆ If G acts transitively, there are no (non-constant) invariants — in which case we need to "prolong" the action to a higher dimensional space.

Joint Invariants

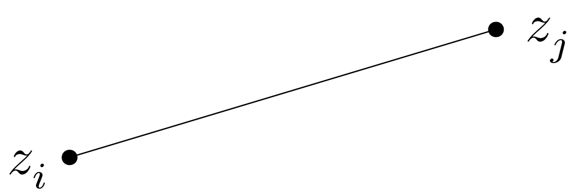
A joint invariant is an invariant of the k-fold Cartesian product action of G on $M \times \cdots \times M$:

$$\boxed{I(g \cdot z_1, \ldots, g \cdot z_k) \;=\; I(z_1, \ldots, z_k)}$$

Joint Euclidean Invariants

Theorem. Every joint Euclidean invariant is a function of the interpoint distances

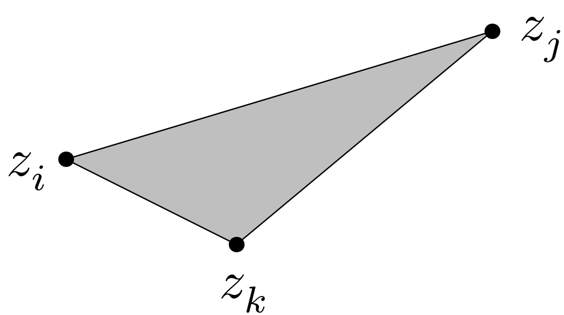
$$d(z_i, z_j) = \| z_i - z_j \|$$



Joint Equi–Affine Invariants

Theorem. Every planar joint equi–affine invariant is a function of the triangular areas

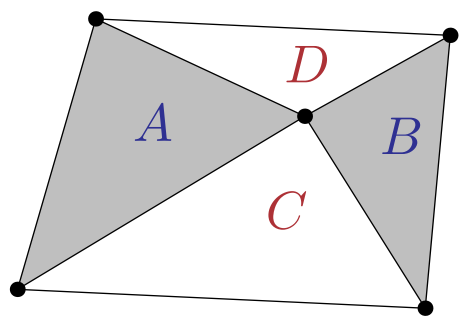
$$A(i,j,k) = \tfrac{1}{2}\,(z_i - z_j) \wedge (z_i - z_k)$$



Joint Projective Invariants

Theorem. Every joint projective invariant is a function of the planar cross-ratios

$$[\, z_i, z_j, z_k, z_l, z_m \,] \;=\; \frac{A\,B}{C\,D}$$



Differential Invariants

Given a submanifold (curve, surface, ...) $N \subset M$, a differential invariant is an invariant of the action of G on N and its derivatives (jets).

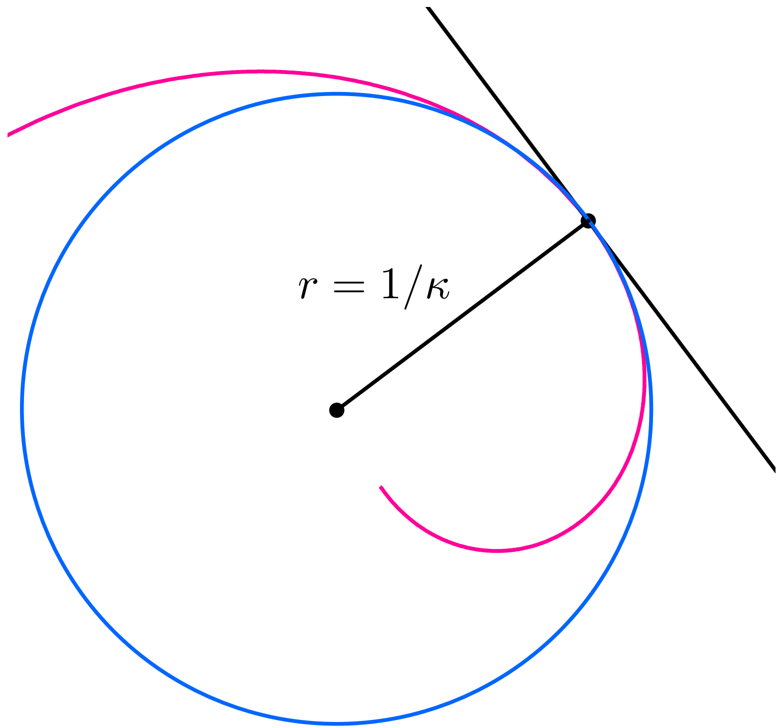
$$\boxed{I(g \cdot z^{(k)}) \;=\; I(z^{(k)})}$$

Euclidean Plane Curves: $G = \mathrm{SE}(2)$

The simplest differential invariant is the curvature, defined as the reciprocal of the radius of the osculating circle:

$$\kappa = \frac{1}{r}$$

Curvature



Euclidean Plane Curves: $G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$

Assume the curve is a graph: $\quad y = u(x)$

Differential invariants:

$$\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}} \qquad \frac{d\kappa}{ds} = \frac{(1+u_x^2)u_{xxx} - 3u_x u_{xx}^2}{(1+u_x^2)^3} \qquad \frac{d^2\kappa}{ds^2} = \cdots$$

Arc length (invariant one-form):

$$ds = \sqrt{1+u_x^2}\;dx, \qquad \frac{d}{ds} = \frac{1}{\sqrt{1+u_x^2}}\,\frac{d}{dx}$$

Theorem. All Euclidean differential invariants are functions of the derivatives of curvature with respect to arc length: $\kappa, \ \kappa_s, \ \kappa_{ss}, \ \cdots$

Equi-affine Plane Curves: $G = \mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$

Equi-affine curvature:

$$\kappa = \frac{5\, u_{xx} u_{xxxx} - 3\, u_{xxx}^2}{9\, u_{xx}^{8/3}} \qquad \frac{d\kappa}{ds} = \cdots \qquad \frac{d^2\kappa}{ds^2} = \cdots$$

Equi-affine arc length:

$$ds = \sqrt[3]{u_{xx}}\; dx \qquad \frac{d}{ds} = \frac{1}{\sqrt[3]{u_{xx}}} \frac{d}{dx}$$

Theorem. All equi-affine differential invariants are functions of the derivatives of equi-affine curvature with respect to equi-affine arc length:

$$\kappa, \qquad \kappa_s, \qquad \kappa_{ss}, \qquad \cdots$$

Projective Plane Curves: $G = \mathrm{PSL}(2)$

Projective curvature:

$$\kappa = K(u^{(7)}, \cdots) \qquad \frac{d\kappa}{ds} = \cdots \qquad \frac{d^2\kappa}{ds^2} = \cdots$$

Projective arc length:

$$ds = L(u^{(5)}, \cdots)\, dx \qquad \frac{d}{ds} = \frac{1}{L}\frac{d}{dx}$$

Theorem. All projective differential invariants are functions of the derivatives of projective curvature with respect to projective arc length:

$$\kappa, \quad \kappa_s, \quad \kappa_{ss}, \quad \cdots$$

Joint Differential Invariants

Given a submanifold (curve, surface, . . .) $N \subset M$, a joint differential invariant or semi-differential invariant is an invariant of the action of G on N and its derivatives (jets) at several points $z_1, \ldots, z_k \in N$:

$$\boxed{I(g \cdot z_1^{(n)}, \ldots, g \cdot z_k^{(n)}) \;=\; I(z_1^{(n)}, \ldots, z_k^{(n)})}$$

Euclidean Joint Differential Invariants — Plane Curves

- One–point

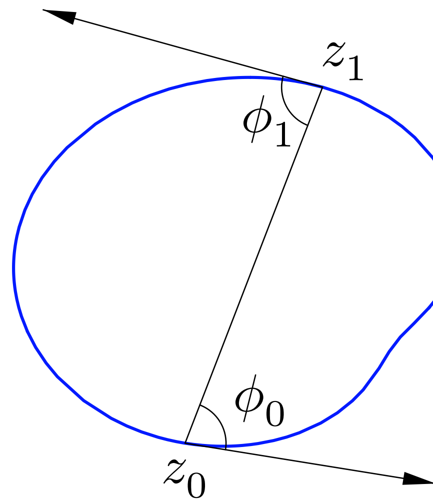
  $\Longrightarrow$ curvature

$$\kappa = \frac{\dot{z} \wedge \ddot{z}}{\| \dot{z} \|^3}$$

- Two–point

$\Rightarrow$ distances $\qquad \| z_1 - z_0 \|$

$\Rightarrow$ tangent angles $\qquad \phi_0 = \sphericalangle (z_1 - z_0, \dot{z}_0)$



Equi–Affine Joint Differential Invariants — Plane Curves

- One–point

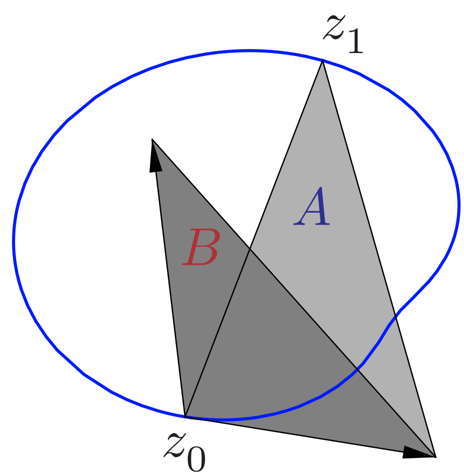
  $\Longrightarrow$ affine curvature

$$\kappa = \frac{(z_t \wedge z_{tttt}) + 4(z_{tt} \wedge z_{ttt})}{3(z_t \wedge z_{tt})^{5/3}} - \frac{5(z_t \wedge z_{ttt})^2}{9(z_t \wedge z_{tt})^{8/3}}$$

$$= z_s \wedge z_{ss}$$

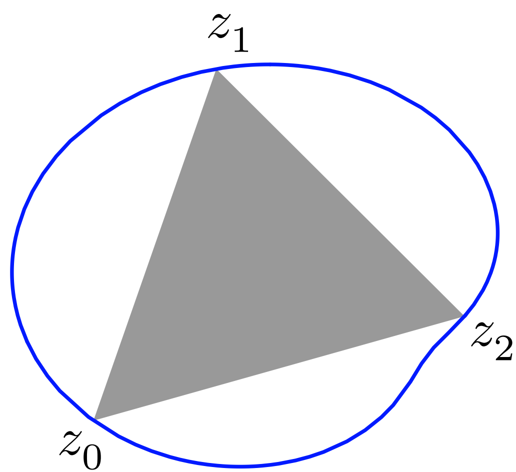
- Two–point $\Longrightarrow$ tangent triangle area ratio

$$\frac{\dot{z}_0 \wedge \ddot{z}_0}{[\,(z_1 - z_0) \wedge \dot{z}_0\,]^3} = \frac{[\,\dot{0}\ \ddot{0}\,]}{[\,0\ 1\ \dot{0}\,]^3} = \frac{A}{B^3}$$



- Three–point $\Longrightarrow$ triangle area

$$\frac{1}{2}\,(z_1 - z_0) \wedge (z_2 - z_0) = \frac{1}{2}\,[\,0\ 1\ 2\,]$$



Projective Joint Differential Invariants — Planar Curves

- One–point

$\Rightarrow$ projective curvature

$$\kappa = K(z^{(7)},\ \cdots\ )$$

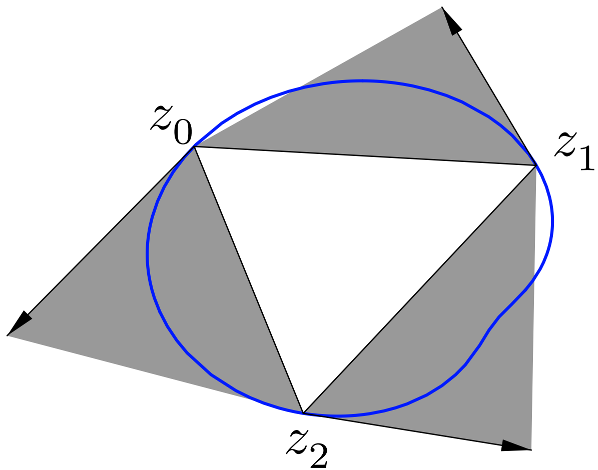
- Two–point

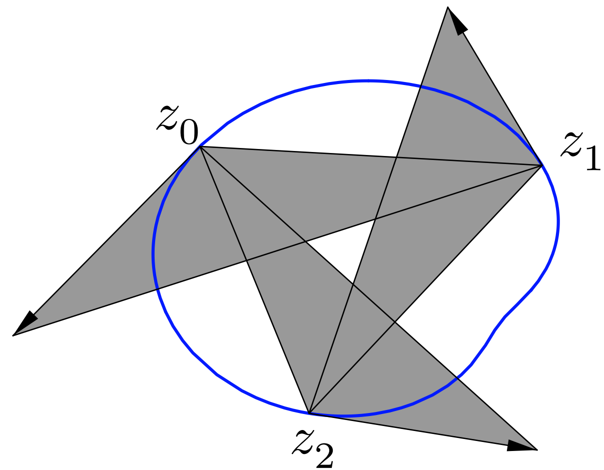
$\Rightarrow$ tangent triangle area ratio

$$\frac{[\,0\;1\;\dot{0}\,]^3\,[\,\dot{1}\;\ddot{1}\,]}{[\,0\;1\;\dot{1}\,]^3\,[\,\dot{0}\;\ddot{0}\,]} = \frac{A_0/B_0^3}{A_1/B_1^3}$$

- Three–point $\Longrightarrow$ triple tangent triangle ratio

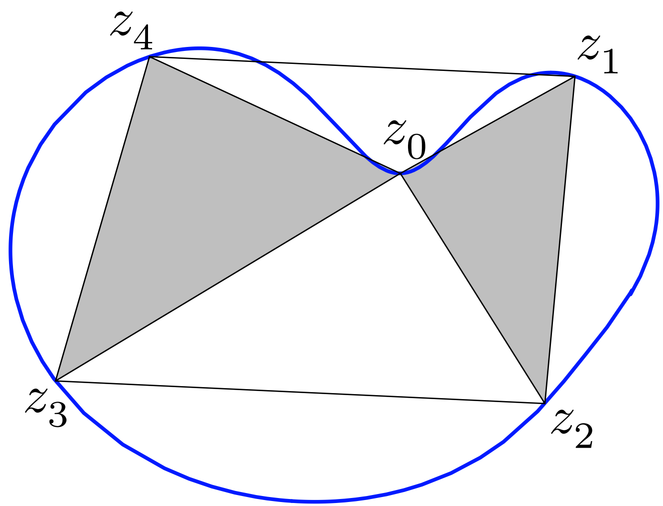
$$\frac{[\,0\;2\;\dot{0}\,]\,[\,0\;1\;\dot{1}\,]\,[\,1\;2\;\dot{2}\,]}{[\,0\;1\;\dot{0}\,]\,[\,1\;2\;\dot{1}\,]\,[\,0\;2\;\dot{2}\,]}\,.$$





- Five–point $\Longrightarrow$ area cross–ratio

$$\frac{[\,0\;1\;2\,]\,[\,0\;3\;4\,]}{[\,0\;1\;3\,]\,[\,0\;2\;4\,]}$$



Moving Frames

The equivariant method of moving frames provides a systematic and algorithmic calculus for determining complete systems of differential invariants, joint invariants, joint differential invariants, invariant differential operators, invariant differential forms, invariant tensors, invariant numerical algorithms, etc.

Symmetry–Preserving Numerical Approximations

$\star$ In practical applications, use invariant numerical approximations, based on joint invariants, to the required differential invariants, joint differential invariants, etc.

♠ Invariantization of numerical integration methods

$\Longrightarrow$ Runge–Kutta, Crank–Nicolson, ...

Equivalence & Invariants

- Equivalent submanifolds $N \approx \overline{N}$
  must have the same invariants: $I = \bar{I}$.

Constant invariants provide immediate information:

$$\text{e.g.} \qquad \kappa = 2 \quad \Longleftrightarrow \quad \overline{\kappa} = 2$$

Non-constant invariants are not useful in isolation, because an equivalence map can drastically alter the dependence on the submanifold parameters:

$$\text{e.g.} \qquad \kappa = x^3 \qquad \text{versus} \qquad \overline{\kappa} = \sinh x$$

However, a functional dependency or syzygy among the invariants *is* intrinsic:

$$\text{e.g.} \qquad \kappa_s = \kappa^3 - 1 \quad \Longleftrightarrow \quad \overline{\kappa}_{\overline{s}} = \overline{\kappa}^3 - 1$$

- Universal syzygies — Gauss–Codazzi
- Distinguishing syzygies.

Theorem. (Cartan)
Two regular submanifolds are (locally) equivalent if and only if they have identical syzygies among *all* their differential invariants.

Proof: Cartan's technique of the graph:
Construct the graph of the equivalence map as the solution to a (Frobenius) integrable differential system, which can be integrated by solving ordinary differential equations.

Finiteness of Generators and Syzygies

♠ There are, in general, an infinite number of differential invariants and hence an infinite number of syzygies must be compared to establish equivalence.

♡ But the higher order differential invariants are always generated by invariant differentiation from a finite collection of basic differential invariants, and the higher order syzygies are all consequences of a finite number of low order syzygies!

Example — Plane Curves

If non-constant, both κ and κ_s depend on a single parameter, and so, locally, are subject to a syzygy:

$$\kappa_s = H(\kappa) \qquad (*)$$

But then

$$\kappa_{ss} = \frac{d}{ds} H(\kappa) = H'(\kappa)\,\kappa_s = H'(\kappa)\,H(\kappa)$$

and similarly for κ_{sss}, etc.

Consequently, all the higher order syzygies are generated by the fundamental first order syzygy $(*)$.

Thus, for Euclidean (or equi-affine or projective or . . .) plane curves we need only know a single syzygy between κ and κ_s in order to establish equivalence!

Signature Curves

Definition. Given an (ordinary) planar action of a Lie group G, the *signature curve* $\Sigma \subset \mathbb{R}^2$ of a plane curve $\mathcal{C} \subset \mathbb{R}^2$ is parametrized by the two lowest order differential invariants

$$\chi : \mathcal{C} \longrightarrow \Sigma = \left\{ \left(\kappa \, , \, \frac{d\kappa}{ds} \right) \right\} \subset \mathbb{R}^2$$

$\Longrightarrow$ *Calabi, PJO, Shakiban, Tannenbaum, Haker*

Theorem. Two regular curves $\mathcal{C}$ and $\overline{\mathcal{C}}$ are (locally) equivalent:

$$\overline{\mathcal{C}} = g \cdot \mathcal{C}$$

if and only if their signature curves are identical:

$$\overline{\Sigma} = \Sigma$$

$\Longrightarrow$ regular: $(\kappa_s, \kappa_{ss}) \neq 0$.

Symmetry and Signature

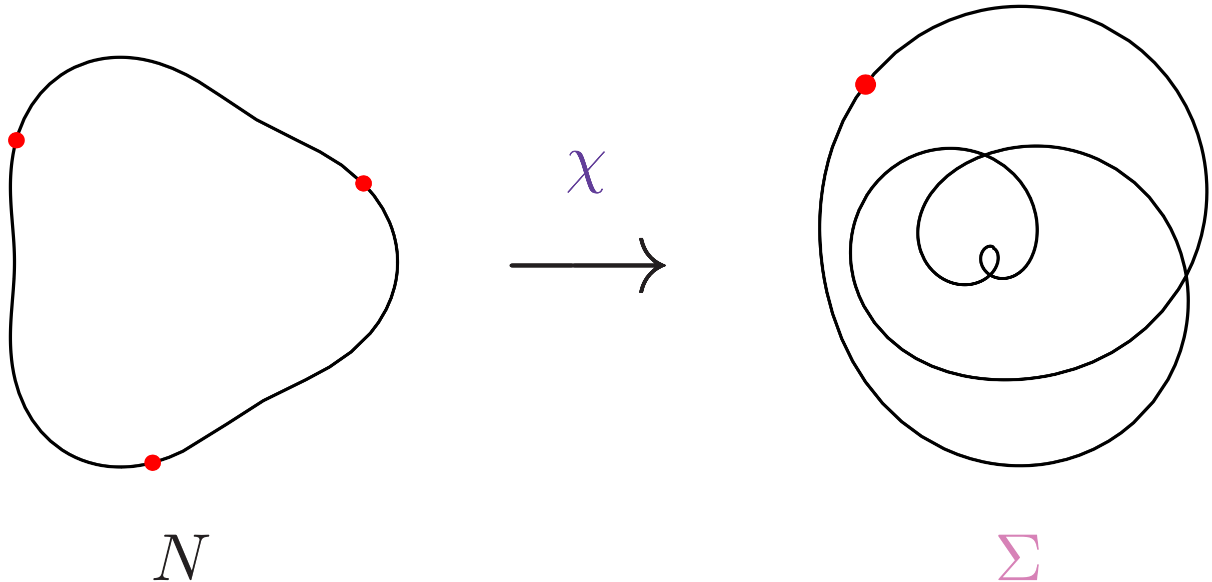
★ For regular p-dimensional submanifolds,
the (local) dimension of the signature equals
the co-dimension of the (local) symmetry group(oid):

$$\dim \Sigma = p - \dim G_S$$

- Maximally symmetric: $\dim \Sigma = 0$
  $\Longleftrightarrow$ all the differential invariants are constant
  $\Longleftrightarrow$ $S \subset H \cdot z_0$ is a piece of
  an orbit of a p-dimensional subgroup $H \subset G$

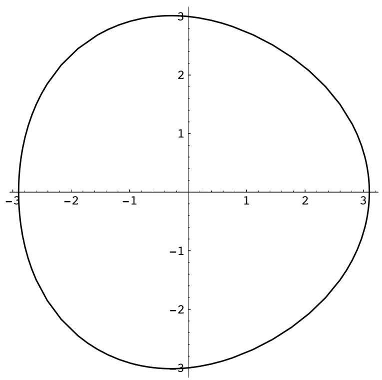
- Discrete symmetries: $\dim \Sigma = p = \dim S$
  the number of discrete (local) symmetries equals the index
  of the signature

The Index

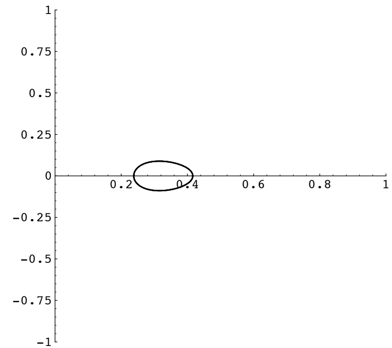


$$\text{index} = 3 = \#\,\text{symmetries}$$

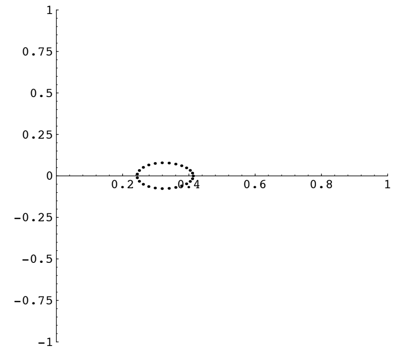
The polar curve $r = 3 + \frac{1}{10}\cos 3\theta$



The Original Curve

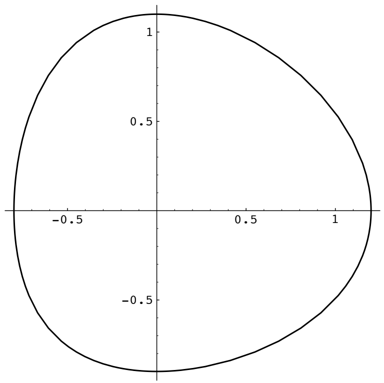


Euclidean Signature

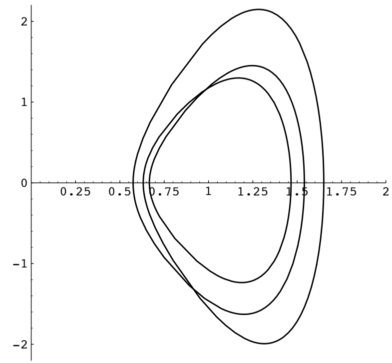


Numerical Signature

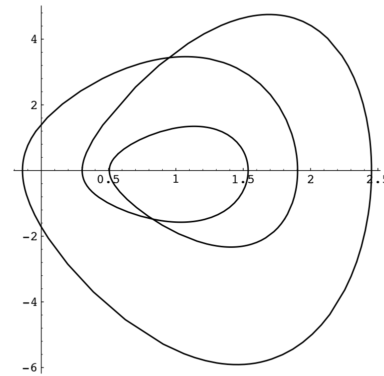
The Curve $x = \cos t + \frac{1}{5}\cos^2 t, \quad y = \sin t + \frac{1}{10}\sin^2 t$



The Original Curve

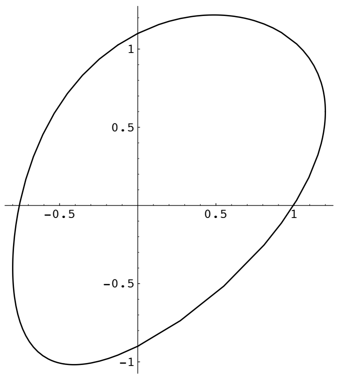


Euclidean Signature

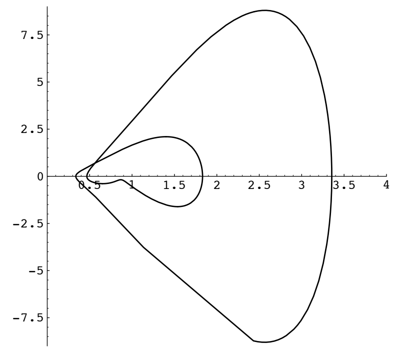


Equi-affine Signature

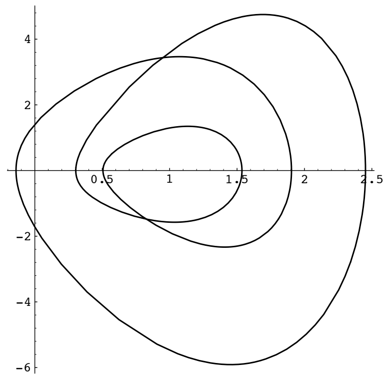
The Curve $x = \cos t + \frac{1}{5}\cos^2 t, \quad y = \frac{1}{2}x + \sin t + \frac{1}{10}\sin^2 t$



The Original Curve

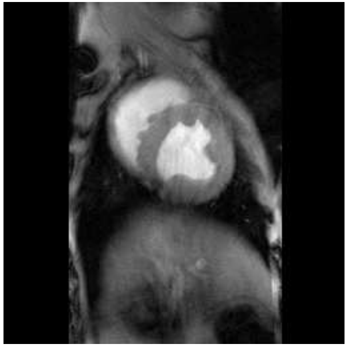


Euclidean Signature

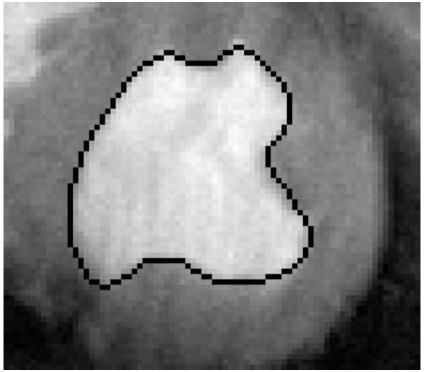


Equi-affine Signature

Canine Left Ventricle Signature

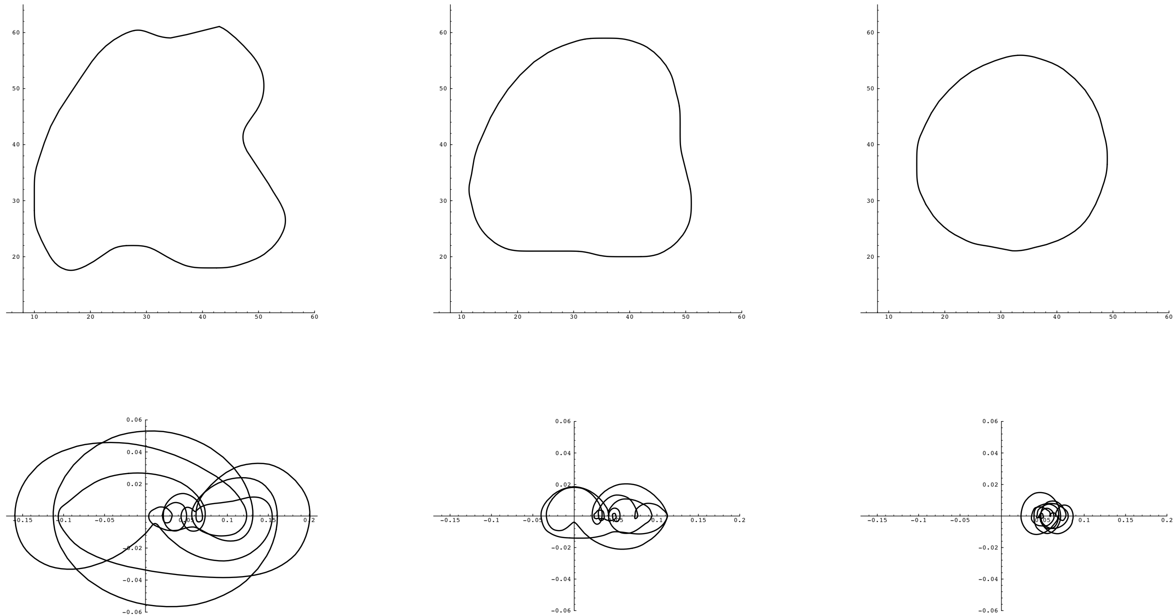


Original Canine Heart
MRI Image



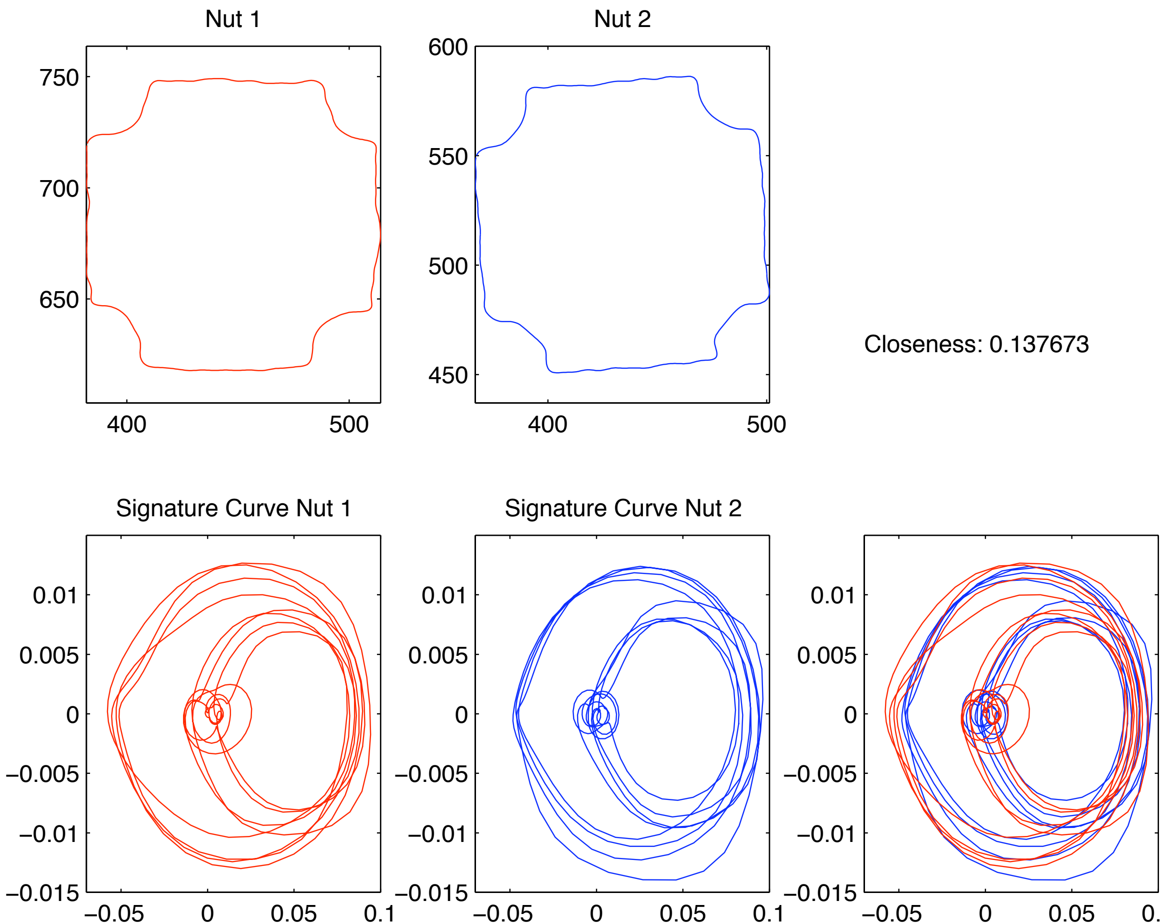
Boundary of Left Ventricle

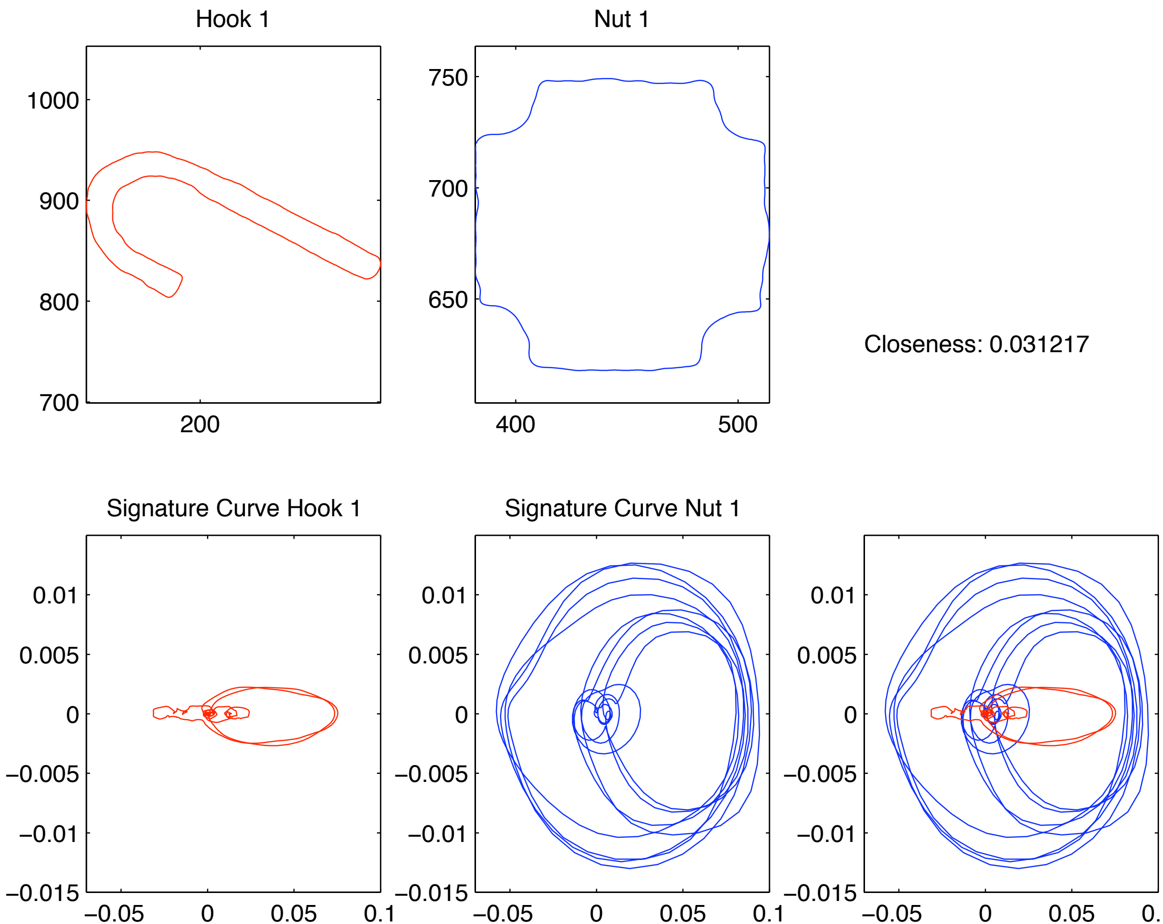
Smoothed Ventricle Signature



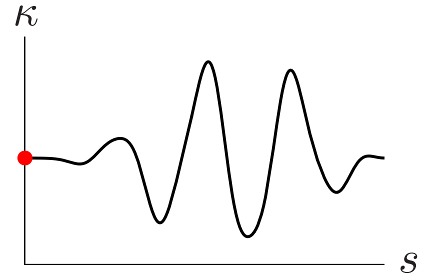
Object Recognition

⟹ Steve Haker

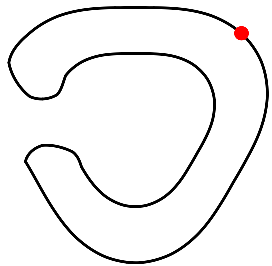

Nut 1
Nut 2
750
700
650
400
500
600
550
500
450
400
500
Closeness: 0.137673
Signature Curve Nut 1
Signature Curve Nut 2
0.01
0.005
0
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1


Hook 1
Nut 1
Closeness: 0.031217
Signature Curve Hook 1
Signature Curve Nut 1
1000
900
800
700
200
750
700
650
400
500
0.01
0.005
0
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1

Signatures

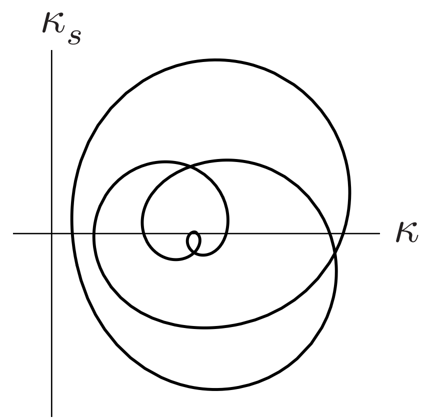


Classical Signature



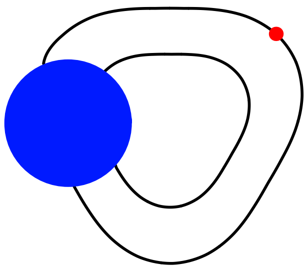
Original curve

$\longrightarrow$



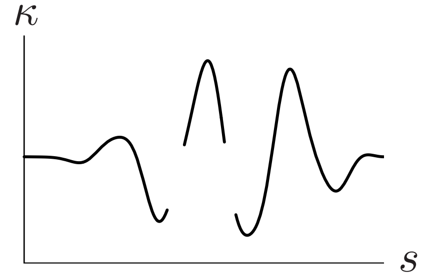
Differential invariant signature

Occlusions

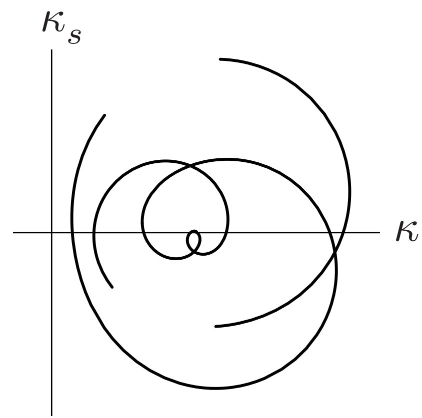


Original curve

$\longrightarrow$



Classical Signature



Differential invariant signature

3D Differential Invariant Signatures

Euclidean space curves: $C \subset \mathbb{R}^3$

$$\Sigma = \{ \, (\, \kappa \, , \, \kappa_s \, , \, \tau \,) \, \} \subset \mathbb{R}^3$$

- κ — curvature, τ — torsion

Euclidean surfaces: $S \subset \mathbb{R}^3$ (generic)

$$\Sigma = \left\{ \left(H \, , \, K \, , \, H_{,1} \, , \, H_{,2} \, , \, K_{,1} \, , \, K_{,2} \right) \right\} \subset \mathbb{R}^6$$

or $$\widehat{\Sigma} = \left\{ \left(H \, , \, H_{,1} \, , \, H_{,2} \, , \, H_{,11} \right) \right\} \subset \mathbb{R}^4$$

- H — mean curvature, K — Gauss curvature

Equi–affine surfaces: $S \subset \mathbb{R}^3$ (generic)

$$\Sigma = \left\{ \left(P \, , \, P_{,1} \, , \, P_{,2} , \, P_{,11} \right) \right\} \subset \mathbb{R}^4$$

- P — Pick invariant

Advantages of the Signature Curve

- Purely local — no ambiguities
- Symmetries and approximate symmetries
- Extends to surfaces and higher dimensional submanifolds
- Occlusions and reconstruction
- Partial matching and puzzles

Main disadvantage: Noise sensitivity due to dependence on high order derivatives.

Localization of Signatures

Generalized vertex: $\kappa_s \equiv 0$

$\Longrightarrow$ critical point; circular arc; straight line segment

---

Bivertex arc: $\kappa_s \neq 0$ everywhere

except $\kappa_s = 0$ at the two endpoints

Bivertex Decomposition of a Curve:

$$\boxed{C = \bigcup_{j=1}^{m} B_j \cup \bigcup_{k=1}^{n} V_k}$$

$B_1, \ldots, B_m$ — bivertex arcs

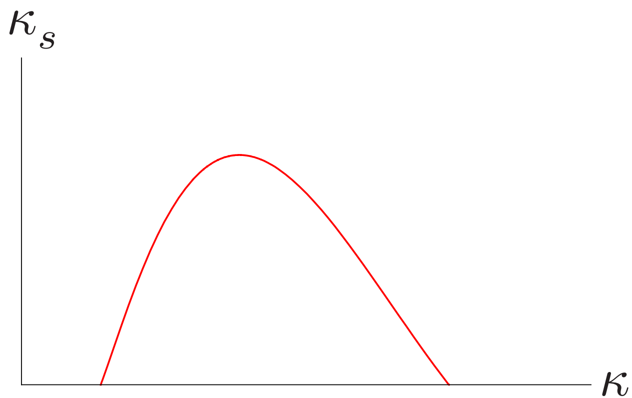
$V_1, \ldots, V_n$ — generalized vertices: $n \geq 4$

⋆ Compare individual bivertex arcs, and then determine whether the rigid equivalences are (approximately) the same.

Dan Hoff & PJO, Extensions of invariant signatures for object recognition,
J. Math. Imaging Vision **45** (2013), 176–185.

Bivertex Arcs

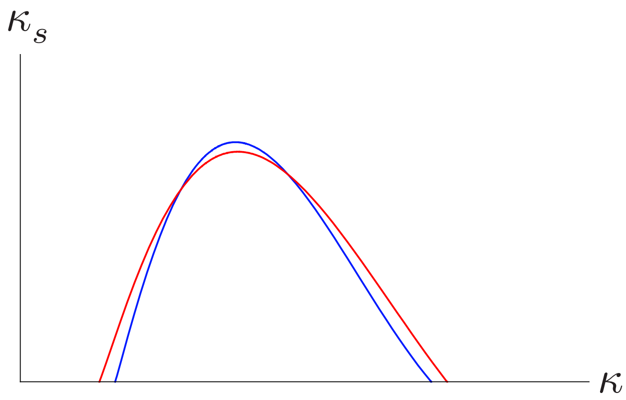
The signature Σ of a bivertex arc is a single arc that starts and ends on the κ–axis.

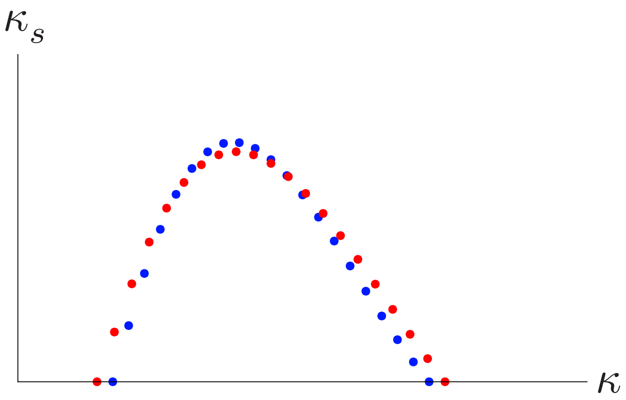


Gravitational/Electrostatic Attraction

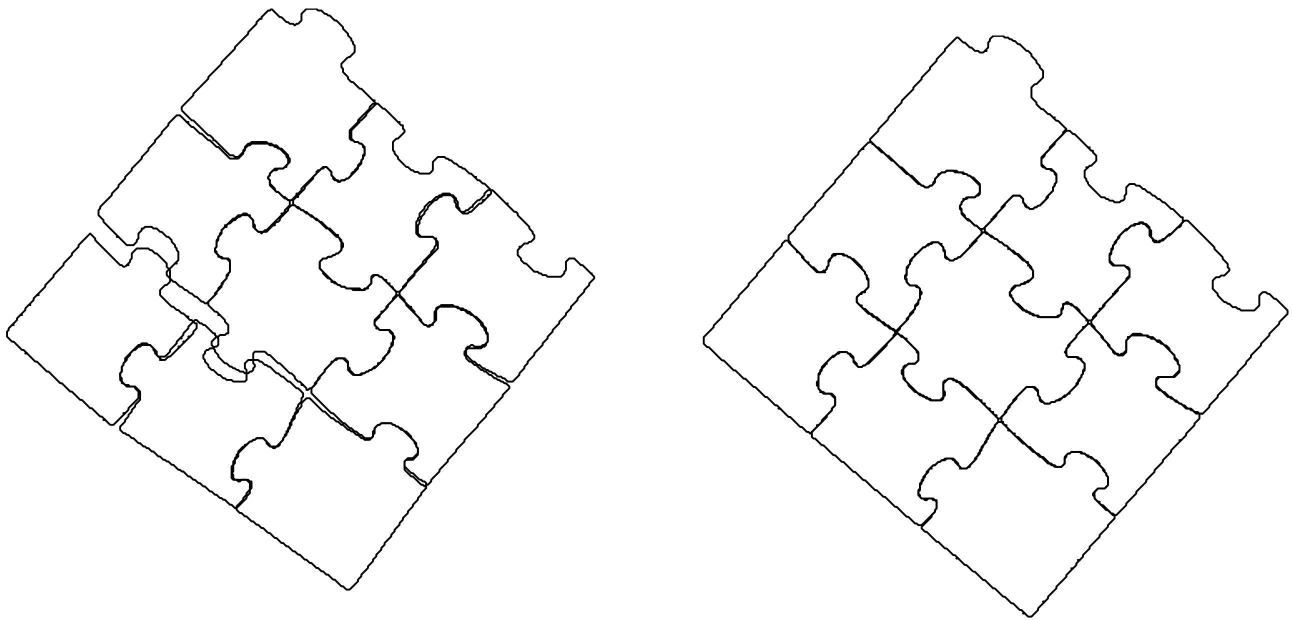
★ Treat the two (signature) curves as masses or as oppositely charged wires. The higher their mutual attraction, the closer they are together.

★ In practice, we are dealing with discrete data (pixels) and so treat the curves and signatures as point masses/charges.



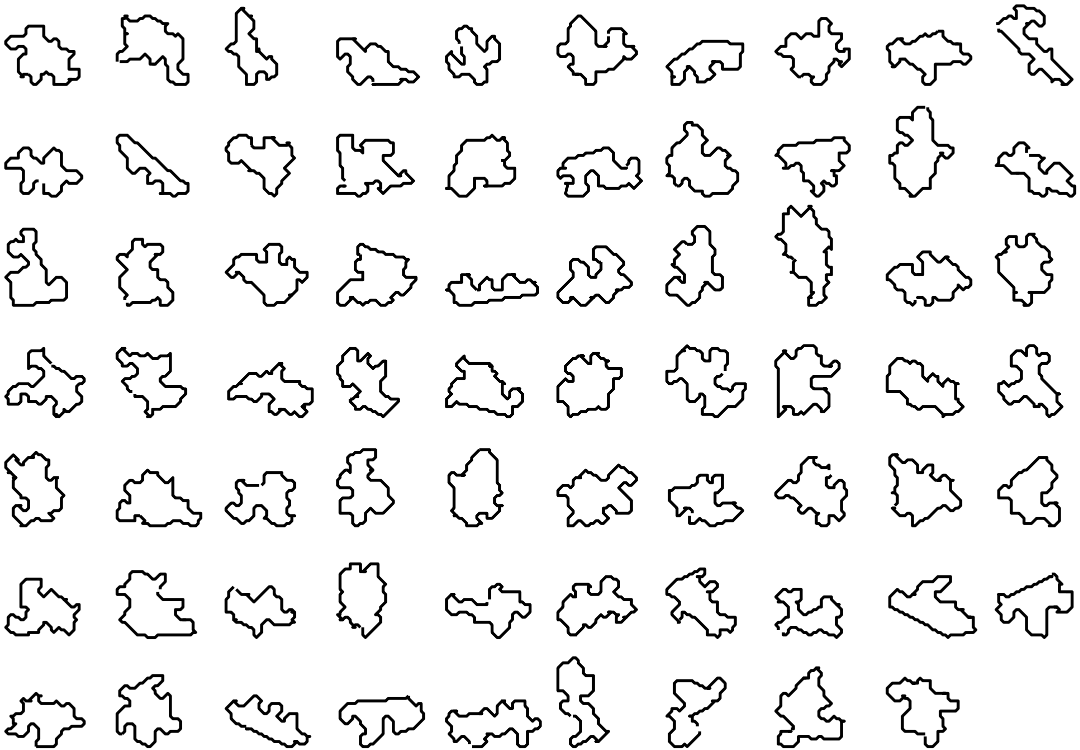


Piece Locking

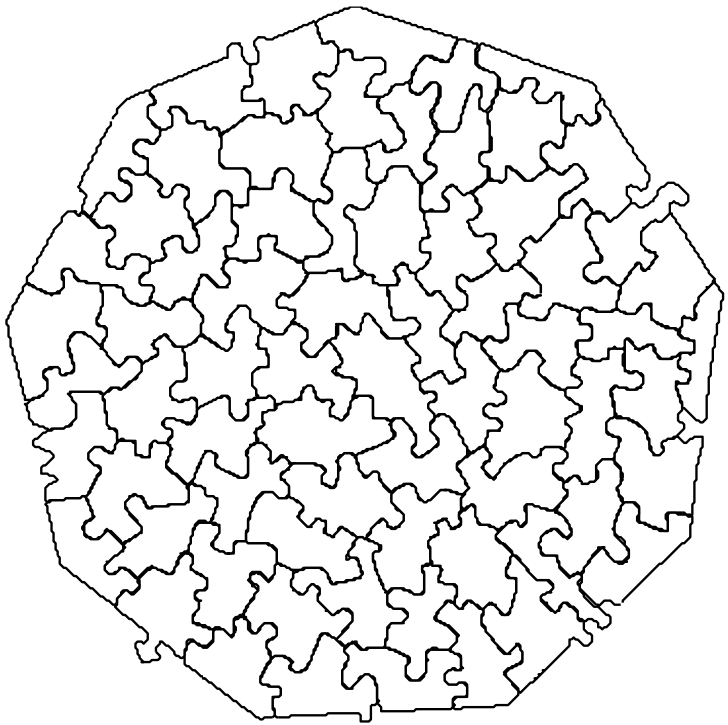


★★ Minimize force and torque based on gravitational attraction of the two matching edges.

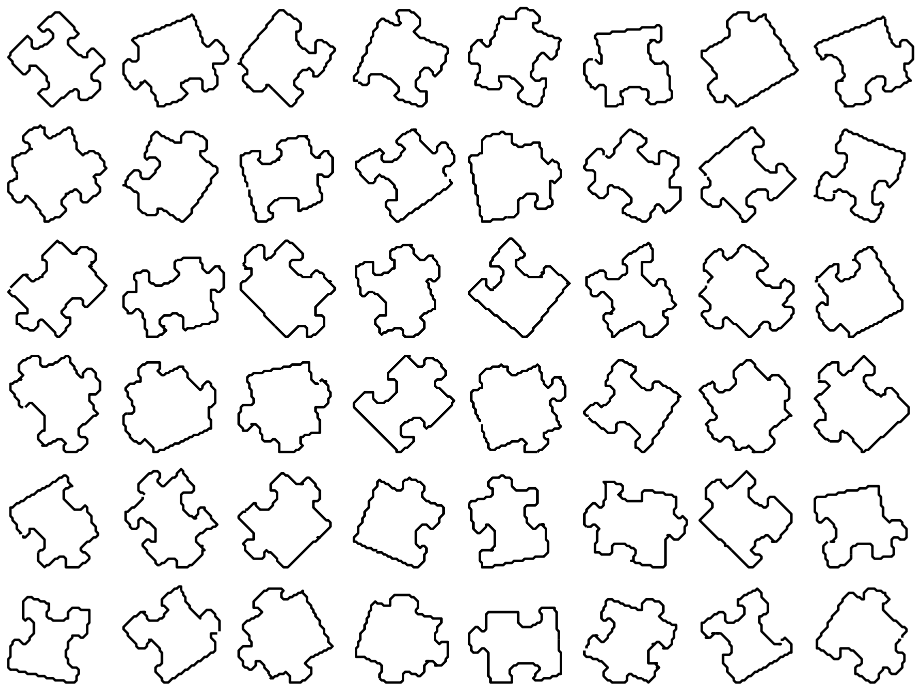
The Baffler Jigsaw Puzzle



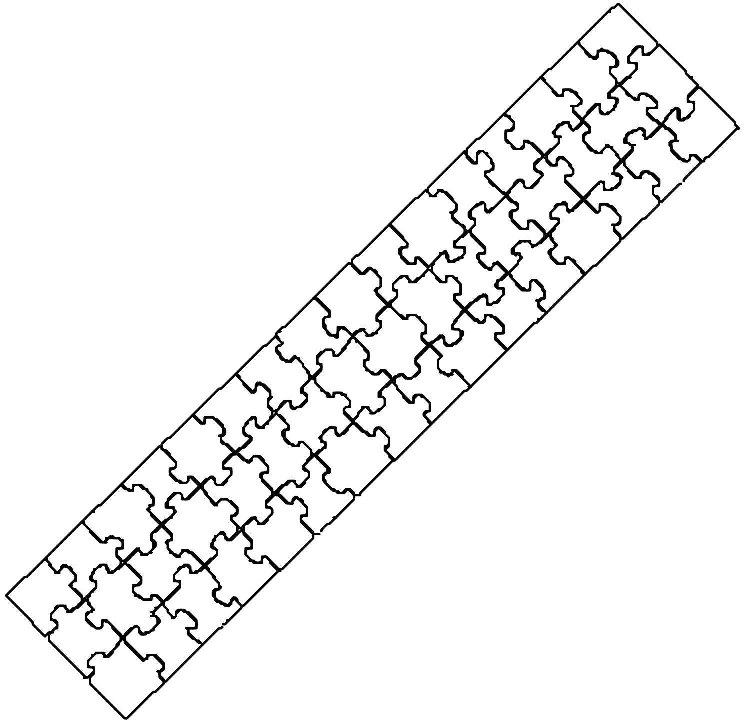
The Baffler Solved



The Rain Forest Giant Floor Puzzle

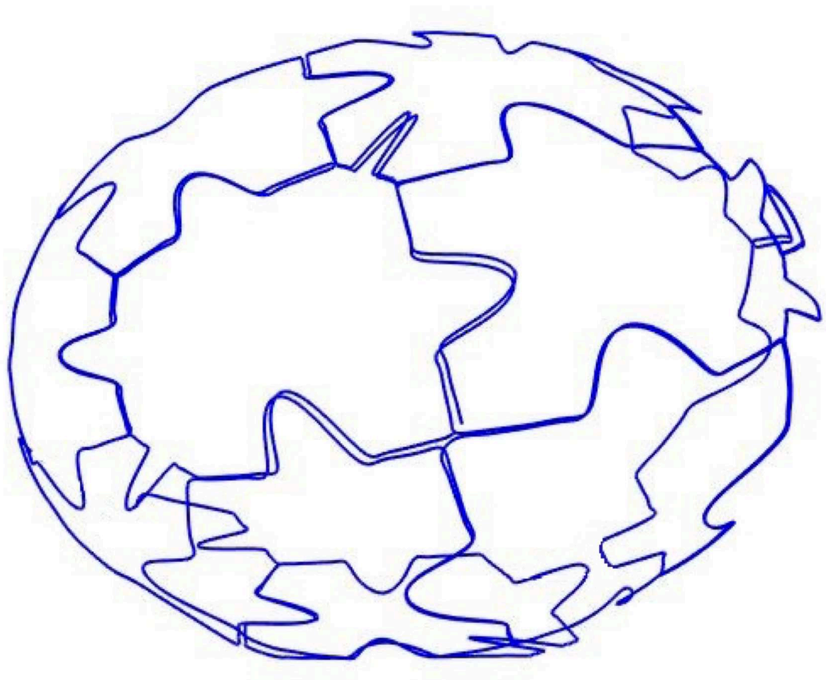


The Rain Forest Puzzle Solved



⟹ Dan Hoff & PJO, Automatic solution of jigsaw puzzles, *J. Math. Imaging Vision*, to appear.

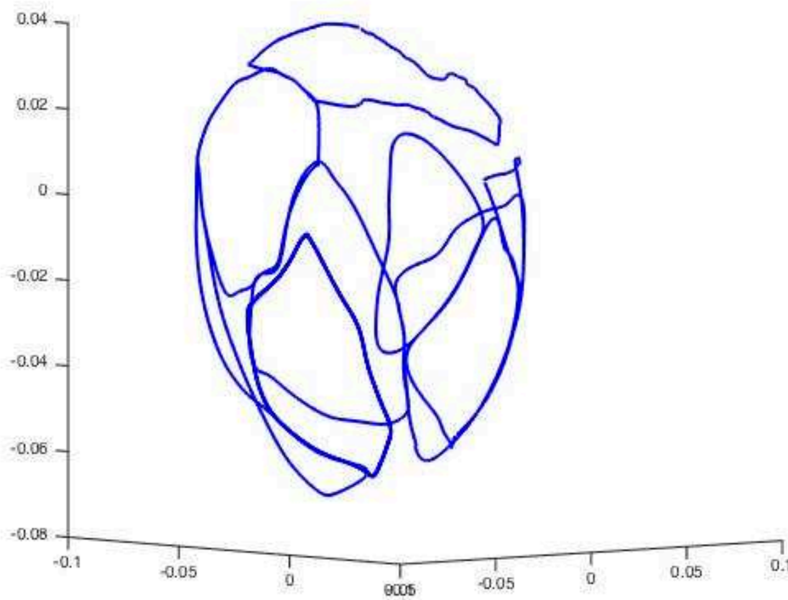
3D Jigsaw Puzzles



$\Longrightarrow$ Anna Grim, Tim O'Connor, Ryan Schlecta

Cheri Shakiban, Rob Thompson, PJO

Reassembling Humpty Dumpty



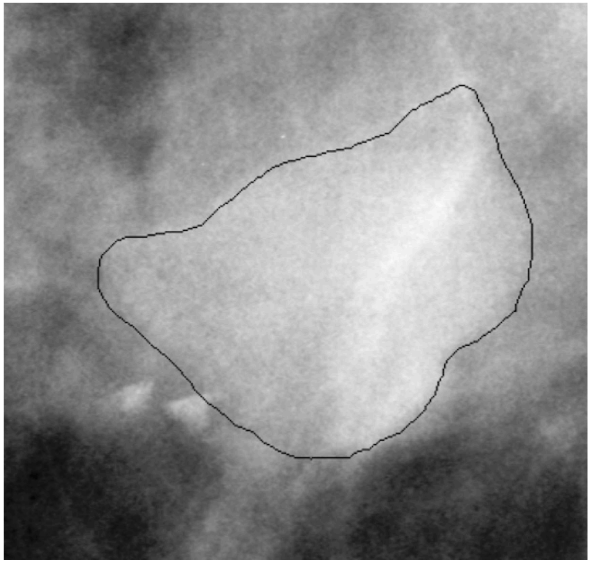
$\Longrightarrow$ Broken ostrich egg shell — Marshall Bern

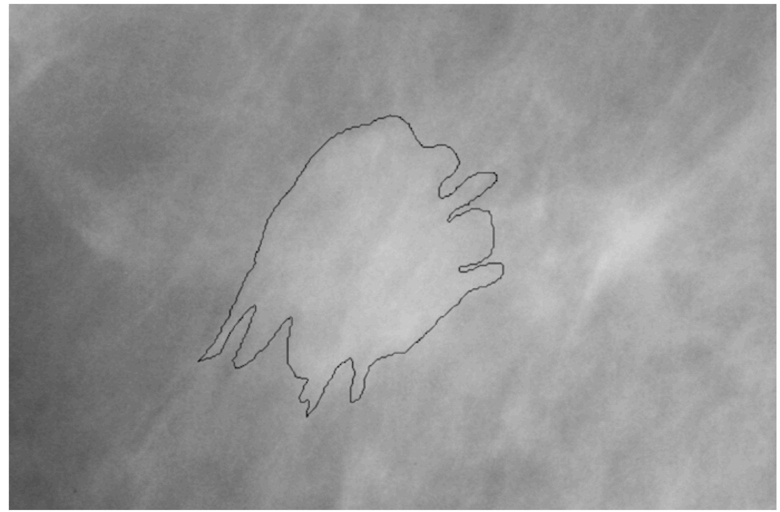
Archaeology

$\Longrightarrow$ **Virtual Archaeology**

Surgery

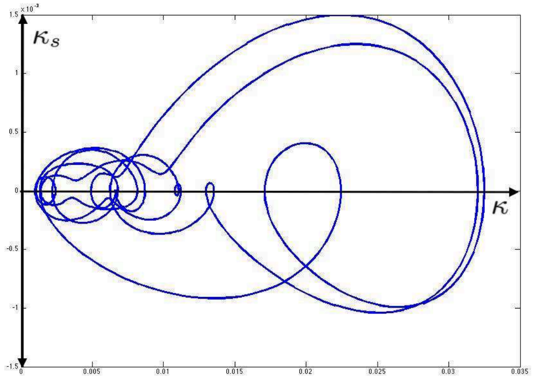
Benign vs. Malignant Tumors

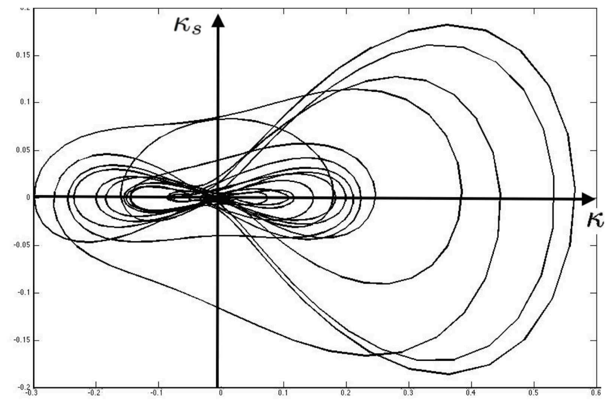




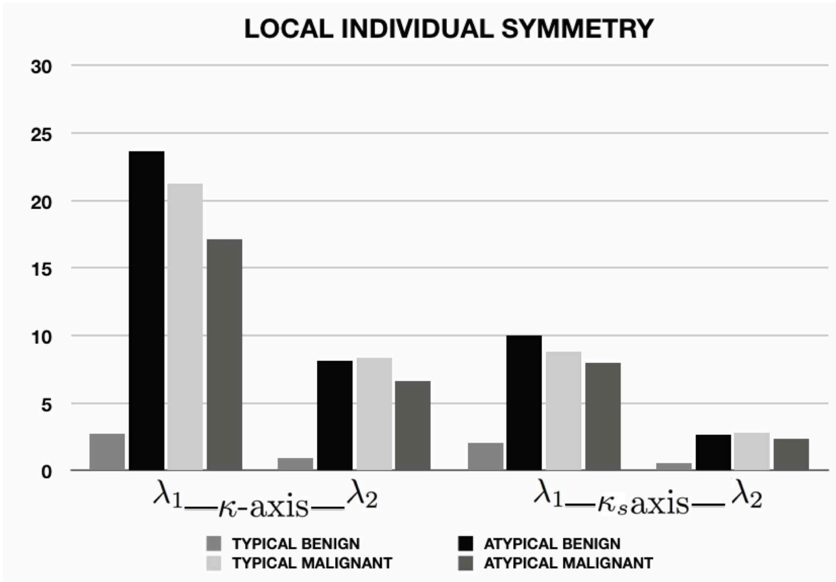
$\Longrightarrow$ Anna Grim, Cheri Shakiban

Benign vs. Malignant Tumors





Benign vs. Malignant Tumors



Noise Resistant Signatures

Use lower order invariants to construct a signature:

- joint invariants
- joint differential invariants
- integral invariants
- topological invariants
- . . .

Joint Euclidean Signature

For the Euclidean group $G = \mathrm{SE}(2)$ acting on curves $\mathcal{C} \subset \mathbb{R}^2$ (or $\mathbb{R}^3$) we need at least four points

$$z_0, z_1, z_2, z_3 \in \mathcal{C}$$

to form a joint signature.

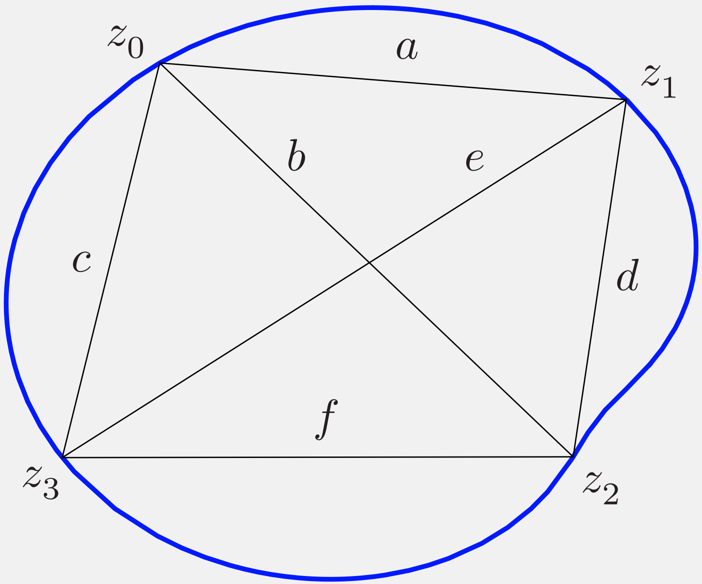
Joint invariants:

$$a = \| z_0 - z_1 \| \qquad b = \| z_0 - z_2 \| \qquad c = \| z_0 - z_3 \|$$

$$d = \| z_1 - z_2 \| \qquad e = \| z_1 - z_3 \| \qquad f = \| z_2 - z_3 \|$$

$\Longrightarrow$ six functions of four variables

Four-Point Euclidean Joint Signature



Joint Euclidean Signature: $\Sigma : \mathcal{C}^{\times 4} \longrightarrow \Sigma \subset \mathbb{R}^6$

$\dim \Sigma = 4 \quad \Longrightarrow \quad \exists$ two syzygies

$$\Phi_1(a,b,c,d,e,f) = 0 \qquad \Phi_2(a,b,c,d,e,f) = 0$$

Universal Cayley–Menger syzygy:

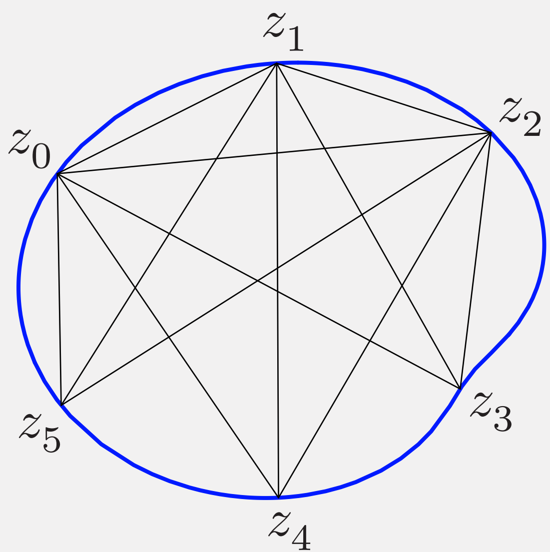
$$\det \begin{vmatrix} 2\,a^2 & a^2+b^2-d^2 & a^2+c^2-e^2 \\ a^2+b^2-d^2 & 2\,b^2 & b^2+c^2-f^2 \\ a^2+c^2-e^2 & b^2+c^2-f^2 & 2\,c^2 \end{vmatrix} = 0$$

$$\Longleftrightarrow \quad \mathcal{C} \subset \mathbb{R}^2$$

Joint Equi–Affine Signature

Requires 7 triangular areas:

$$[\,0\ 1\ 2\,],\ [\,0\ 1\ 3\,],\ [\,0\ 1\ 4\,],\ [\,0\ 1\ 5\,],\ [\,0\ 2\ 3\,],\ [\,0\ 2\ 4\,],\ [\,0\ 2\ 5\,]$$



Joint Invariant Signatures

- The joint invariant signature subsumes other signatures, but resides in a higher dimensional space and contains a lot of redundant information.
- Identification of landmarks can significantly reduce the redundancies
- Includes the differential invariant signature and joint differential invariant signatures as its "coalescent boundaries".
- Invariant numerical approximations to differential invariants and semi-differential invariants are constructed (using moving frames) near these coalescent boundaries.
- Integral invariants are alternative "projections" thereof

The Distance Histogram

Definition. The distance histogram of a finite set of points $P = \{z_1, \ldots, z_n\} \subset V$ is the function

$$\eta_P(r) = \#\left\{ (i,j) \;\middle|\; 1 \le i < j \le n,\ d(z_i, z_j) = r \right\}.$$

The Distance Set

The support of the histogram function,

$$\operatorname{supp} \eta_P = \Delta_P \subset \mathbb{R}^+$$

is the distance set of P.

Erdös' distinct distances conjecture (1946):

If $P \subset \mathbb{R}^m$, then $\# \Delta_P \geq c_{m,\varepsilon} (\# P)^{2/m-\varepsilon}$

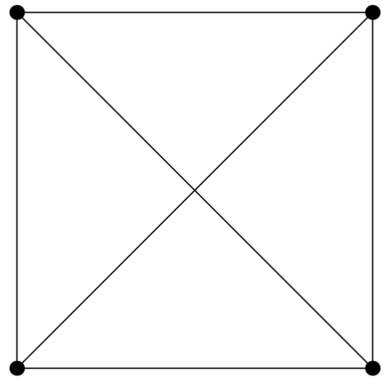
Characterization of Point Sets

Note: If $\widetilde{P} = g \cdot P$ is obtained from $P \subset \mathbb{R}^m$ by a rigid motion $g \in \mathrm{E}(n)$, then they have the same distance histogram: $\eta_P = \eta_{\widetilde{P}}$.

Question: Can one uniquely characterize, up to rigid motion, a set of points $P\{z_1, \ldots, z_n\} \subset \mathbb{R}^m$ by its distance histogram?

$\Longrightarrow$ Tinkertoy problem.

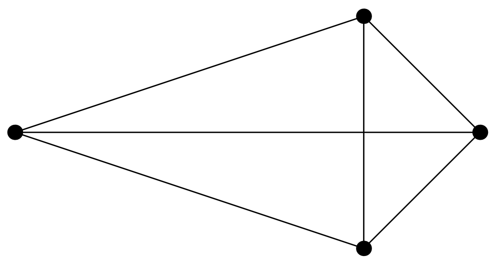
Yes:



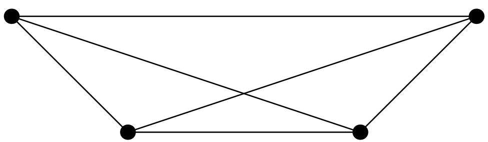
$$\eta = 1, \; 1, \; 1, \; 1, \; \sqrt{2}, \; \sqrt{2}.$$

No:

Kite



Trapezoid



$$\eta = \sqrt{2}, \quad \sqrt{2}, \quad 2, \quad \sqrt{10}, \quad \sqrt{10}, \quad 4.$$

No:

$$\left.\begin{aligned} P &= \{0, 1, 4, 10, 12, 17\} \\ Q &= \{0, 1, 8, 11, 13, 17\} \end{aligned}\right. \subset \mathbb{R}$$

$$\eta = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 17$$

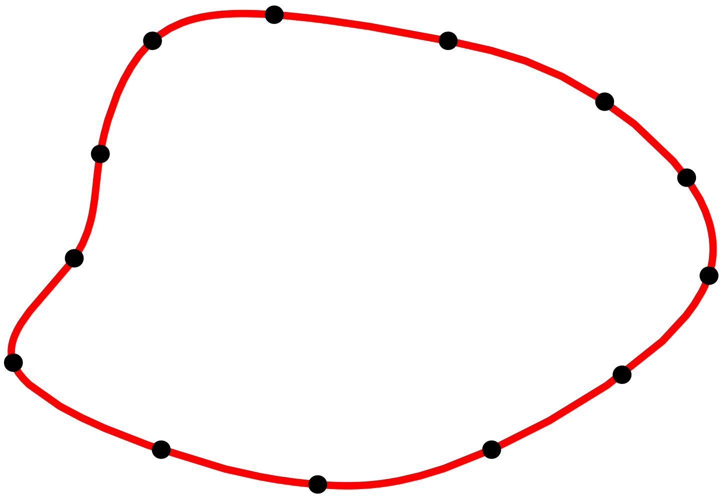
$\Longrightarrow$ G. Bloom, *J. Comb. Theory, Ser. A* **22** (1977) 378–379

Characterizing Point Sets by their Distance Histograms

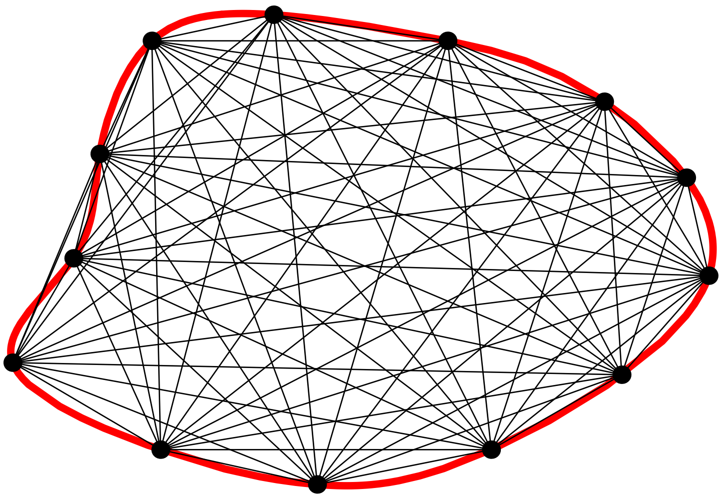
Theorem. Suppose $n \leq 3$ or $n \geq m+2$.
Then there is a Zariski dense open subset in the space of n point configurations in $\mathbb{R}^m$ that are uniquely characterized, up to rigid motion, by their distance histograms.

$\Longrightarrow$ M. Boutin & G. Kemper, *Adv. Appl. Math.* **32** (2004) 709–735

Limiting Curve Histogram



Limiting Curve Histogram



Sample Point Histograms

Cumulative distance histogram: $\quad n = \#P$:

$$\Lambda_P(r) = \frac{1}{n} + \frac{2}{n^2} \sum_{s \leq r} \eta_P(s) = \frac{1}{n^2} \# \left\{ (i,j) \;\middle|\; d(z_i, z_j) \leq r \right\},$$

Note:

$$\eta_P(r) = \tfrac{1}{2} n^2 [\, \Lambda_P(r) - \Lambda_P(r - \delta) \,] \qquad \delta \ll 1.$$

Local cumulative distance histogram:

$$\lambda_P(r, z) = \frac{1}{n} \# \left\{ j \;\middle|\; d(z, z_j) \leq r \right\} = \frac{1}{n} \#(P \cap B_r(z))$$

$$\Lambda_P(r) = \frac{1}{n} \sum_{z \in P} \lambda_P(r, z) = \frac{1}{n^2} \sum_{z \in P} \#(P \cap B_r(z)).$$

Ball of radius r centered at z:

$$B_r(z) = \{ v \in V \mid d(v, z) \leq r \}$$

Limiting Curve Histogram Functions

Length of a curve

$$l(C) = \int_C ds < \infty$$

Local curve distance histogram function

$$h_C(r, z) = \frac{l(C \cap B_r(z))}{l(C)}$$

$\Longrightarrow$ The fraction of the curve contained in the ball of radius r centered at z.

Global curve distance histogram function:

$$H_C(r) = \frac{1}{l(C)} \int_C h_C(r, z(s))\, ds.$$

Convergence of Histograms

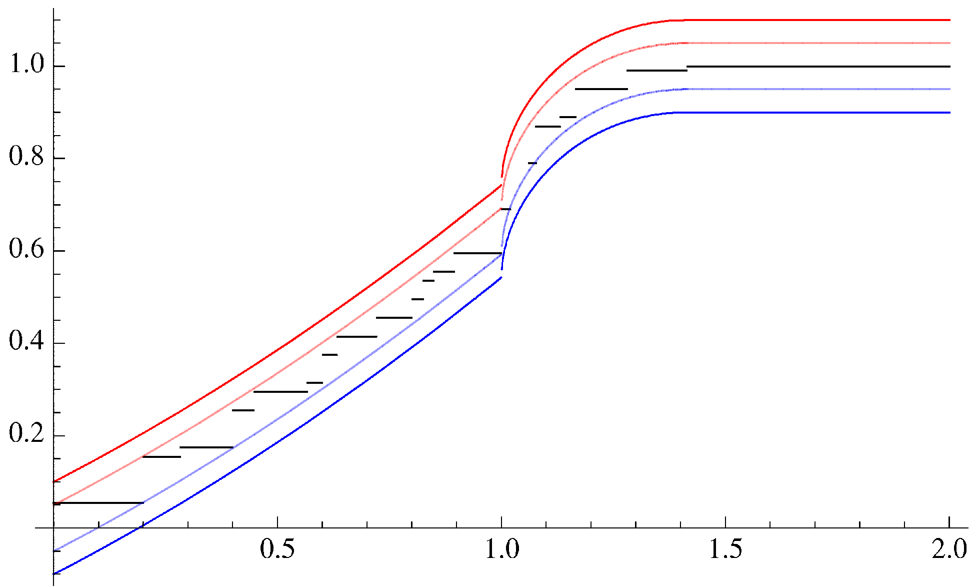
Theorem. Let C be a regular plane curve. Then, for both uniformly spaced and randomly chosen sample points $P \subset C$, the cumulative local and global histograms converge to their continuous counterparts:

$$\lambda_P(r,z) \longrightarrow h_C(r,z), \qquad \Lambda_P(r) \longrightarrow H_C(r),$$

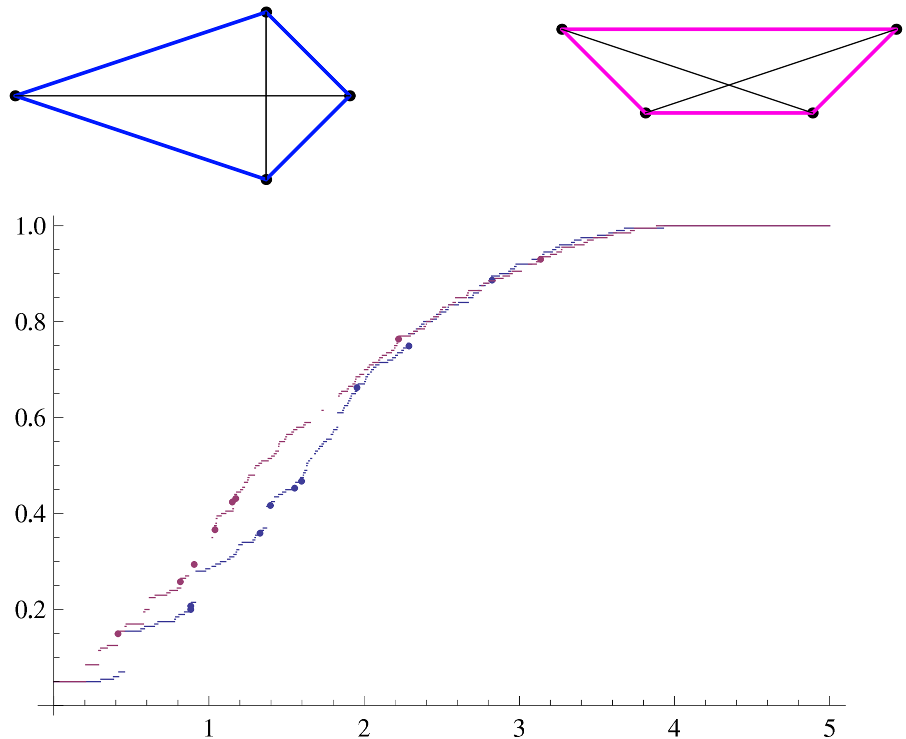
as the number of sample points goes to infinity.

Dan Brinkman & PJO, Invariant histograms,
Amer. Math. Monthly **118** (2011) 2–24.

Square Curve Histogram with Bounds



Kite and Trapezoid Curve Histograms

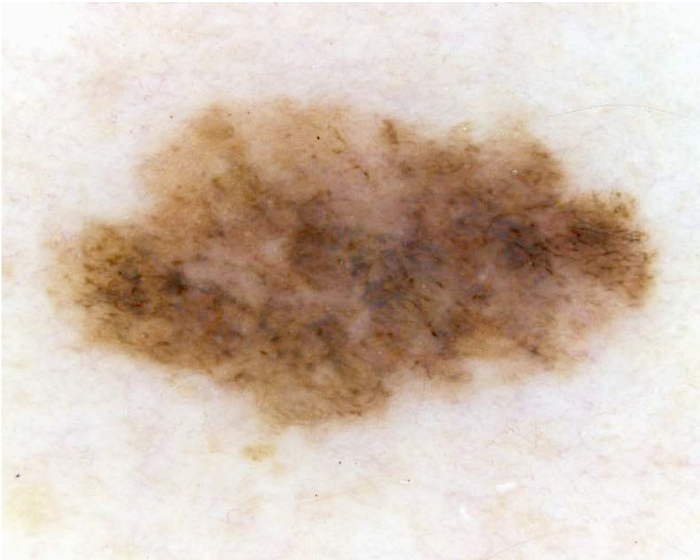


Histogram–Based Shape Recognition

500 sample points

Shape	(a)	(b)	(c)	(d)	(e)	(f)
(a) triangle	2.3	20.4	66.9	81.0	28.5	76.8
(b) square	28.2	.5	81.2	73.6	34.8	72.1
(c) circle	66.9	79.6	.5	137.0	89.2	138.0
(d) 2×3 rectangle	85.8	75.9	141.0	2.2	53.4	9.9
(e) 1×3 rectangle	31.8	36.7	83.7	55.7	4.0	46.5
(f) star	81.0	74.3	139.0	9.3	60.5	.9

Distinguishing Melanomas from Moles

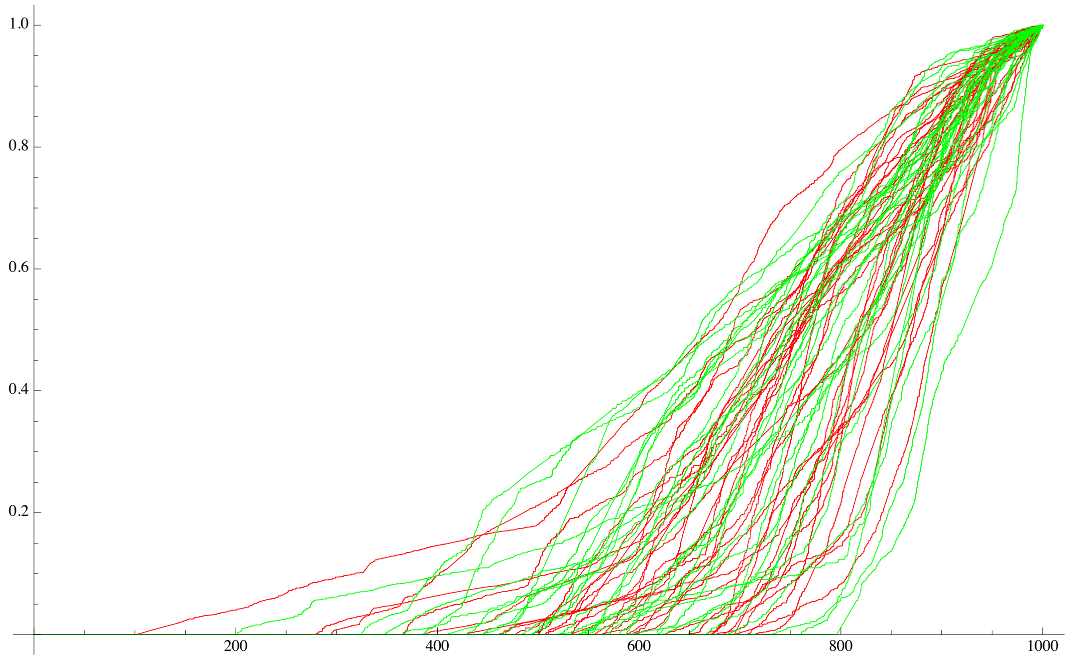


Melanoma

Mole

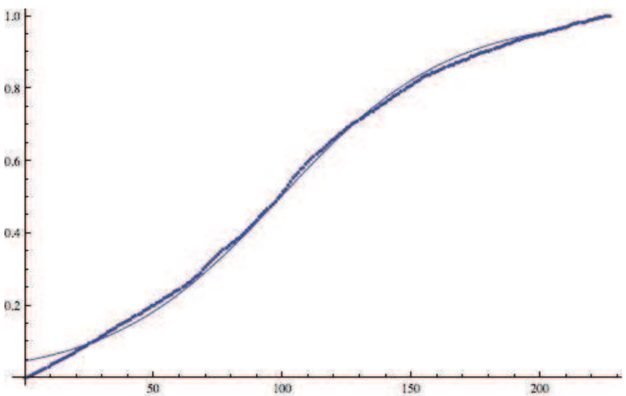
$\Longrightarrow$ A. Rodriguez, J. Stangl, C. Shakiban

Cumulative Global Histograms

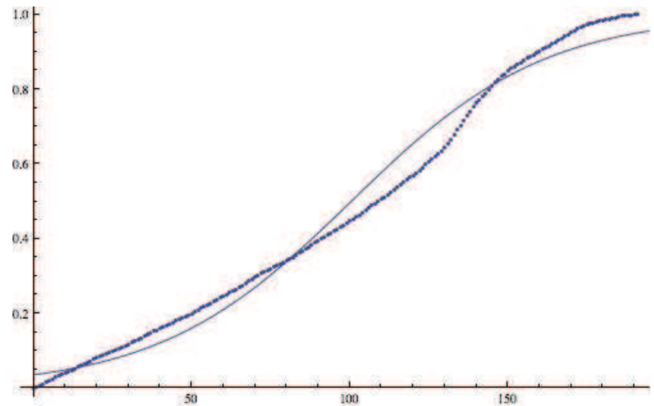


Red: melanoma Green: mole

Logistic Function Fitting

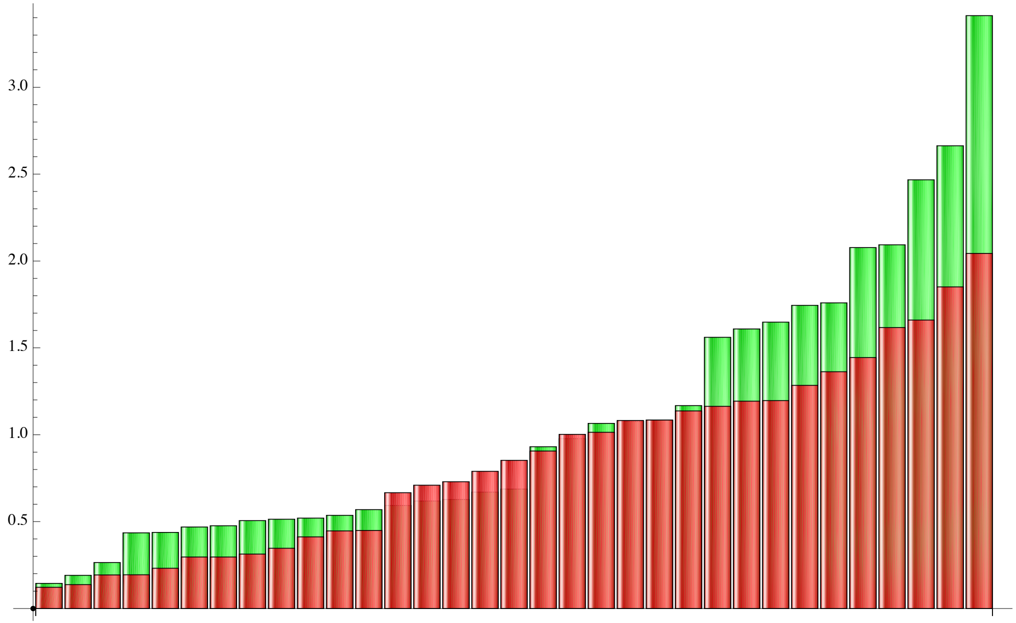


Melanoma



Mole

Logistic Function Fitting — Residuals



$$\text{Melanoma} = 17.1336 \ \pm \ 1.02253$$

$$\text{Mole} = 19.5819 \ \pm \ 1.42892$$

58.7% Confidence

Curve Histogram Conjecture

Two sufficiently regular plane curves C and $\widetilde{C}$ have identical global distance histogram functions, so $H_C(r) = H_{\widetilde{C}}(r)$ for all $r \geq 0$, if and only if they are rigidly equivalent: $C \simeq \widetilde{C}$.

Possible Proof Strategies

- Show that any polygon obtained from (densely) discretizing a curve does not lie in the Boutin–Kemper exceptional set.
- Polygons with obtuse angles: taking r small, one can recover (i) the set of angles and (ii) the shortest side length from $H_C(r)$. Further increasing r leads to further geometric information about the polygon ...
- Expand $H_C(r)$ in a Taylor series at $r = 0$ and show that the corresponding integral invariants characterize the curve.

Taylor Expansions

Local distance histogram function:

$$L\, h_C(r, z) = 2r + \tfrac{1}{12}\kappa^2 r^3 + \left(\tfrac{1}{40}\kappa\kappa_{ss} + \tfrac{1}{45}\kappa_s^2 + \tfrac{3}{320}\kappa^4\right) r^5 + \cdots .$$

Global distance histogram function:

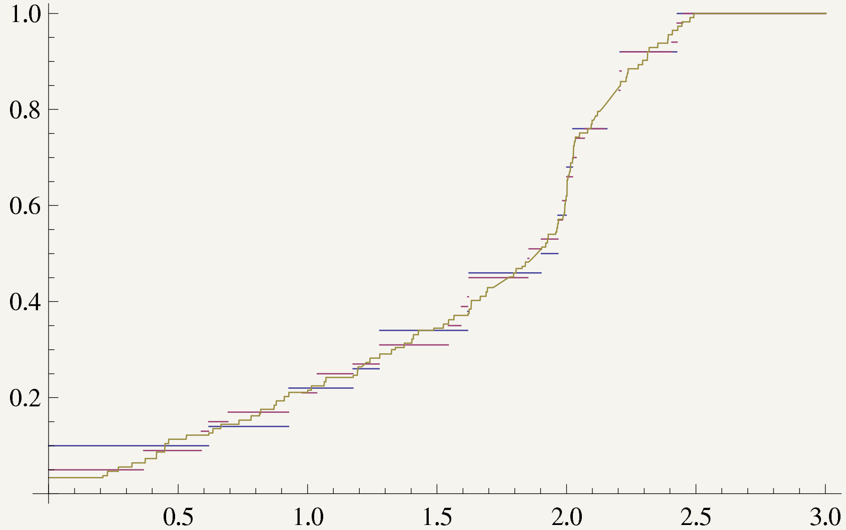
$$H_C(r) = \frac{2r}{L} + \frac{r^3}{12L^2}\oint_C \kappa^2\, ds + \frac{r^5}{40L^2}\oint_C \left(\tfrac{3}{8}\kappa^4 - \tfrac{1}{9}\kappa_s^2\right) ds + \cdots .$$

Space Curves

Saddle curve:

$$z(t) = (\cos t, \sin t, \cos 2t), \qquad 0 \leq t \leq 2\pi.$$

Convergence of global curve distance histogram function:

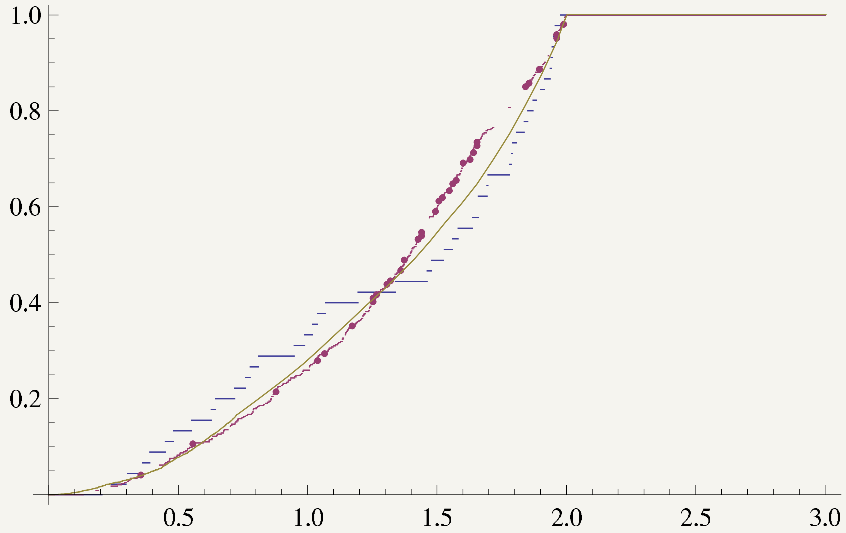


Surfaces

Local and global surface distance histogram functions:

$$h_S(r, z) = \frac{\text{area}\,(S \cap B_r(z))}{\text{area}\,(S)}, \qquad H_S(r) = \frac{1}{\text{area}\,(S)} \iint_S h_S(r, z)\, dS.$$

Convergence for sphere:



Area Histograms

Rewrite global curve distance histogram function:

$$H_C(r) = \frac{1}{L} \oint_C h_C(r, z(s))\, ds = \frac{1}{L^2} \oint_C \oint_C \chi_r(d(z(s), z(s'))\, ds\, ds'$$

$$\text{where} \qquad \chi_r(t) = \begin{cases} 1, & t \leq r, \\ 0, & t > r, \end{cases}$$

Global curve area histogram function:

$$A_C(r) = \frac{1}{L^3} \oint_C \oint_C \oint_C \chi_r(\text{area}\,(z(\hat{s}), z(\hat{s}'), z(\hat{s}''))\, d\,\hat{s}\, d\,\hat{s}'\, d\,\hat{s}'',$$

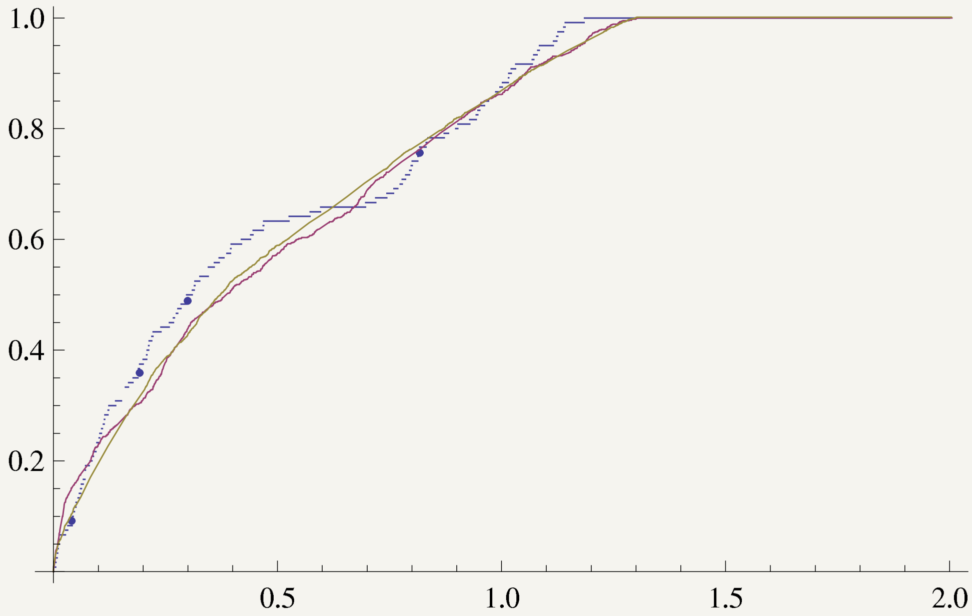
$d\,\hat{s}$ — equi-affine arc length element $\qquad L = \int_C d\,\hat{s}$

Discrete cumulative area histogram

$$A_P(r) = \frac{1}{n(n-1)(n-2)} \sum_{z \neq z' \neq z'' \in P} \chi_r(\text{area}\,(z, z', z'')),$$

Boutin & Kemper: The area histogram uniquely determines generic point sets $P \subset \mathbb{R}^2$ up to equi-affine motion.

Area Histogram for Circle



★ ★ Joint invariant histograms — convergence???

Triangle Distance Histograms

$Z = (\dots z_i \dots) \subset M$ —
sample points on a subset $M \subset \mathbb{R}^n$ (curve, surface, etc.)

$T_{i,j,k}$ — triangle with vertices z_i, z_j, z_k.

Side lengths:

$$\sigma(T_{i,j,k}) = (\, d(z_i, z_j), d(z_i, z_k), d(z_j, z_k) \,)$$

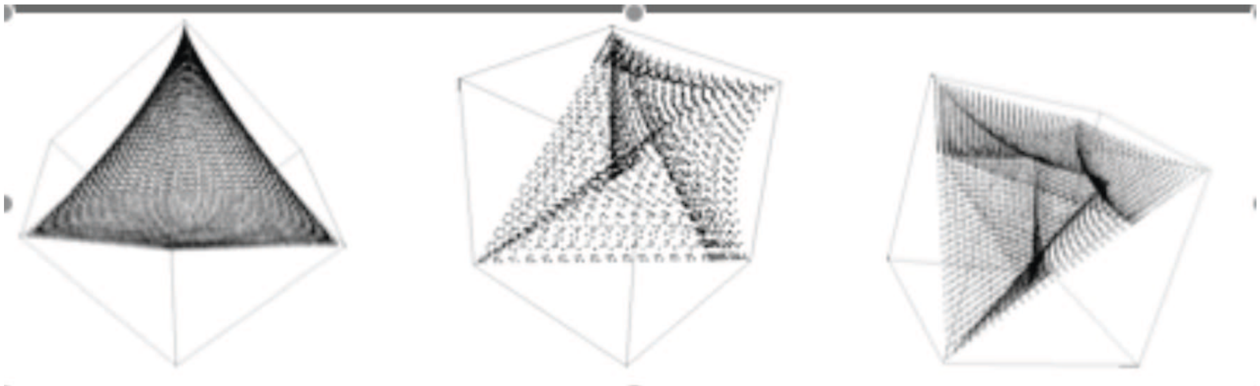
Discrete triangle histogram:

$$\mathcal{S} = \sigma(\mathcal{T}) \subset K$$

Triangle inequality cone:

$$K = \{ \, (x, y, z) \mid \; x, y, z \geq 0, \; x + y \geq z, \; x + z \geq y, \; y + z \geq x \, \} \subset \mathbb{R}^3.$$

Triangle Histogram Distributions



Circle Triangle Square

Convergence to measures ...

$\Longrightarrow$ Madeleine Kotzagiannidis

Practical Object Recognition

- Scale-invariant feature transform (SIFT) (Lowe)
- Shape contexts (Belongie–Malik–Puzicha)
- Integral invariants (Krim, Kogan, Yezzi, Pottman, . . .)
- Shape distributions (Osada–Funkhouser–Chazelle–Dobkin)

  Surfaces: distances, angles, areas, volumes, etc.
- Gromov–Hausdorff and Gromov-Wasserstein distances (Mémoli)

  $\Longrightarrow$ lower bounds & stability