Origins and Applications of Signatures

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In the beginning ...

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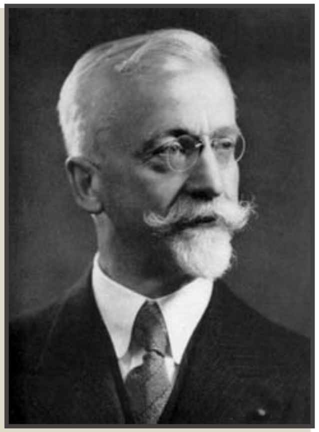


The Cartan Equivalence Method

⋆ Élie Cartan, Les sous-groupes des groupes continus de transformations, *Ann. Sci. École Normale Supérieur*, 3e sér., **25** (1908), 57–194.

LES SOUS-GROUPES

DES

GROUPES CONTINUS DE TRANSFORMATIONS;

PAR M. E. CARTAN.

Ce Mémoire peut être considéré comme une suite au Mémoire précédemment paru en deux Parties dans ces mêmes *Annales* (1), où est exposée une théorie de la structure des groupes continus de transformations s'appliquant aussi bien aux groupes infinis qu'aux groupes finis. Dans la théorie classique de S. Lie, la structure d'un groupe fini est définie par ce qu'il appelle les *constantes de structure*, et ces constantes s'introduisent lorsqu'on compose entre elles les transformations infinitésimales du groupe; c'est donc la notion de transformation infinitésimale qui est à la base de cette théorie classique de la structure; mais en restant à ce point de vue cette théorie devait se borner aux groupes finis et il a été impossible de l'étendre aux groupes infinis. Au contraire, dans la théorie que j'ai proposée, on prend pour point de départ les *équations de définition* des équations finies du groupe et ce sont ces équations de définition qui donnent naissance à des constantes que j'appelle les *constantes de structure* du groupe; ces

(1) E. CARTAN, *Sur la structure des groupes infinis de transformations* (*Annales de l'École Normale*, 3e série, t. XXI, 1904, p. 153-156; 3e série, t. XXII, 1905, p. 219-308).

à 1; l'un d'eux, par exemple, donne naissance jusqu'à 98 groupes différents. D'ailleurs l'énumération complète ne semble pas devoir présenter un grand intérêt, elle ne fournirait aucun groupe simple transitif nouveau; mais peut-être y aurait-il lieu d'étudier, parmi les groupes intransitifs, ceux que j'ai appelés improprement *simples* et qui semblent rendre difficile le problème si important de la réduction d'un groupe à une série normale de sous-groupes (1).

CHAPITRE I.

LE PROBLÈME GÉNÉRAL DE L'ÉQUIVALENCE.

1. Considérons deux systèmes de n expressions aux différentielles totales linéairement indépendantes à n variables : l'un aux n variables $x_1, x_2, \ldots, x_n$,

$$(1)\qquad \begin{cases} \omega_1 = a_{11}\,dx_1 + a_{12}\,dx_2 + \ldots + a_{1n}\,dx_n, \\ \omega_2 = a_{21}\,dx_1 + a_{22}\,dx_2 + \ldots + a_{2n}\,dx_n, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \omega_n = a_{n1}\,dx_1 + a_{n2}\,dx_2 + \ldots + a_{nn}\,dx_n; \end{cases}$$

l'autre aux n variables $X_1, X_2, \ldots, X_n$,

$$(2)\qquad \begin{cases} \Omega_1 = A_{11}\,dX_1 + A_{12}\,dX_2 + \ldots + A_{1n}\,dX_n, \\ \Omega_2 = A_{21}\,dX_1 + A_{22}\,dX_2 + \ldots + A_{2n}\,dX_n, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \Omega_n = A_{n1}\,dX_1 + A_{n2}\,dX_2 + \ldots + A_{nn}\,dX_n; \end{cases}$$

les a_{ik} désignant des fonctions données des x et les A_{ik} des fonctions données des X. Le problème que nous allons résoudre est celui de *reconnaître si l'on peut trouver pour les X des fonctions indépendantes*

(1) *Voir* E. CARTAN, *Annales de l'École Normale*, 3e série, t. XXII, 1905, p. 284-285. *Cf.* la Note, parue postérieurement à la rédaction de ce Mémoire, dans les *Comptes rendus de l'Académie des Sciences de Paris*, t. CXLIV, 21 mai 1907, sous le titre : *Les groupes de transformations continus, infinis, simples*.

Some (Personal) History

Cartan's remarkable solution to the general equivalence problem relied on his theory of exterior differential systems (EDS), including the Cartan–Kähler Existence Theorem.

Owing to its difficulty, it remained under-appreciated and rarely used, except by some of his disciples such as S.S. Chern, R. Debever, M. Kuranishi, and D.C. Spencer.

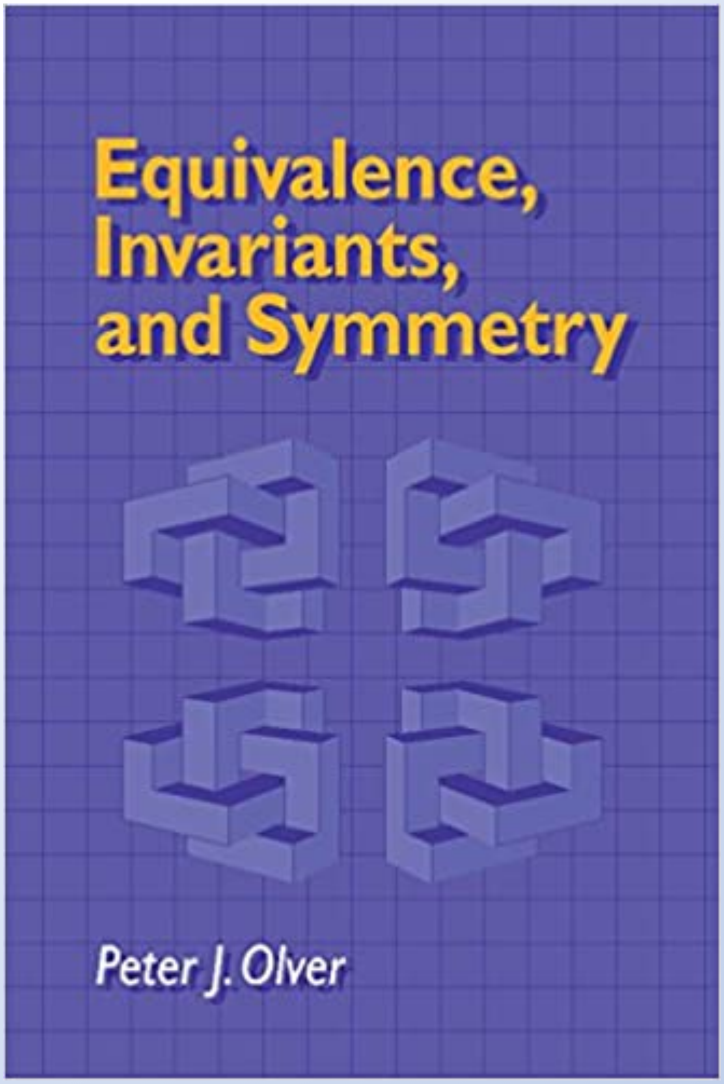
In the 1980's, several researchers, notably Robby Gardner, Robert Bryant, Niky Kamran, and their collaborators and students, realized that the Cartan equivalence method could be made algorithmic and had significant potential in applications, particularly to equivalence problems arising in ordinary and partial differential equations, the calculus of variations, differential operators, including those arising in quantum mechanics, and control theory.

Through attendance at meetings and interactions with them, I also became convinced of the potential of the equivalence method, and ended up writing a series of papers with Niky Kamran on the topic.

My first individual success was applying it to the basic equivalence problem of classical invariant theory using an observation that it was isomorphic to an already solved equivalence problem for first order variational problems.

After learning how to use and justify Cartan's methods, I was inspired (or tricked) to write my second book (1995). The theme was Lie versus Cartan, or, rather, reconciling Lie and Cartan. Of course, Cartan was directly inspired by Lie, but the two approaches had subsequently gone in rather different directions.

This was where the idea of a differential invariant signature, then called a "classifying manifold" first arose in my reformulation of Cartan's solution to the equivalence problem.



As I was putting the finishing touches on the book, my long time friend and, at that time, colleague Allen Tannenbaum convinced me of the importance of differential invariants in image processing and computer vision. We ended up writing a series of papers with Anthony Yezzi, Guillermo Sapiro, Satya Kichenassamy, and others on applications of Lie groups and differential invariants to issues in computer vision, particularly denoising and segmentation. This culminated in

Calabi, E., Olver, P.J., Shakiban, C., Tannenbaum, A., Haker, S., Differential and numerically invariant signature curves applied to object recognition, *Int. J. Computer Vision* **26** (1998), 107–135.

where we proposed the use of differential invariant signatures and their invariant numerical approximations for solving equivalence problems arising in image processing. The term signature was already in use in image processing, although not rigorously backed up by the Cartan machinery, and I chose to start using it in general.

And the rest is history . . .

The Basic Equivalence Problem

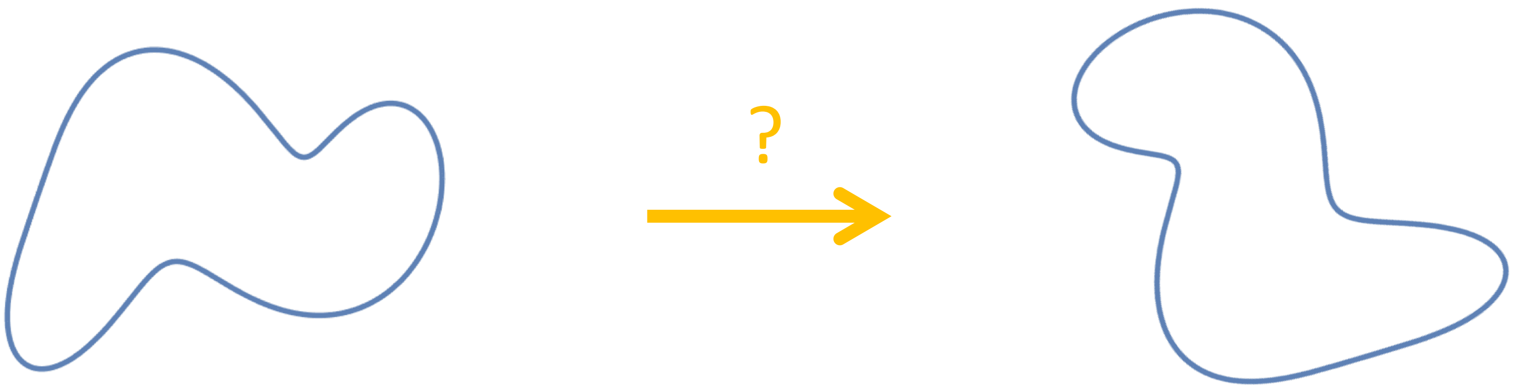
Given a transformation group acting on a space, determine when two subsets can be mapped to each other by a transformation in the group.

Symmetry

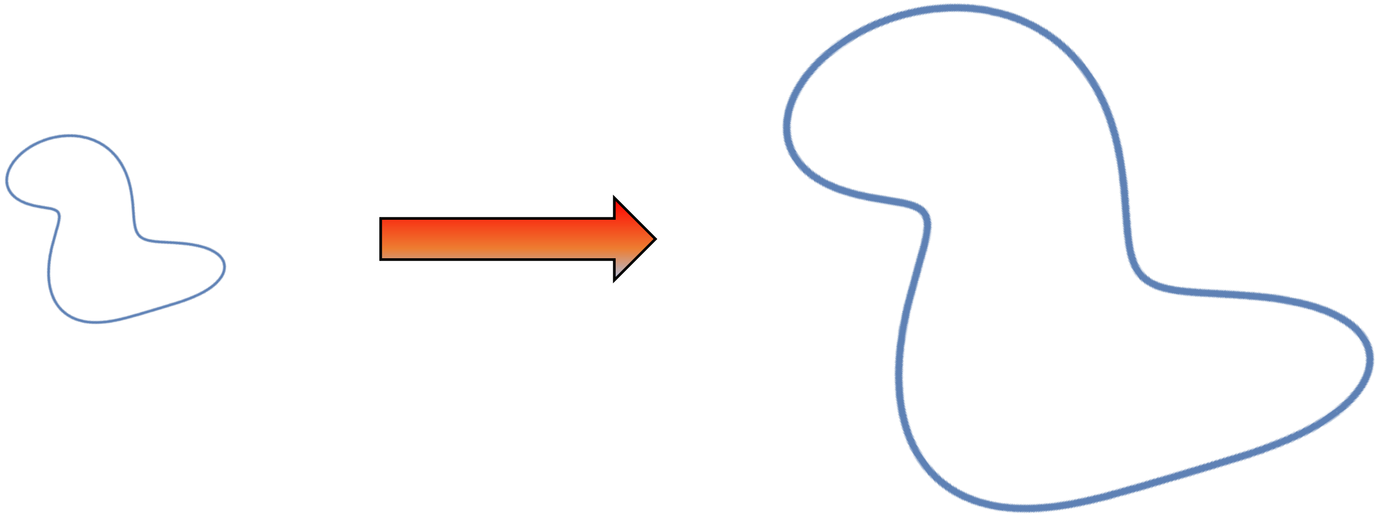
A symmetry of a subset is a self-equivalence.

Rigid equivalence

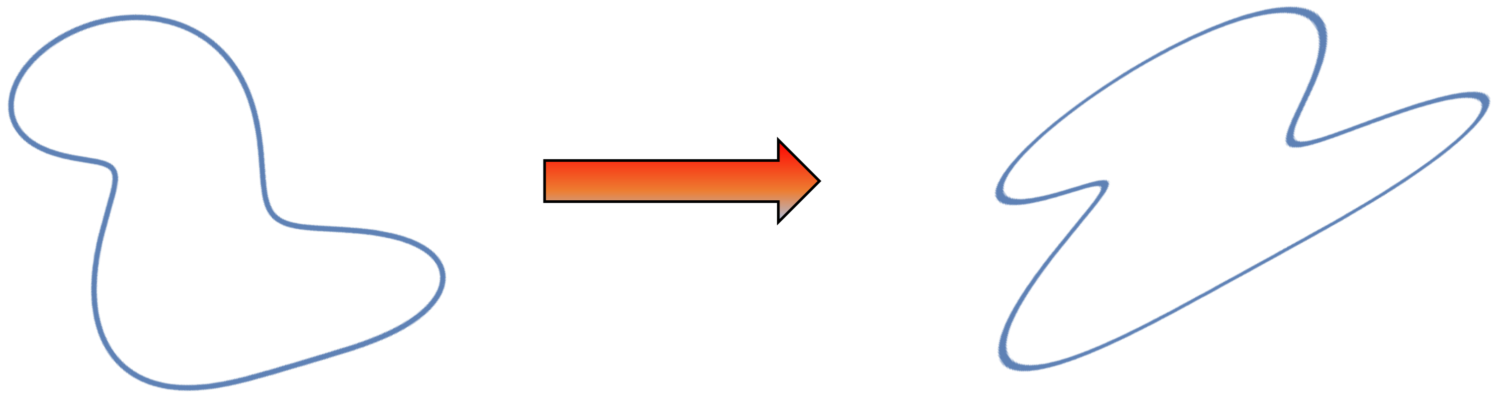
When are two shapes related by a rigid motion?



Scaling (similarity) equivalence



Projective and Equiaffine Equivalence

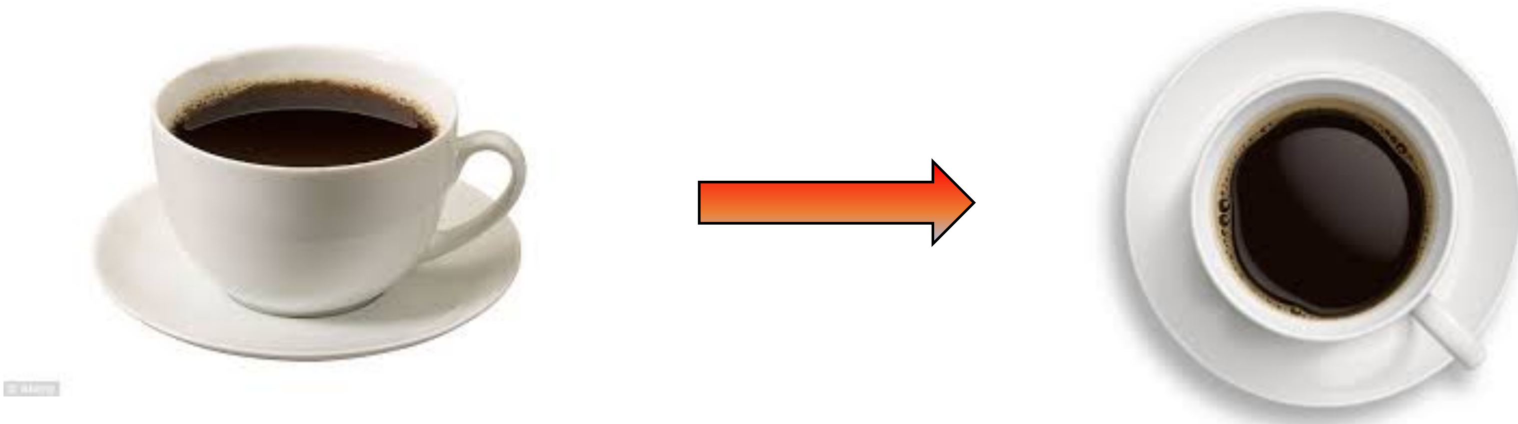


Transformation groups

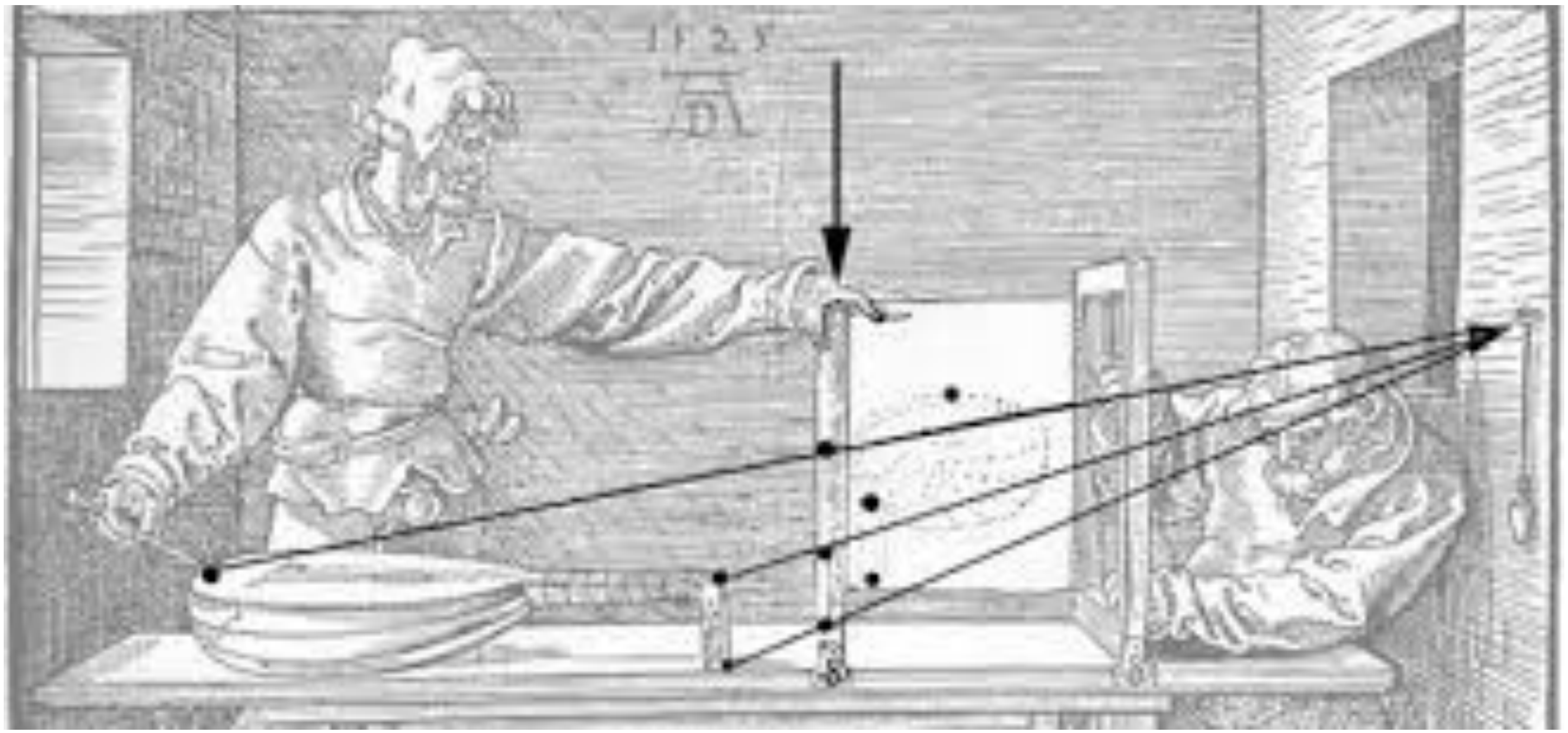
Projective Transformation

Transformation groups

Projective Transformation



Projective transformations in art and photography



Albrecht Durer — 1500

Tennis, anyone?

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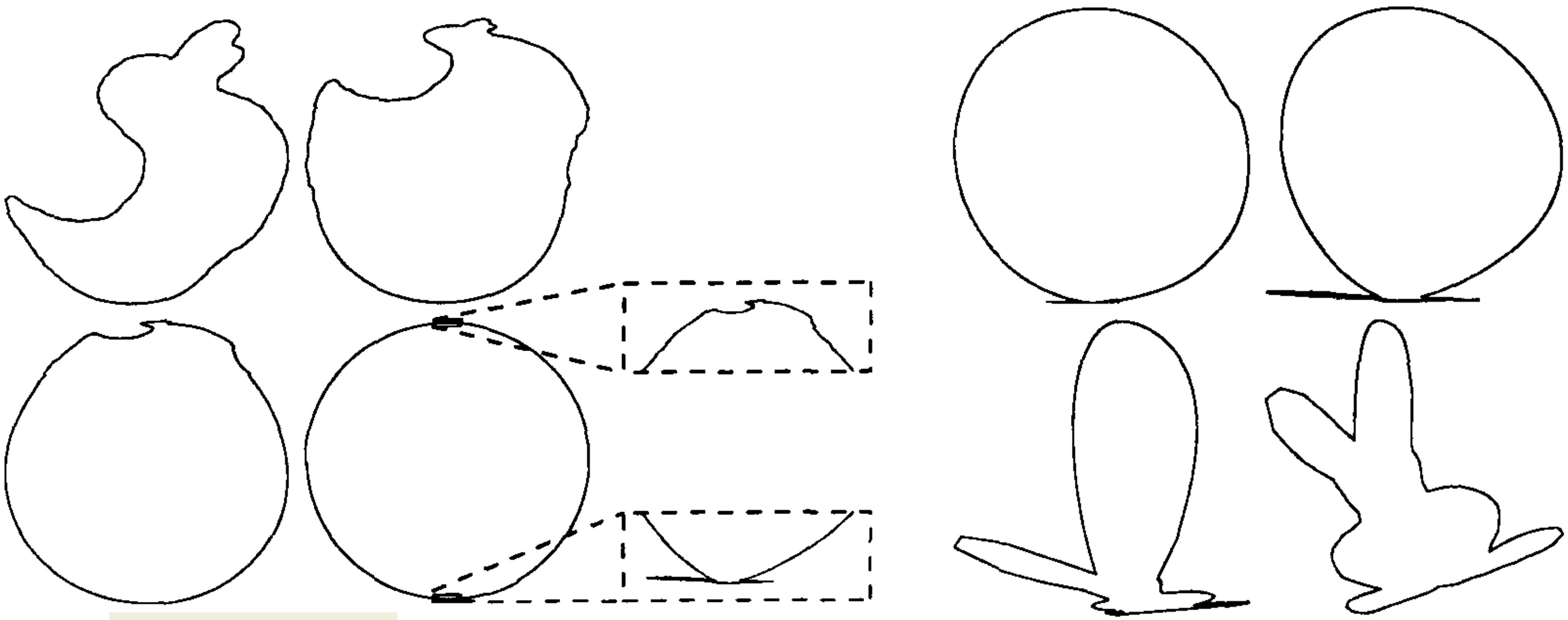
★ Projective or equi-affine equivalence & symmetry

Duck = Rabbit?

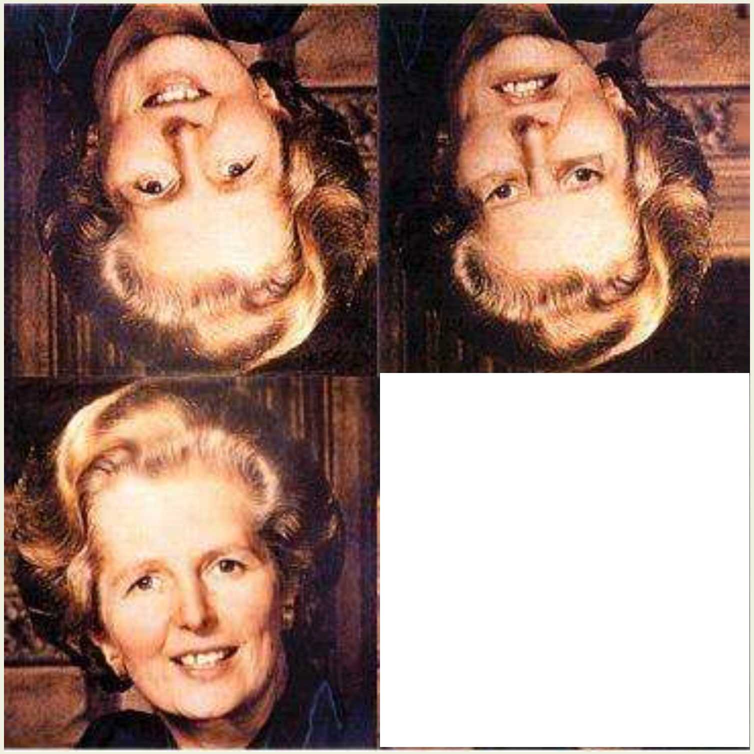
Limitations of Projective Equivalence



$\Longrightarrow$ K. Åström (1995)

Duck or Rabbit?

Thatcher Illusion

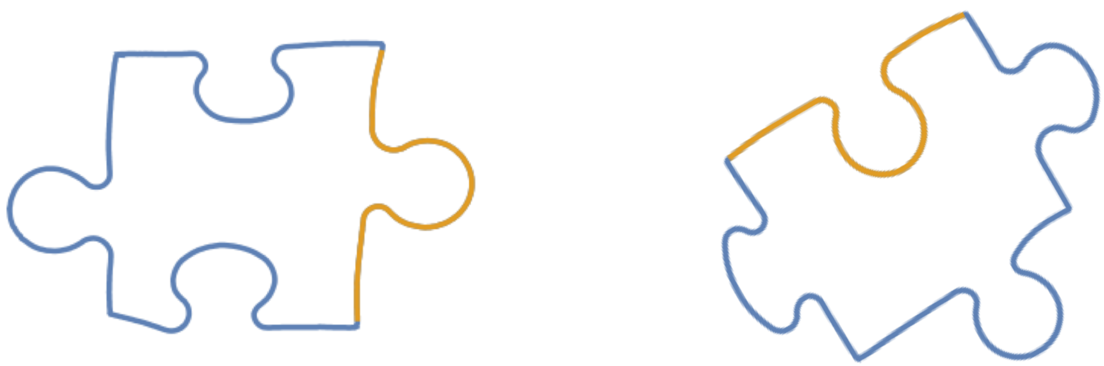


Thatcher Illusion

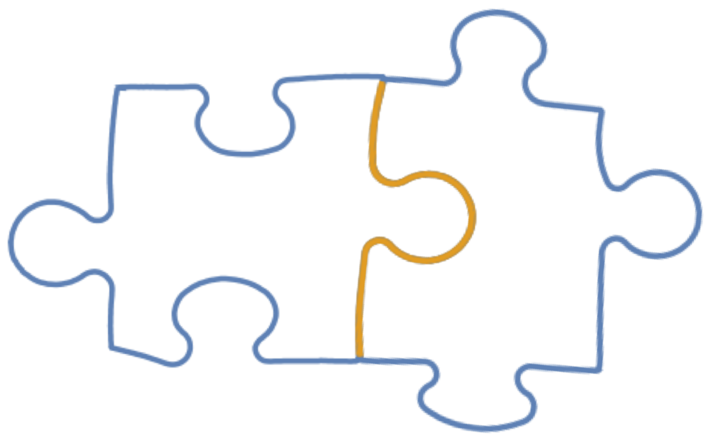
Thatcher Illusion

Local equivalence and symmetry — groupoids?

Local equivalence of puzzle pieces



Local equivalence of puzzle pieces



Classical Invariant Theory

Binary form:

$$Q(x) = \sum_{k=0}^{n} \binom{n}{k} a_k \, x^k$$

Equivalence of polynomials (binary forms):

$$Q(x) = (\gamma x + \delta)^n \, \overline{Q} \left(\frac{\alpha x + \beta}{\gamma x + \delta} \right) \qquad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2)$$

- multiplier representation of $\mathrm{GL}(2)$
- modular forms

$$Q(x) = (\gamma x + \delta)^n \, \overline{Q} \left(\frac{\alpha x + \beta}{\gamma x + \delta} \right)$$

Transformation group:

$$g \colon (x, u) \longmapsto \left(\frac{\alpha x + \beta}{\gamma x + \delta} \, , \, \frac{u}{(\gamma x + \delta)^n} \right)$$

Equivalence of functions $\iff$ equivalence of graphs

$$\Gamma_Q = \{ \, (x, u) = (x, Q(x)) \, \} \subset \mathbb{C}^2$$

$\Longrightarrow$ *I. Kogan*

Cartan's Key Idea

⋆ Recast the equivalence problem for submanifolds under a (pseudo-)group action, in the geometric language of differential forms.

Then reduce the equivalence problem to the most fundamental equivalence problem:

⋆ Equivalence of coframes.

Coframes

A coframe on an m-dimensional manifold M are m one-forms that forms a basis for the cotangent space T^*M at each point:

$$\theta^i = \sum_{j=1}^{m} h^i_j(x)\, dx^j \qquad i = 1, \ldots, m \qquad \det(\, h^i_j(x)\,) \neq 0$$

- Equivalence of Coframes: $\Phi^*\overline{\theta}^i = \theta^i \implies \Phi^*(d\overline{\theta}^i) = d\theta^i$
- Structure equations: $d\theta^i = \sum_{j<k} I^i_{jk}\, \theta^j \wedge \theta^k$
- Invariants: $I^i_{j,k} \qquad \dfrac{\partial I^i_{j,k}}{\partial \theta^l} \qquad \dfrac{\partial^2 I^i_{j,k}}{\partial \theta^l\, \partial \theta^n} \qquad \ldots$
- Rank $= r = \#$ functionally independent invariants
- Order $= s =$ order of derivatives where rank is achieved
- Invariants of order $\leq\ s+1$ parametrize the signature of the coframe

Equivalence of Coframes

Cartan's Theorem:

Two coframes are equivalent if and only if

- Their ranks are the same
- Their signature manifolds are identical

Cartan's Graphical Proof Technique

The graph of the equivalence map

$$\psi : M \longrightarrow \overline{M}$$

can be viewed as a transverse m-dimensional integral submanifold

$$\Gamma_\psi \subset M \times \overline{M}$$

for the involutive differential system generated by the one-forms and functions

$$\overline{\theta}^i - \theta^i \qquad\qquad \bar{I}_j - I_j$$

Existence of suitable integrable submanifolds determining equivalence maps is guaranteed by the Frobenius Theorem, which is, at its heart, an existence theorem for ordinary differential equations, and hence valid in the smooth category.

Determining the Invariant (Extended) Coframe

There are now two methods for explicitly determining the invariant (extended) coframe associated with a given equivalence problem.

- The Cartan Equivalence Method
- Equivariant Moving Frames

Either will produce the invariant coframe and the fundamental differential invariants required to construct a signature and thereby effectively solve the equivalence problem.

$\Longrightarrow$ F. Valiquette

$\Longrightarrow$ Ö. Arnaldsson

The Cartan Equivalence Method

(1) Reformulate the problem as an equivalence problem for G-valued coframes, for some structure group G

(2) Calculate the structure equations by applying d

(3) Use absorption of torsion to determine the essential torsion

(4) Normalize the group-dependent essential torsion coefficients to reduce the structure group

(5) Repeat the process until the essential torsion coefficients are all invariant

(6) Test for involutivity

(7) If not involutive, prolong (à la EDS) and repeat until involutive

The result is an invariant coframe that completely encodes the equivalence problem, perhaps on some higher dimensional space. The structure invariants for the coframe are used to parametrize the signature.

Equivariant Moving Frames

$\Longrightarrow$ Fels and Olver, 1999

(1) Prolong (à la jet bundle) the (pseudo-)group action to the jet bundle of order n where the action becomes (locally) free

(2) Choose a cross-section to the group orbits and solve the normalization equations to determine an equivariant moving frame map $\rho \colon \mathrm{J}^n \to G$

(3) Use invariantization to determine the normalized differential invariants of order $\leq n+1$ and invariant differential forms; invariant differential operators; ...

(4) Apply the recurrence formulae to determine higher order differential invariants, and the structure of the differential invariant algebra

★ Step (4) can be done completely symbolically, using only linear algebra, independent of the explicit formulae in step (*3*)

The key to understanding and solving an equivalence problem lies in the invariants

For Cartan, the differential invariants are fundamental.

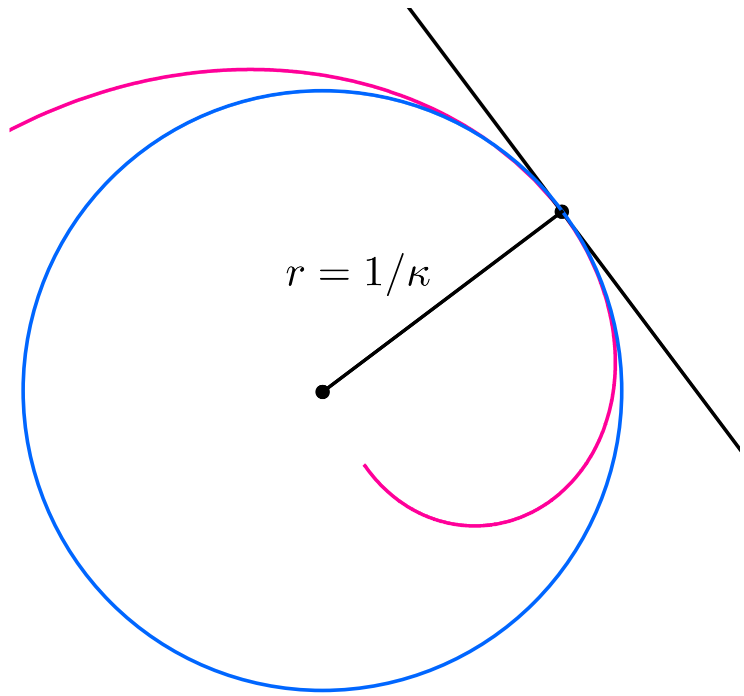
Differential Invariants

Given a submanifold (curve, surface, . . .)

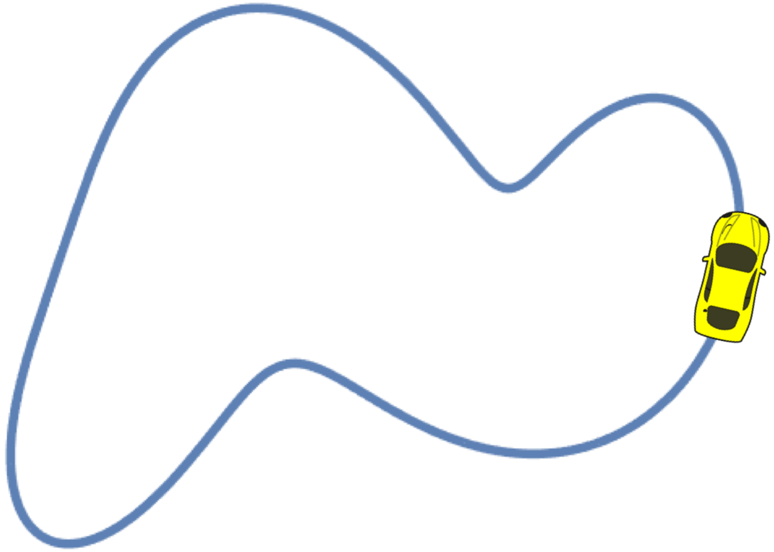
$$S \subset M$$

a differential invariant is an invariant of the prolonged action of G on its derivatives (jets):

$$\boxed{I(g \cdot z^{(k)}) \;=\; I(z^{(k)})}$$



Curvature = reciprocal of radius of osculating circle



“... the theory of differential invariants is to the theory of curvature as projective geometry is to elementary geometry.”

— Poincaré

Euclidean Plane Curves: $G = \mathrm{SE}(2)$

Differentiation with respect to the Euclidean-invariant arc length element ds is an invariant differential operator, meaning that it maps differential invariants to differential invariants.

Thus, starting with curvature κ, we can generate an infinite collection of higher order Euclidean differential invariants:

$$\kappa, \quad \frac{d\kappa}{ds}, \quad \frac{d^2\kappa}{ds^2}, \quad \frac{d^3\kappa}{ds^3}, \quad \cdots$$

Theorem. All Euclidean differential invariants are functions of the derivatives of curvature with respect to arc length: $\kappa, \ \kappa_s, \ \kappa_{ss}, \ \cdots$

Euclidean Plane Curves: $G = \mathrm{SE}(2)$

Assume the curve $C \subset M$ is a graph: $\quad y = u(x)$

Differential invariants:

$$\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}, \qquad \frac{d\kappa}{ds} = \frac{(1+u_x^2)u_{xxx} - 3u_x u_{xx}^2}{(1+u_x^2)^3}, \qquad \frac{d^2\kappa}{ds^2} = \cdots$$

Arc length (invariant one-form):

$$ds = \sqrt{1+u_x^2}\; dx, \qquad \frac{d}{ds} = \frac{1}{\sqrt{1+u_x^2}}\,\frac{d}{dx}$$

Equi-affine Plane Curves: $G = \mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$

Equi-affine curvature:

$$\kappa = \frac{5\, u_{xx} u_{xxxx} - 3\, u_{xxx}^2}{9\, u_{xx}^{8/3}} \qquad \frac{d\kappa}{ds} = \cdots$$

Equi-affine arc length:

$$ds = \sqrt[3]{u_{xx}}\; dx \qquad \frac{d}{ds} = \frac{1}{\sqrt[3]{u_{xx}}}\, \frac{d}{dx}$$

Theorem. All equi-affine differential invariants are functions of the derivatives of equi-affine curvature with respect to equi-affine arc length: $\kappa, \quad \kappa_s, \quad \kappa_{ss}, \quad \cdots$

Projective Plane Curves: $G = \mathrm{PSL}(2)$

Projective curvature:

$$\kappa = K(u^{(7)}, \ \cdots\ ) \qquad \frac{d\kappa}{ds} = \ \cdots \qquad \frac{d^2\kappa}{ds^2} = \ \cdots$$

Projective arc length:

$$ds \ = \ L(u^{(5)}, \ \cdots\ ) \ dx \qquad \frac{d}{ds} = \frac{1}{L}\,\frac{d}{dx}$$

Theorem. All projective differential invariants are functions of the derivatives of projective curvature with respect to projective arc length:

$$\kappa, \quad \kappa_s, \quad \kappa_{ss}, \quad \cdots$$

Euclidean Space Curves $\quad C \subset \mathbb{R}^3$

- κ — curvature: order $= 2$
- τ — torsion: order $= 3$
- $\kappa_s, \tau_s, \kappa_{ss}, \ldots$ — derivatives w.r.t. arc length ds

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Theorem. Every Euclidean differential invariant of a space curve $C \subset \mathbb{R}^3$ can be written

$$I = H(\kappa, \tau, \kappa_s, \tau_s, \kappa_{ss}, \ \ldots\ )$$

Thus, κ and τ *generate* the differential invariants of space curves under the Euclidean group.

Euclidean Surfaces $\quad S \subset \mathbb{R}^3$

- $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ — mean curvature: order $= 2$
- $K = \kappa_1 \kappa_2$ — Gauss curvature: order $= 2$
- $\mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1 K, \mathcal{D}_2 K, \mathcal{D}_1^2 H, \ \ldots$ — derivatives with respect to the equivariant Frenet frame on S

Theorem. Every Euclidean differential invariant of a non-umbilic surface $S \subset \mathbb{R}^3$ can be written

$$I = \Phi(H, K, \mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1 K, \mathcal{D}_2 K, \mathcal{D}_1^2 H, \ \ldots\ )$$

Thus, H, K *generate* the differential invariant algebra of (generic) Euclidean surfaces.

Euclidean Surfaces

Theorem.

The algebra of Euclidean differential invariants for suitably non-degenerate surfaces is generated by only the mean curvature through invariant differentiation.

In particular:

$$K = \Phi(H, \mathcal{D}_1 H, \mathcal{D}_2 H, \ \ldots \ )$$

The Basis Theorem

Theorem. The differential invariant algebra $\mathcal{I}(G)$ is locally generated by a finite number of differential invariants

$$I_1, \ \ldots \ , I_\ell$$

and $p = \dim S$ invariant differential operators

$$\mathcal{D}_1, \ \ldots \ , \mathcal{D}_p$$

meaning that *every* differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives:

$$\mathcal{D}_J I_\kappa = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_n} I_\kappa .$$

★ Lie groups: *Lie, Ovsiannikov, Fels–PJO*

★ Lie pseudo-groups: *Tresse, Kumpera, Pohjanpelto–PJO, Kruglikov–Lychagin*

Moving Frames

The mathematical theory is all based on the equivariant method of moving frames (Fels+PJO, 1999) which provides a systematic and algorithmic calculus for constructing complete systems of differential invariants, joint invariants, joint differential invariants, invariant differential operators, invariant differential forms, invariant variational problems, invariant conservation laws, invariant numerical algorithms, invariant signatures, etc., etc.

Equivalence & Invariants

- Equivalent submanifolds $N \approx \overline{N}$
  must have the same invariants: $I = \bar{I}$.

Constant invariants provide immediate information:

$$\text{e.g.} \qquad \kappa = 2 \quad \Longleftrightarrow \quad \overline{\kappa} = 2$$

Non-constant invariants are not useful in isolation, because an equivalence map can drastically alter the dependence on the submanifold parameters:

$$\text{e.g.} \qquad \kappa = x^3 \qquad \text{versus} \qquad \overline{\kappa} = \sinh x$$

However, a functional dependency or syzygy among the invariants *is* intrinsic:

$$\text{e.g.} \qquad \kappa_s = \kappa^3 - 1 \quad \Longleftrightarrow \quad \overline{\kappa}_{\overline{s}} = \overline{\kappa}^3 - 1$$

- Universal syzygies — Gauss–Codazzi
- Distinguishing syzygies.

Theorem. (Cartan)
Two regular submanifolds are (locally) equivalent if and only if they have identical syzygies among *all* their differential invariants.

Finiteness of Generators and Syzygies

♠ There are, in general, an infinite number of differential invariants and hence an infinite number of syzygies must be compared to establish equivalence.

♡ But the higher order differential invariants are always generated by invariant differentiation from a finite collection of basic differential invariants, and the higher order syzygies are all consequences of a finite number of low order syzygies!

Example — Plane Curves

If non-constant, both κ and κ_s depend on a single parameter, and so, locally, are subject to a syzygy:

$$\kappa_s = H(\kappa) \qquad (*)$$

But then

$$\kappa_{ss} = \frac{d}{ds} H(\kappa) = H'(\kappa)\,\kappa_s = H'(\kappa)\,H(\kappa)$$

and similarly for κ_{sss}, etc.

Consequently, all the higher order syzygies are generated by the fundamental first order syzygy $(*)$.

Thus, for Euclidean (or equi-affine or projective or ...) plane curves we need only know a single syzygy between κ and κ_s in order to establish equivalence!

The Signature Map

The generating syzygies are encoded by the signature map

$$\chi : N \longrightarrow \Sigma$$

of the submanifold N, which is parametrized by the fundamental differential invariants:

$$\chi(x) = (I_1(x), \ldots, I_m(x))$$

The image

$$\Sigma = \mathrm{Im}\, \chi$$

is the signature subset (or submanifold) of N.

Equivalence & Signature

Theorem. Two regular submanifolds are equivalent:

$$\overline{N} = g \cdot N$$

if and only if their signatures are identical:

$$\overline{\Sigma} = \Sigma$$

Signature Curves

Definition. Given an (ordinary) planar action of a Lie group G, the *signature curve* $\Sigma \subset \mathbb{R}^2$ of a plane curve $\mathcal{C} \subset \mathbb{R}^2$ is parametrized by the two lowest order differential invariants

$$\chi : \mathcal{C} \longrightarrow \Sigma = \left\{ \left(\kappa \, , \, \frac{d\kappa}{ds} \right) \right\} \subset \mathbb{R}^2$$

$\Longrightarrow$ *Calabi, PJO, Shakiban, Tannenbaum, Haker*

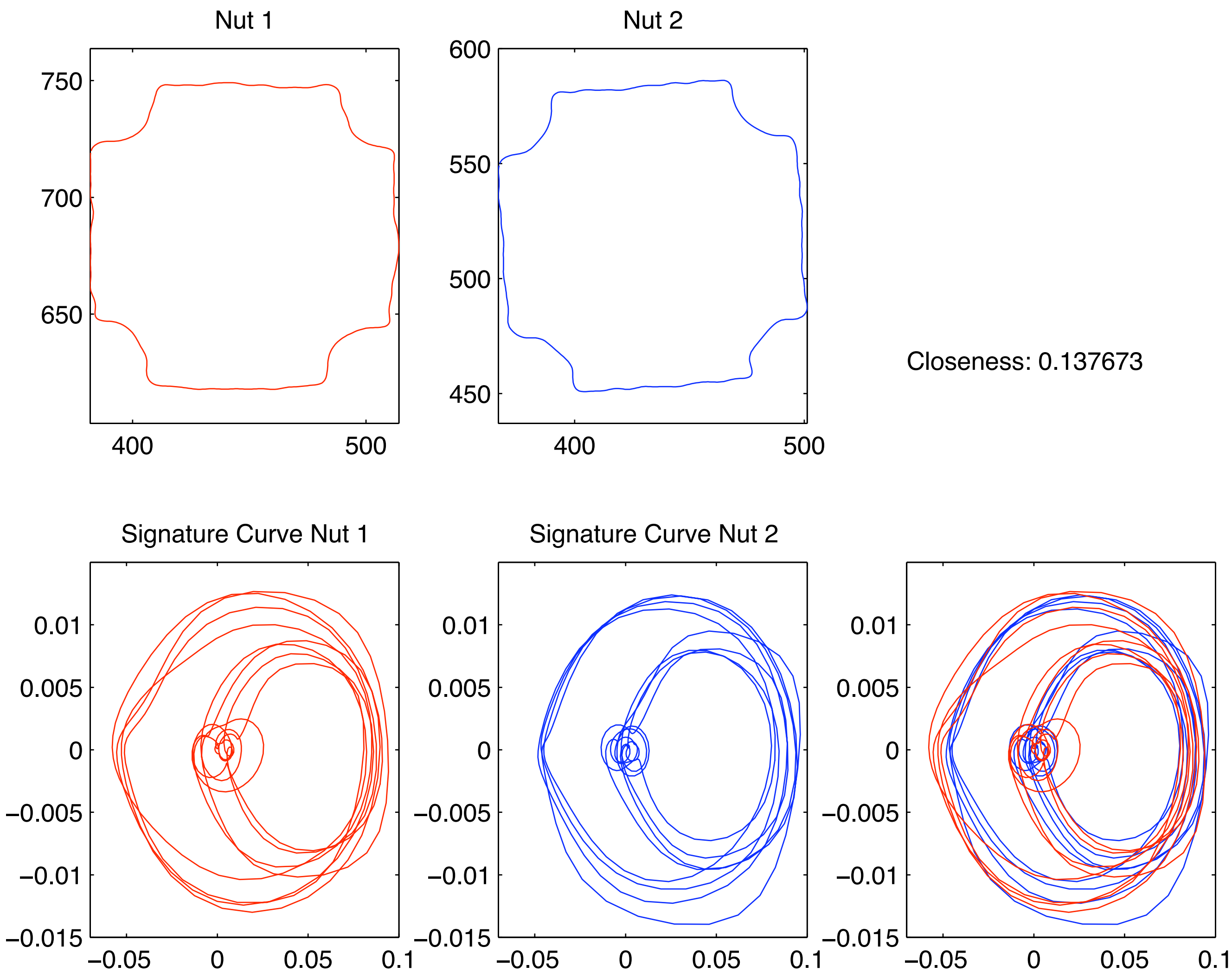
Theorem. Two regular curves $\mathcal{C}$ and $\overline{\mathcal{C}}$ are (locally) equivalent:

$$\overline{\mathcal{C}} = g \cdot \mathcal{C}$$

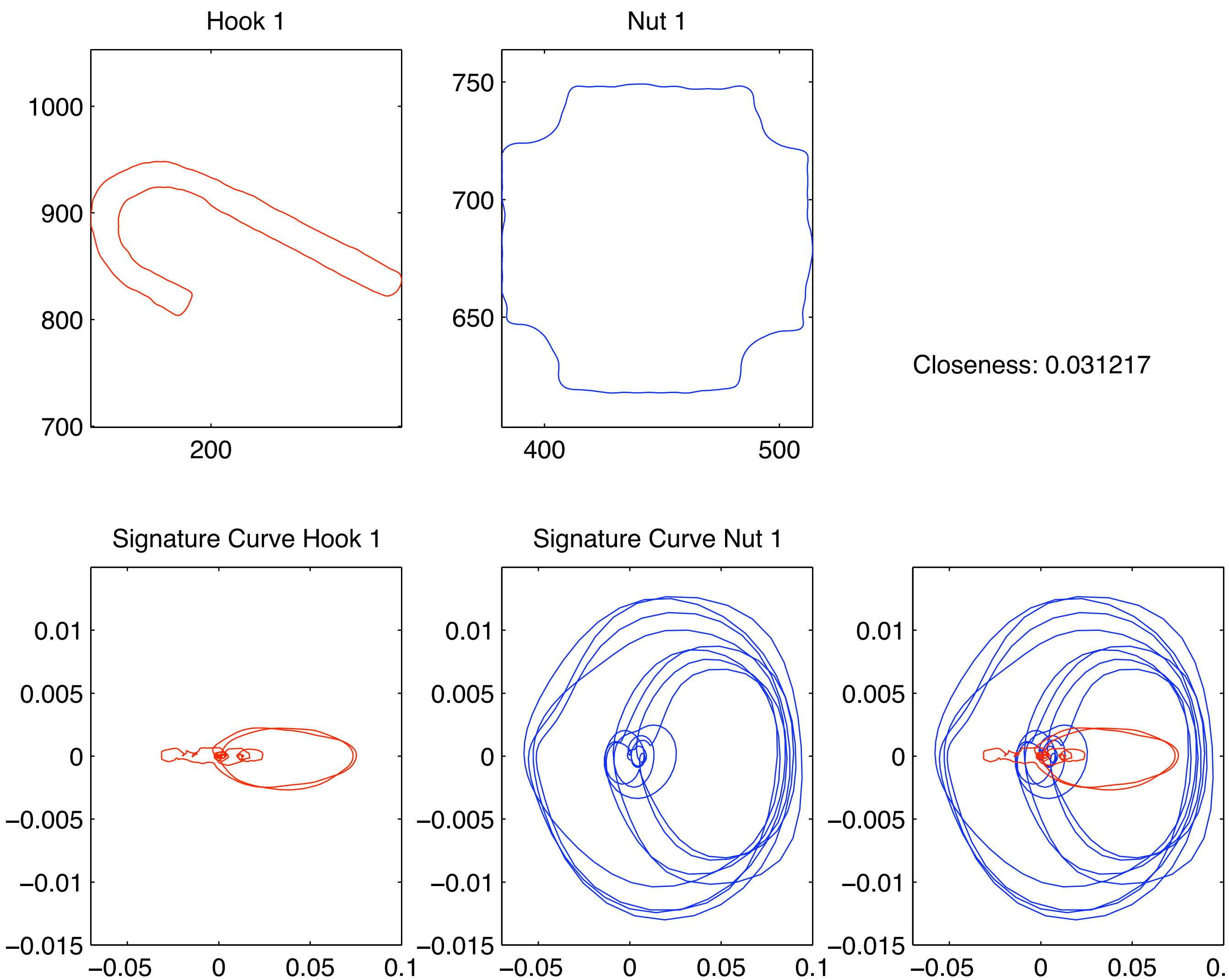
if and only if their signature curves are identical:

$$\overline{\Sigma} = \Sigma$$

$\Longrightarrow$ regular: $(\kappa_s, \kappa_{ss}) \neq 0$.

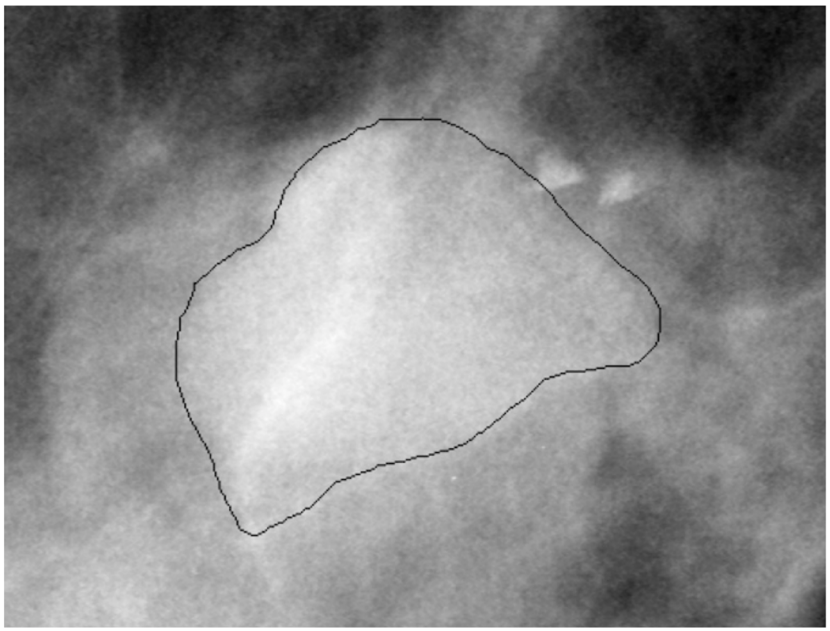


Nut 1
750
700
650
400
500
Nut 2
600
550
500
450
400
500
Closeness: 0.137673
Signature Curve Nut 1
Signature Curve Nut 2
0.01
0.005
0
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1

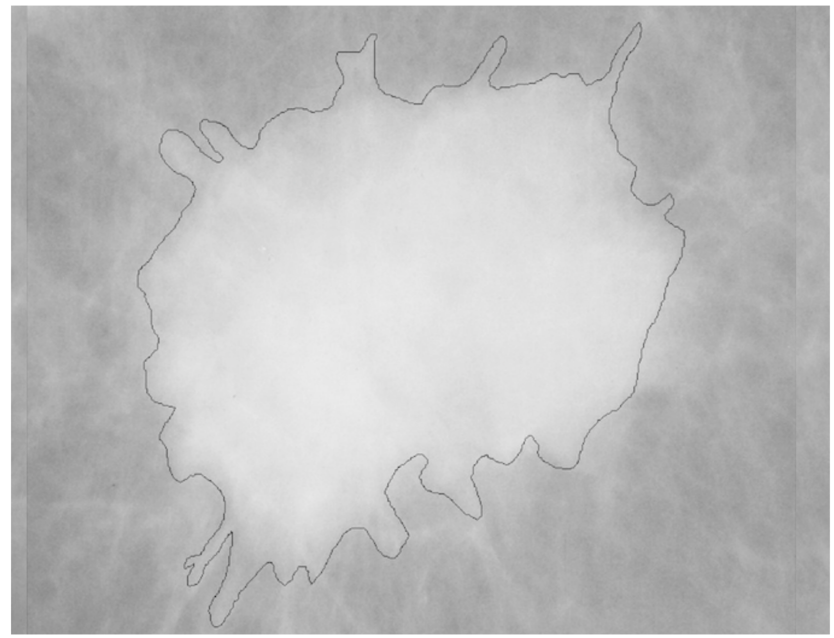

Hook 1
Nut 1
Closeness: 0.031217
Signature Curve Hook 1
Signature Curve Nut 1
1000
900
800
700
200
750
700
650
400
500
0.01
0.005
0
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1

Diagnosing breast tumors

Anna Grim, Cheri Shakiban



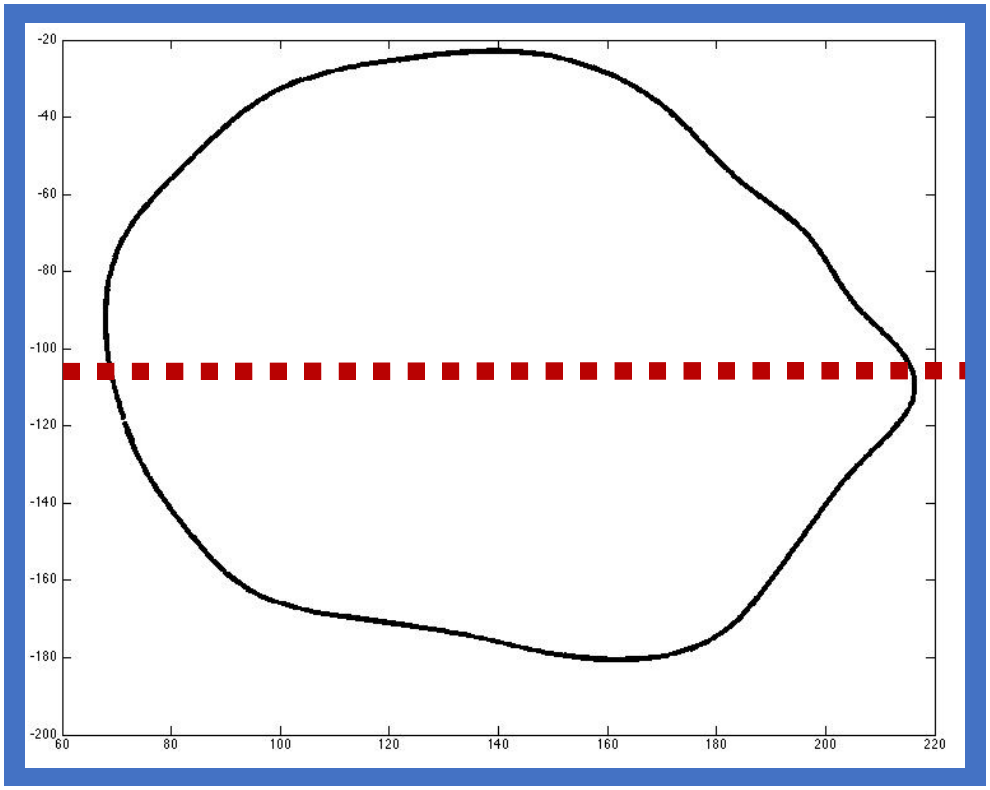
Benign — cyst



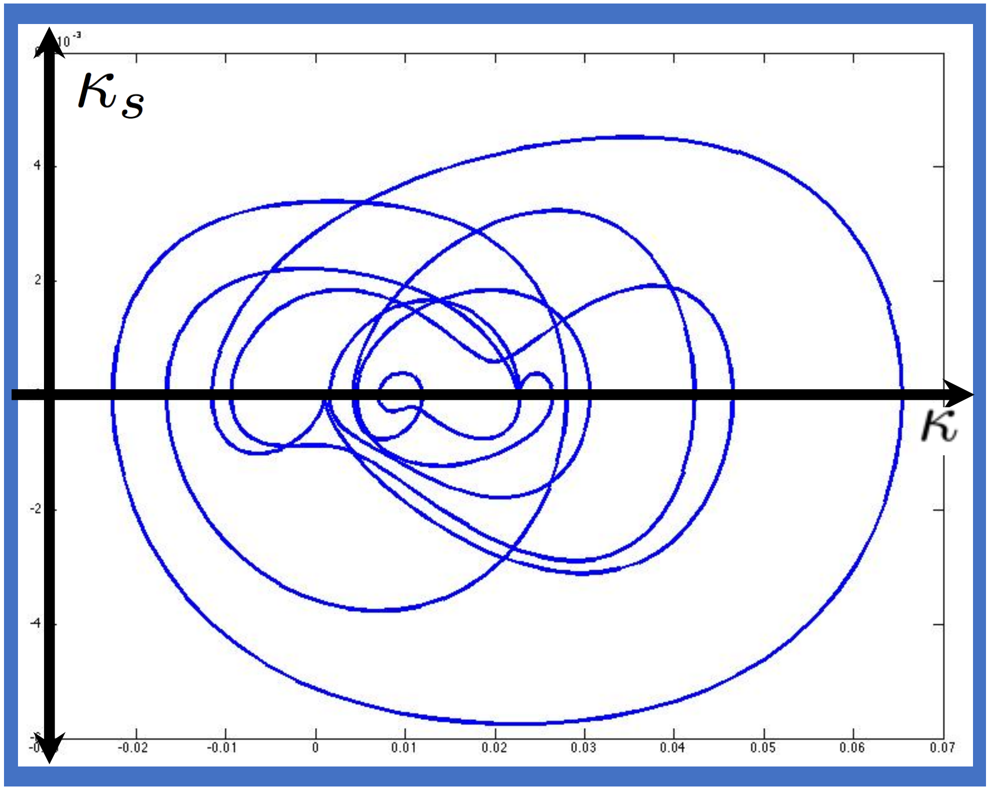
Malignant — cancerous

A BENIGN TUMOR

Contour

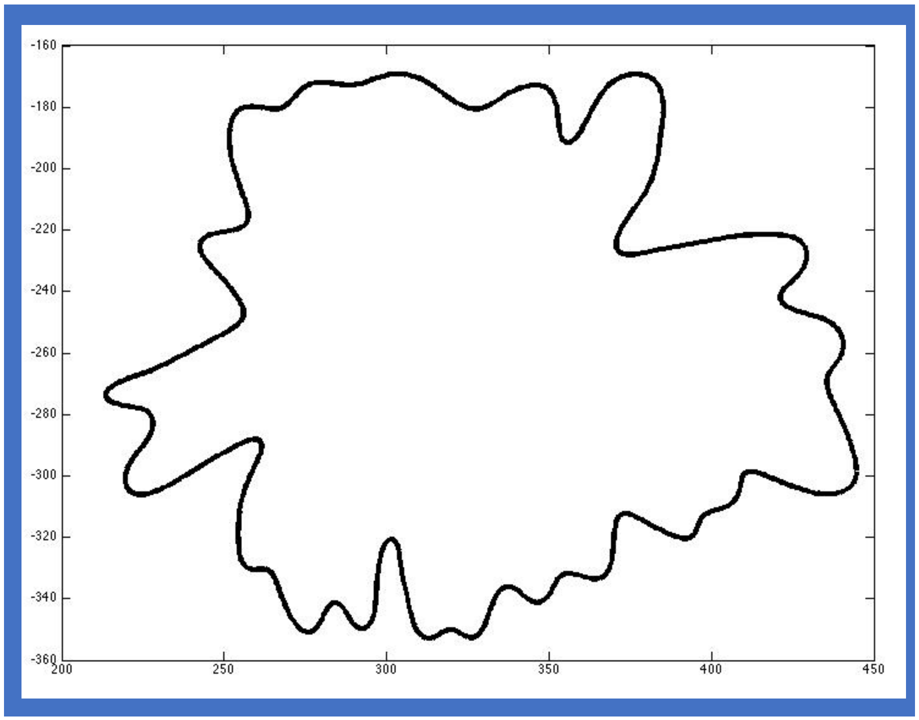


Signature Curve

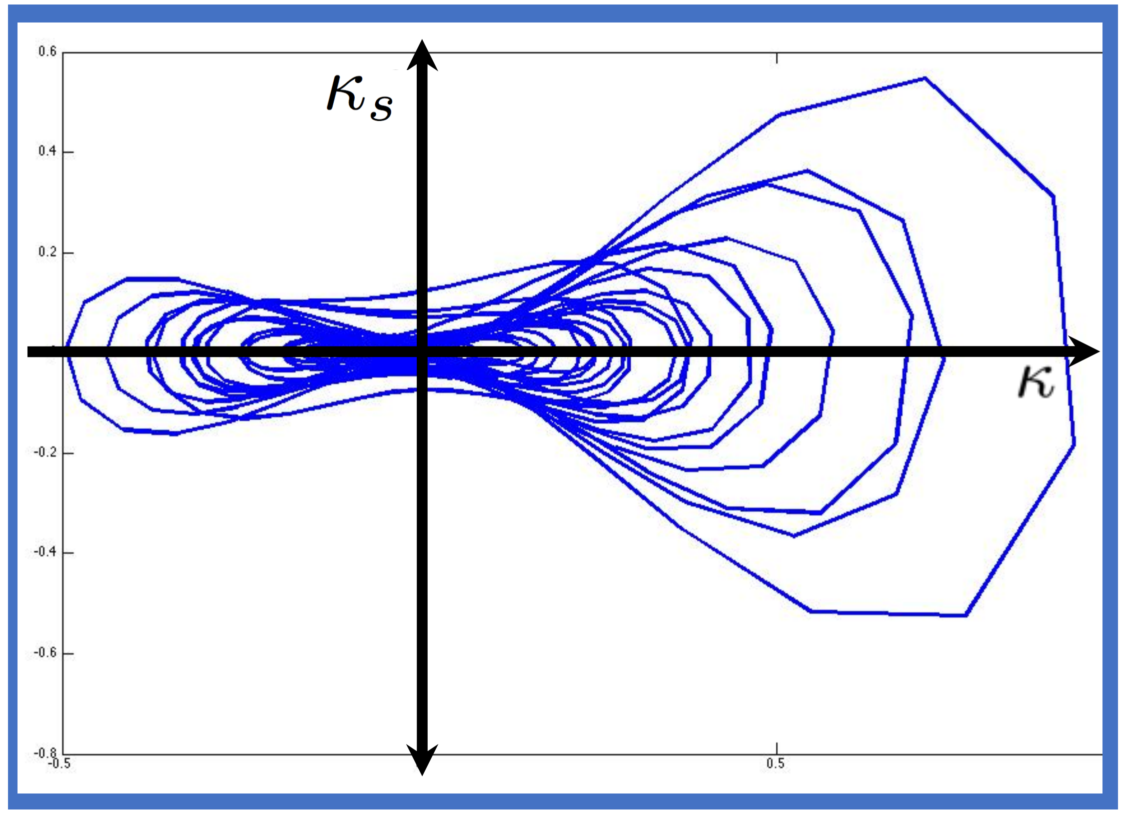


A MALIGNANT TUMOR

Contour



Signature Curve



3D Differential Invariant Signatures

Euclidean space curves: $C \subset \mathbb{R}^3$

$$\Sigma = \{ \, (\, \kappa \, , \, \kappa_s \, , \, \tau \,) \, \} \; \subset \; \mathbb{R}^3$$

- κ — curvature, τ — torsion

Euclidean surfaces: $S \subset \mathbb{R}^3$ (generic)

$$\Sigma = \left\{ \, \left(\, H \, , \, K \, , \, H_{,1} \, , \, H_{,2} \, , \, K_{,1} \, , \, K_{,2} \right) \, \right\} \; \subset \; \mathbb{R}^6$$

or $$\widehat{\Sigma} = \left\{ \, \left(\, H \, , \, H_{,1} \, , \, H_{,2} \, , \, H_{,11} \right) \, \right\} \; \subset \; \mathbb{R}^4$$

- H — mean curvature, K — Gauss curvature

Classical Invariant Theory

$$M = \mathbb{R}^2 \setminus \{\, u = 0 \,\}$$

$$G = \mathrm{GL}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \;\middle|\; \Delta = \alpha\,\delta - \beta\,\gamma \neq 0 \right\}$$

$$(x, u) \longmapsto \left(\frac{\alpha x + \beta}{\gamma x + \delta}\,,\; \frac{u}{(\gamma x + \delta)^n} \right) \qquad n \neq 0, 1$$

Hessian:

$$H = n(n-1)u\,u_{xx} - (n-1)^2 u_x^2 \neq 0$$

Note: $H \equiv 0$ if and only if $Q(x) = (a\,x+b)^n$

$\Longrightarrow$ Totally singular forms

Differential invariants:

$$v_{yyy} \longmapsto \frac{J}{n^2(n-1)} \approx \kappa \qquad v_{yyyy} \longmapsto \frac{K + 3(n-2)}{n^3(n-1)} \approx \frac{d\kappa}{ds}$$

Absolute rational covariants:

$$J^2 = \frac{T^2}{H^3} \qquad K = \frac{U}{H^2}$$

$$H = \tfrac{1}{2}(Q,\, Q)^{(2)} = n(n-1)QQ'' - (n-1)^2 Q'^2 \sim Q_{xx}Q_{yy} - Q_{xy}^2$$

$$T = (Q,\, H)^{(1)} = (2n-4)Q'H - nQH' \sim Q_x H_y - Q_y H_x$$

$$U = (Q,\, T)^{(1)} = (3n-6)Q'T - nQT' \sim Q_x T_y - Q_y T_x$$

$$\deg Q = n \quad \deg H = 2n-4 \quad \deg T = 3n-6 \quad \deg U = 4n-8$$

Signatures of Binary Forms

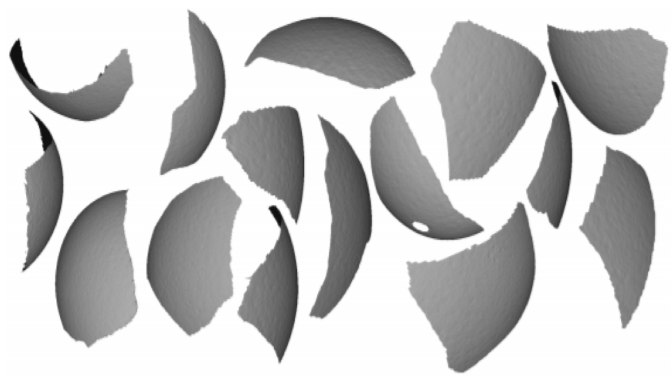
Signature curve of a nonsingular binary form $Q(x)$:

$$\Sigma_Q = \left\{ (J(x)^2, K(x)) = \left(\frac{T(x)^2}{H(x)^3}, \frac{U(x)}{H(x)^2} \right) \right\}$$

Nonsingular: $\quad H(x) \neq 0 \;$ and $\; (J'(x), K'(x)) \neq 0.$

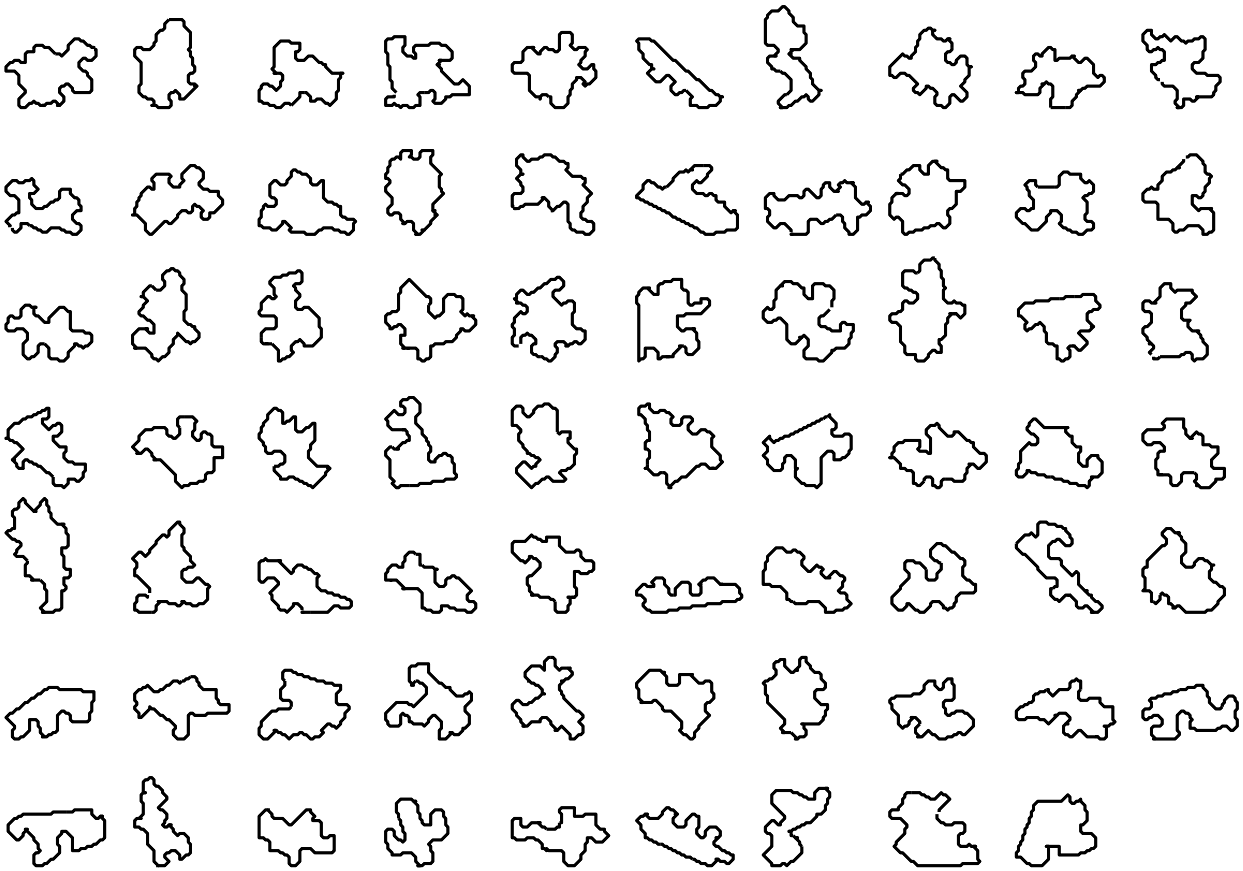
Theorem. Two nonsingular binary forms are equivalent if and only if their signature curves are identical.

Reassembly of Broken Objects

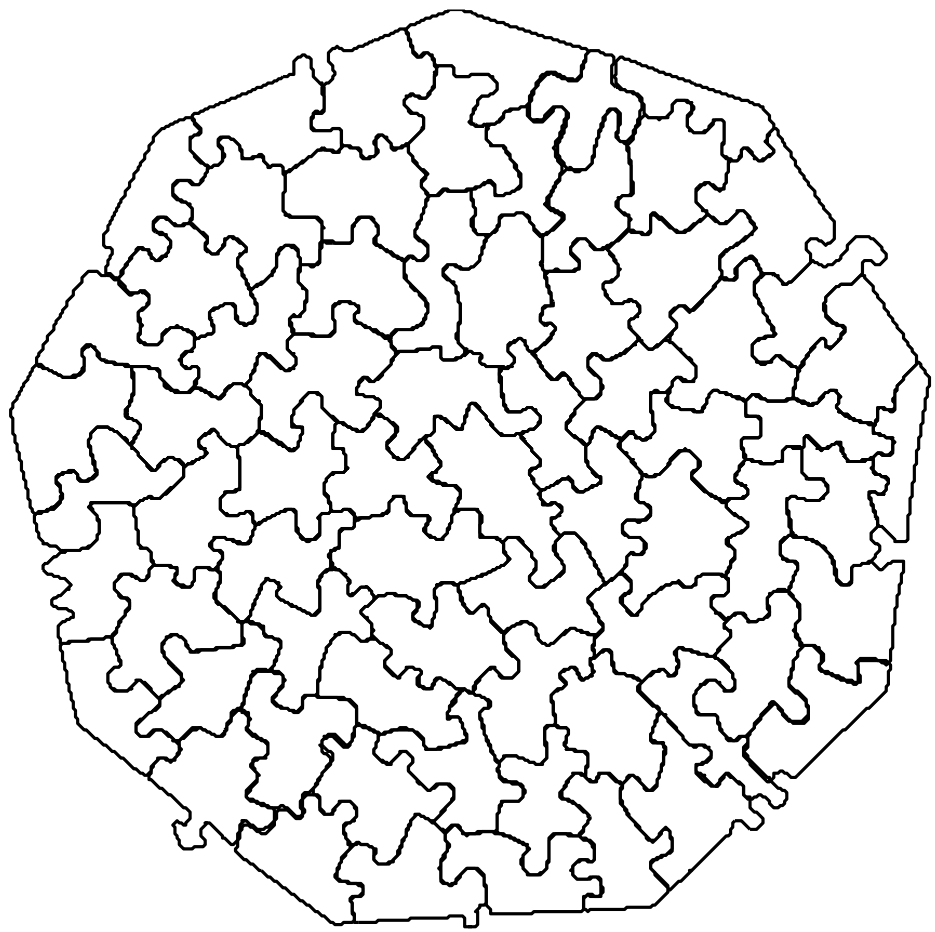



the most unique
puzzle ever
the BAFFLER™
by CHRIS YATES
The Nonagon
67 pieces

The Baffler Nonagon

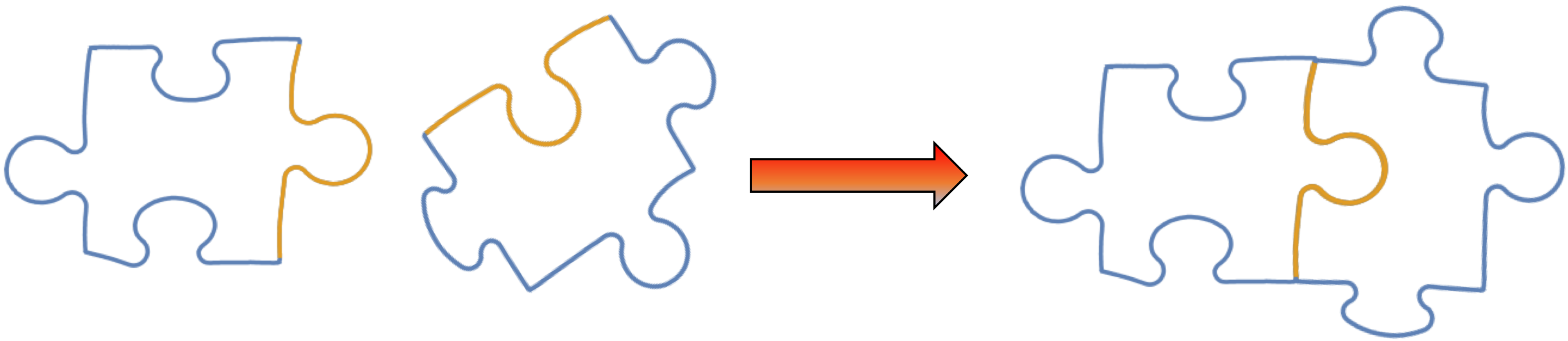


The Baffler Nonagon — Solved



Dan Hoff

Automatic puzzle reassembly



Step 0. Digitally photograph and smooth the puzzle pieces.

Step 1. Numerically compute invariant signatures of (parts of) pieces.

Step 2. Compare signatures to find potential fits.

Step 3. Put them together, if they fit, as closely as possible.

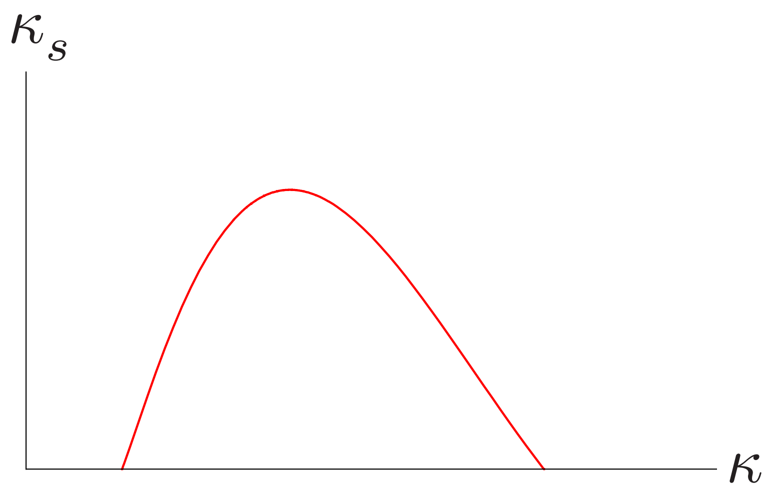
Repeat steps 1–3 until puzzle is assembled....

Localization of Signatures

Bivertex arc: $\kappa_s \neq 0$ everywhere

except $\kappa_s = 0$ at the two endpoints

The signature Σ of a bivertex arc is a single arc that starts and ends on the κ–axis.



Bivertex Decomposition

v-regular curve — finitely many generalized vertices

$$C = \bigcup_{j=1}^{m} B_j \cup \bigcup_{k=1}^{n} V_k$$

$B_1, \ldots, B_m$ — bivertex arcs

$V_1, \ldots, V_n$ — generalized vertices: $n \geq 4$

Main Idea: Compare individual bivertex arcs, and then decide whether the rigid equivalences are (approximately) the same.

D. Hoff & PJO, Extensions of invariant signatures for object recognition, *J. Math. Imaging Vision* **45** (2013), 176–185.

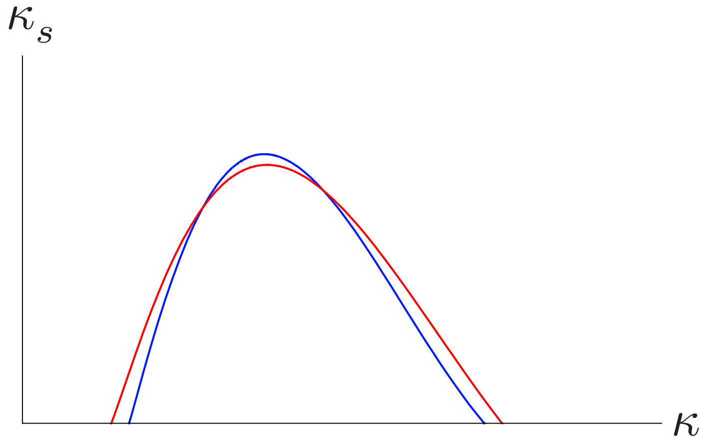
Signature Metrics

Used to compare signatures:

- Hausdorff
- Monge–Kantorovich transport
- Electrostatic/gravitational attraction
- Latent semantic analysis
- Histograms
- Geodesic distance
- Diffusion metric
- Gromov–Hausdorff & Gromov–Wasserstein

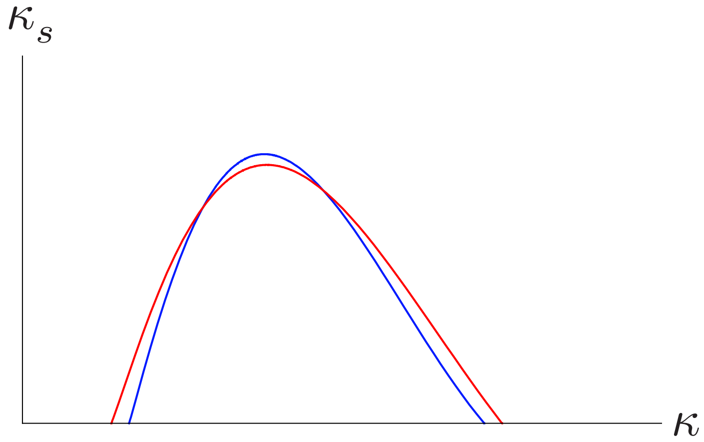
Gravitational/Electrostatic Attraction

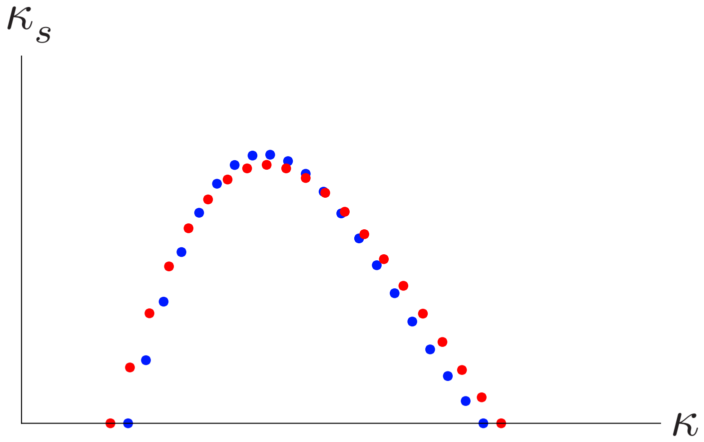
- ⋆ Treat the two (signature) curves as masses or as oppositely charged wires. The higher their mutual attraction, the closer they are together.



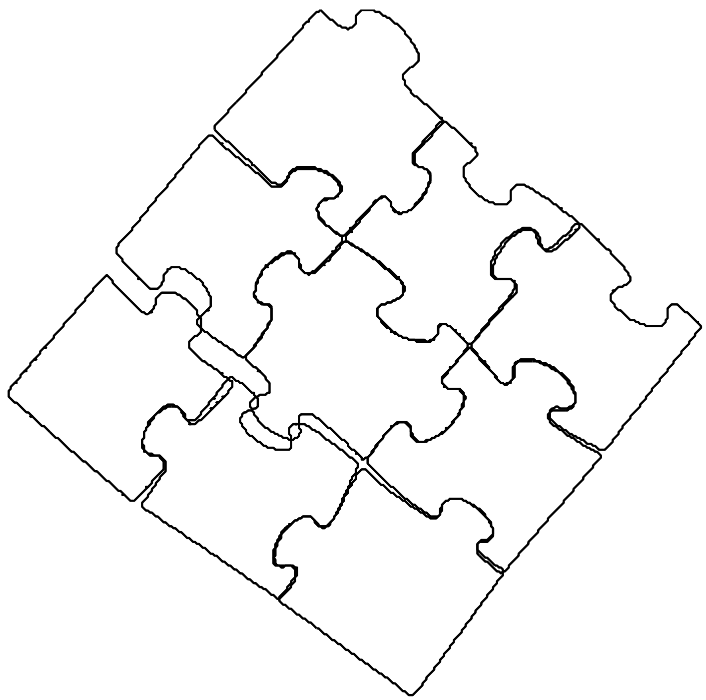
Gravitational/Electrostatic Attraction

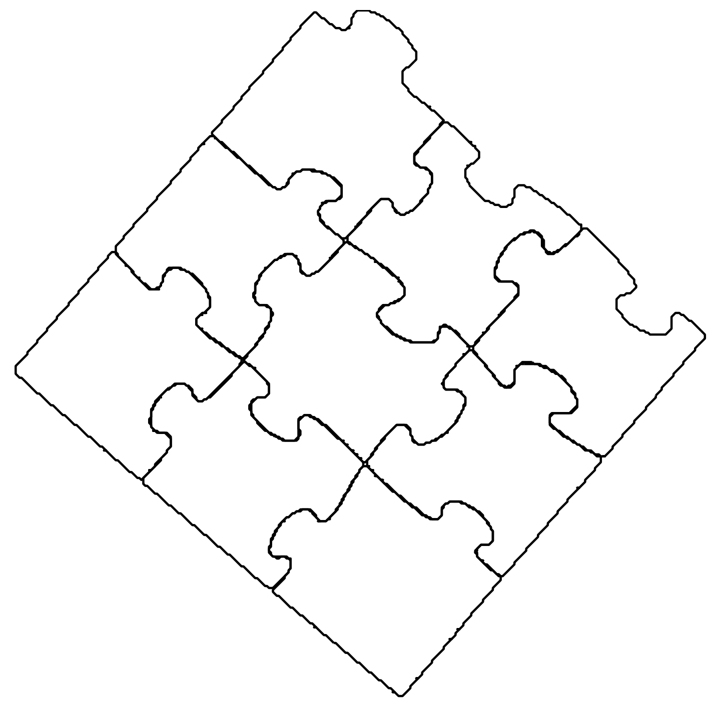
- ⋆ Treat the two (signature) curves as masses or as oppositely charged wires. The higher their mutual attraction, the closer they are together.
- ⋆ In practice, we are dealing with discrete data (pixels) and so treat the curves and signatures as point masses/charges.





Piece Locking





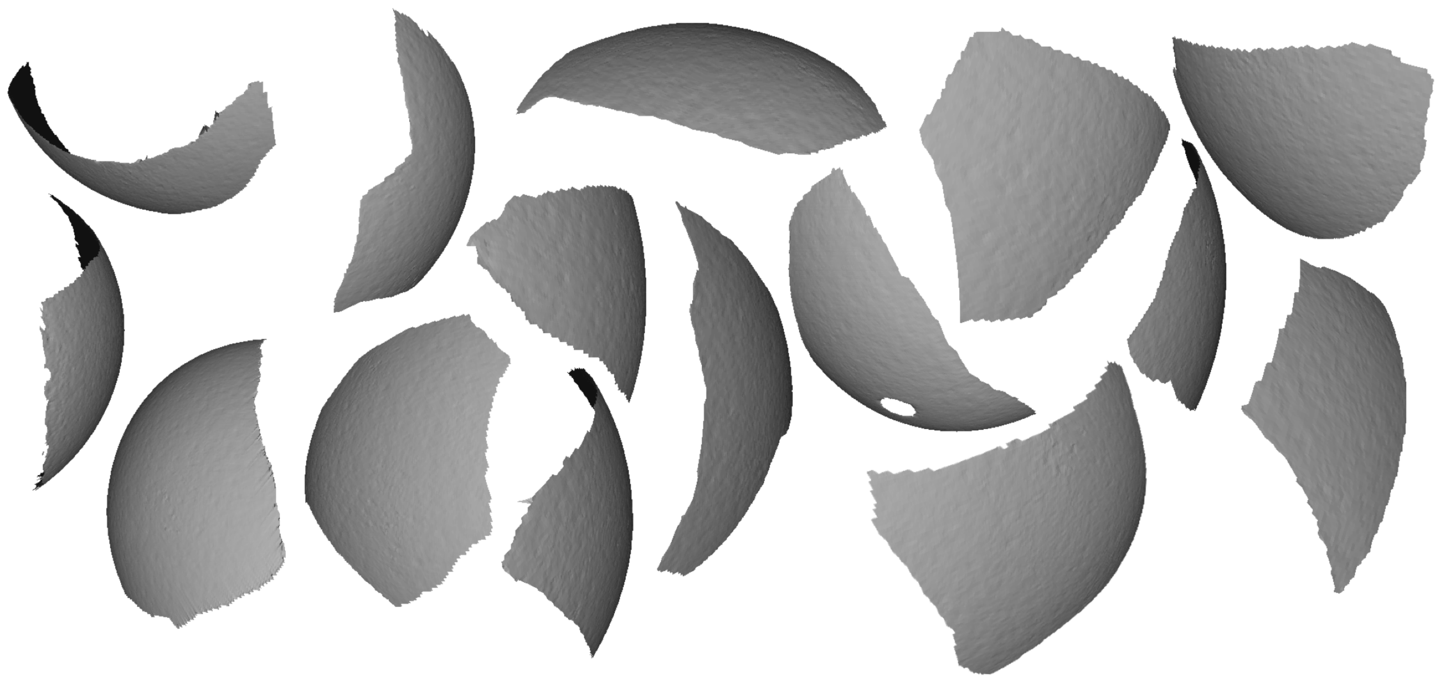
★★ Minimize force and torque based on gravitational attraction of the two matching edges.

Automatic Solution of Jigsaw Puzzles

Putting Humpty Dumpty Together Again

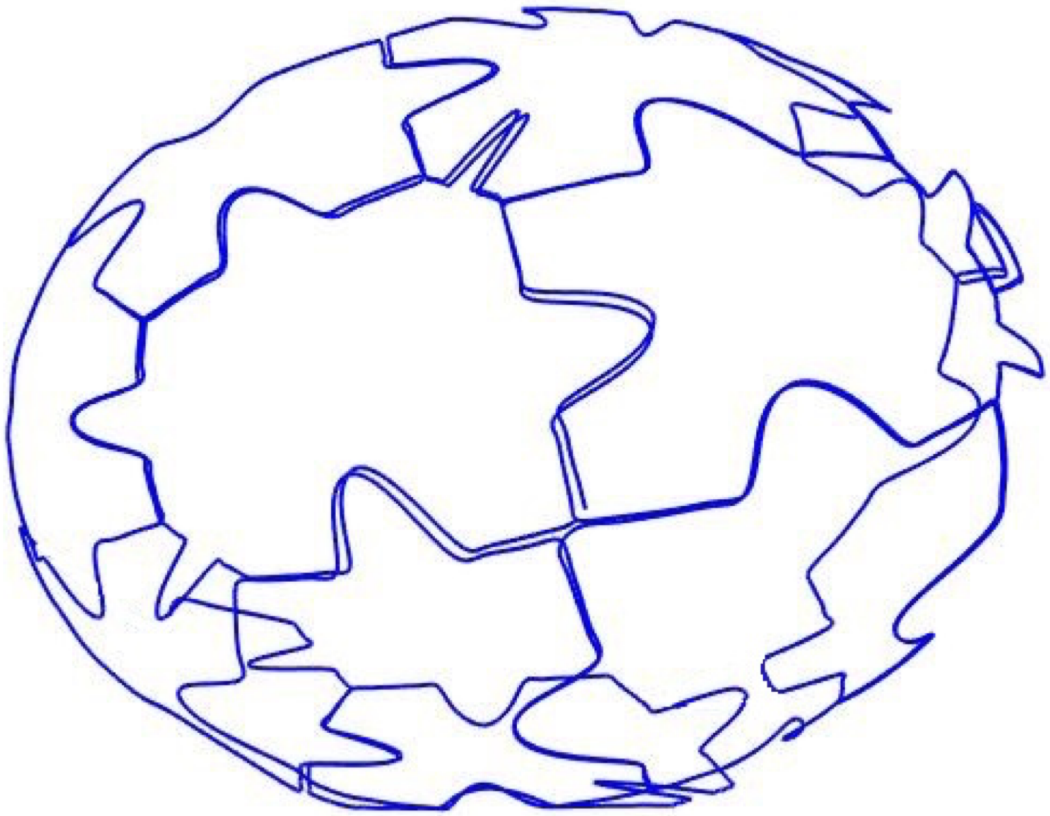
→ Anna Grim, Ryan Slechta, Tim O'Connor, Rob Thompson, Cheri Shakiban, PJO

A broken ostrich egg



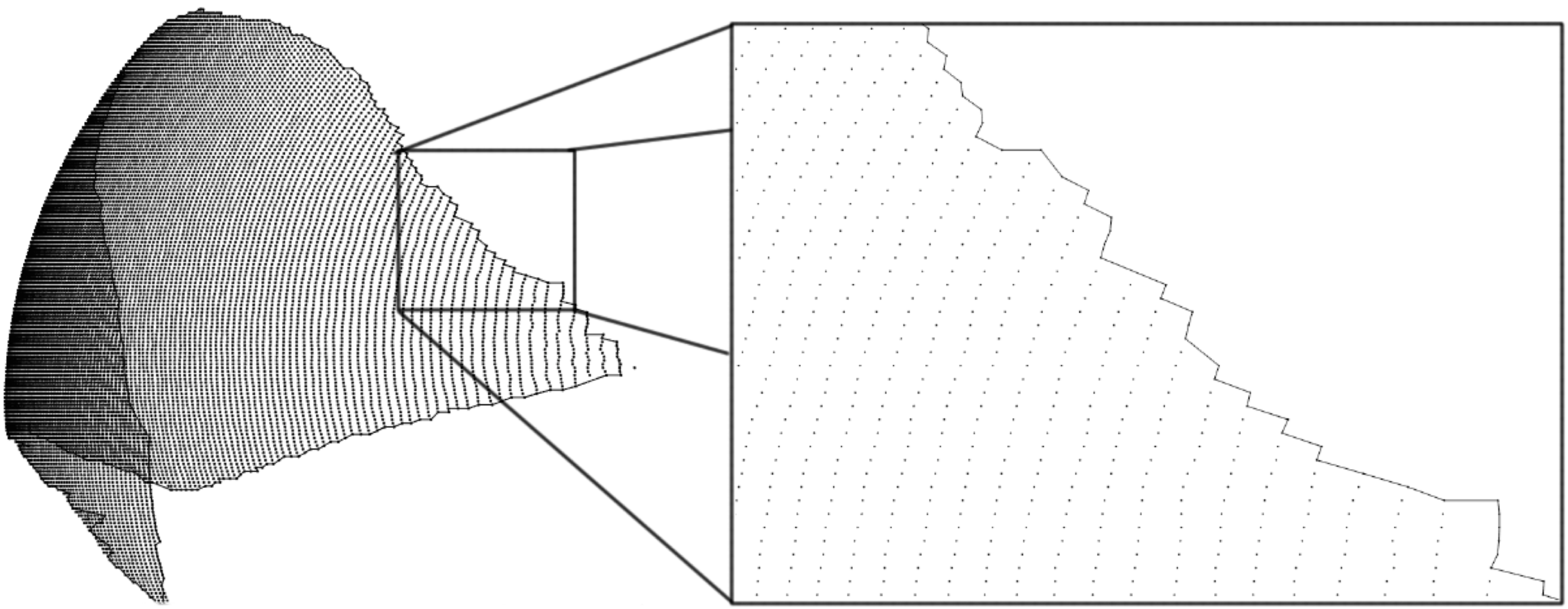
(Scanned by M. Bern, Xerox PARC)

Assembly of Synthetic Ellipsoidal Puzzle

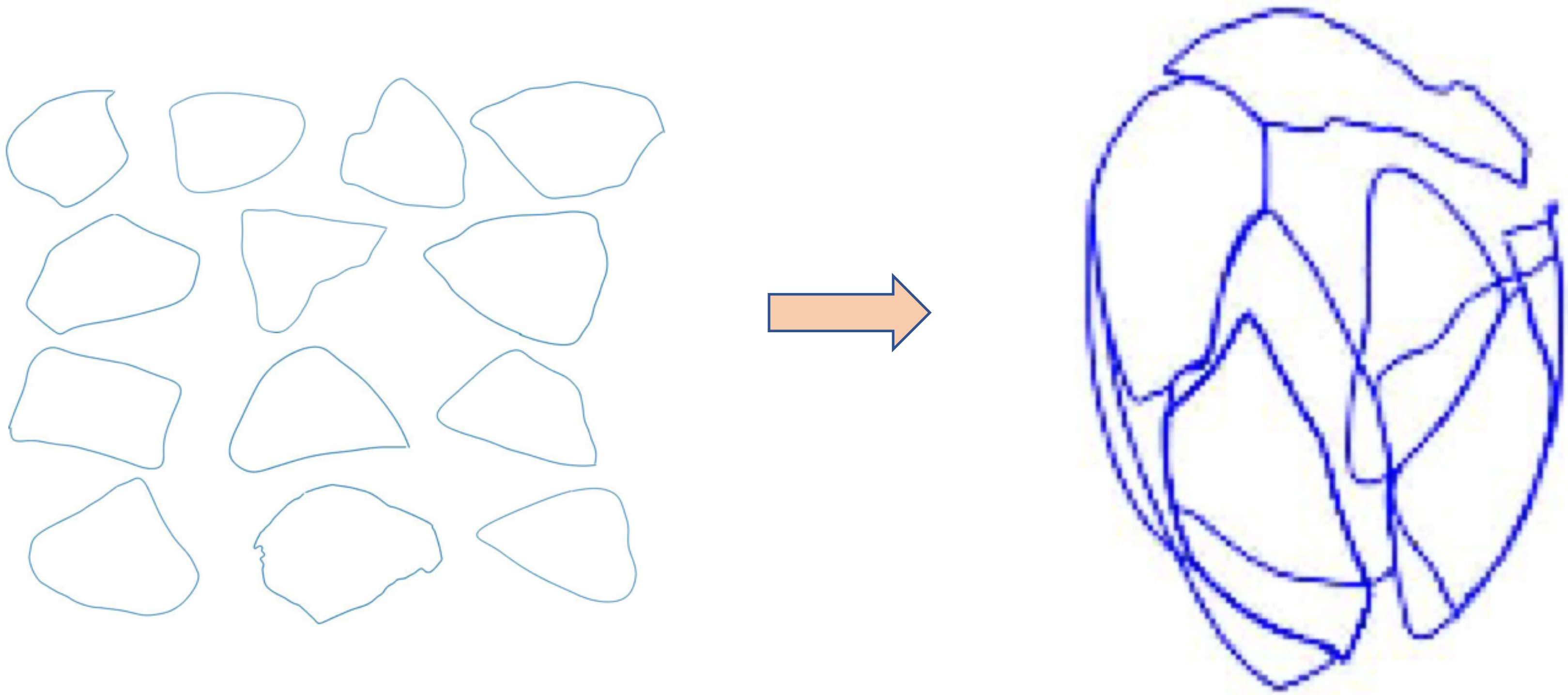


- Uses curvature and torsion invariants

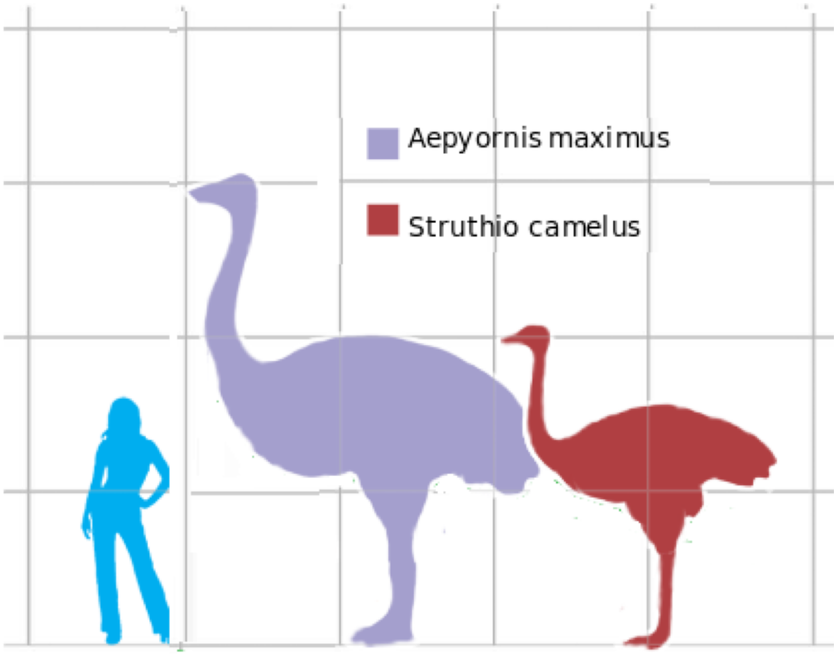
An egg piece



All the king's horses and men



The elephant bird of Madagascar



(Image from wikipedia.org)

- more than 3 meters tall
- extinct by the 1700's
- one egg could make about 160 omelets

The elephant bird of Madagascar

(Image from Tennant's Auctioneers)

- pictured egg is 70% complete
- complete egg recently sold for $100,000

Puzzles in archaeology

Puzzles in archaeology

Puzzles in surgery

A little more history

In November, 2016, I gave a couple of invited talks at Georgia Tech and then had lunch with Tony Yezzi (see above), where I told him about my work on jigsaw puzzles and egg shells. And he said well, my sister Katrina is a graduate student in Anthropology at the University of Minnesota and she is very interested in putting together broken bones.

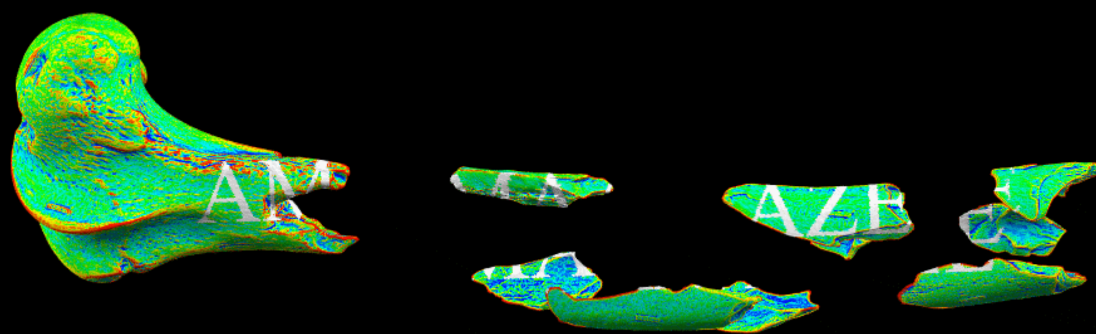
And so, almost 4 years later . . .

AMAAZE

Anthropological and Mathematical Analysis of

Archaeological and Zooarchaeological Evidence

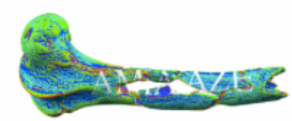
https://amaaze.umn.edu

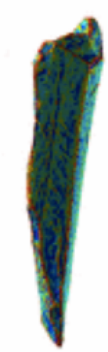


Anthropological and Mathematical Analysis of Archaeological and Zooarchaeological Evidence

AMAAZE

Home





The Anthropological and Mathematical Analysis of Archaeological and Zooarchaeological Evidence (AMAAZE) is an international consortium of anthropologists, mathematicians, and computer scientists who are working together to advance analytical methods and to use advanced mathematical methods to address important questions within archaeology and zooarchaeology.

Whether studying fossils, lithics, pottery, or other remnants of the past, archaeological analysis is grounded in identifying patterns and frequencies, which is inherently mathematical. Early research was founded on the observation and qualitative description of these patterns. Over the last several decades, the discipline has increasingly sought quantitative data analytical methods. Powerful tools such as 3d modeling, geometric morphometrics, and machine learning allows us to quickly capture and process massive amounts of information that cannot practically be gathered from physical measurements. Together, anthropologists, mathematicians, and computer scientists leverage their expertise to truly optimize these tools, the implications of which are expected to impact the current understanding of early human prehistory, culture, and origins.

Current projects include the Geometric Analysis for Classification and Reassembly of Broken Bones which uses mathematical techniques based on invariant signatures and

The AMAAZE Broken Bones Team

Jeff Calder

Katrina Yezzi-Woodley

Cheri Shakiban

Reed Coil

Grad Students

Cora Brown
Carter Chain
Annie Melton
Samantha Porter

Undergrad Students

Owen Cody
David Floeder
Thomas Huffstutler
Jiafeng Li
Riley O'Neill
Meredith Shipp
Chloe Siewart
Alexander Terwilliger
Jacob Theis

Pedro Angulo-Umaña
Jacob Elafandi
Bo Hessburg

Breaking Bones

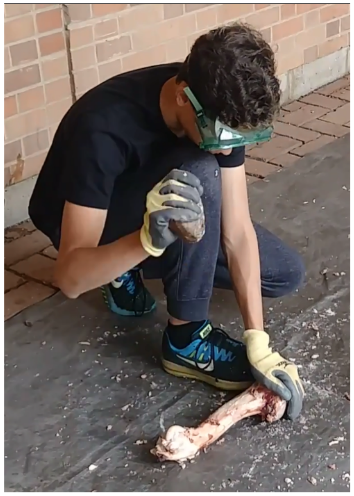
Carnivore

Crocuta crocuta = *hyena*

Hominin

Batting

Hammerstone and anvil



Hammerstone only

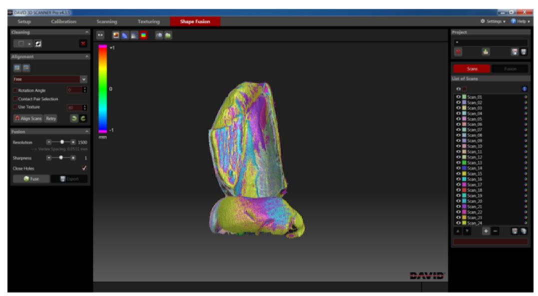
Geological

Rock fall

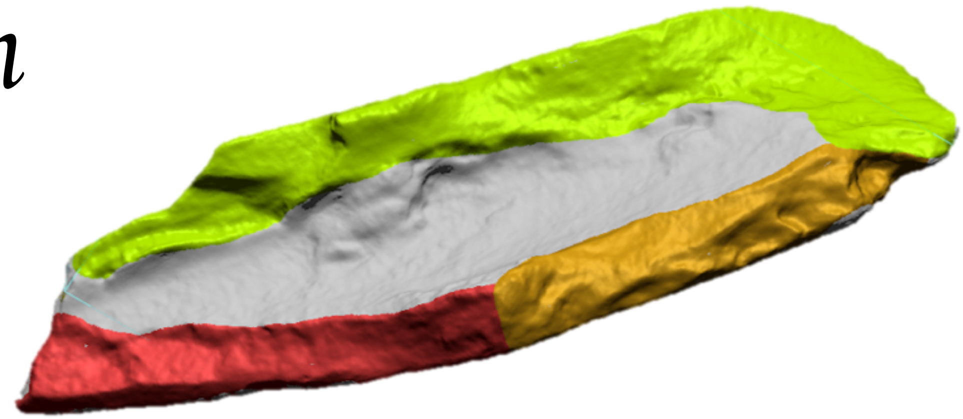
Working Hypothesis

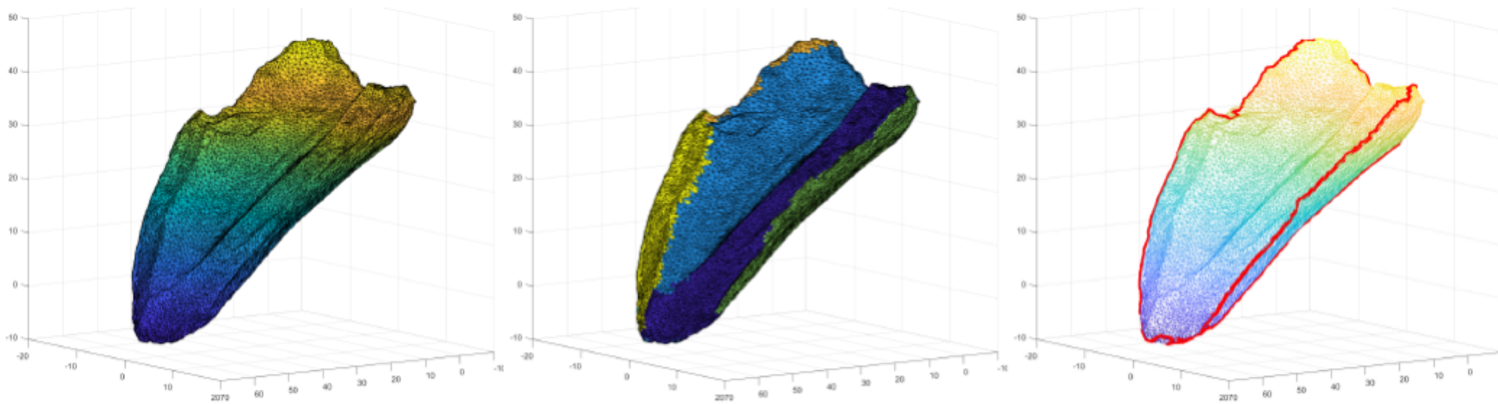
The geometry of the bone fragments,
their identity (taxon and element),
and how they are reassembled
can tell us the actor of breakage

Broken Bone Fragments



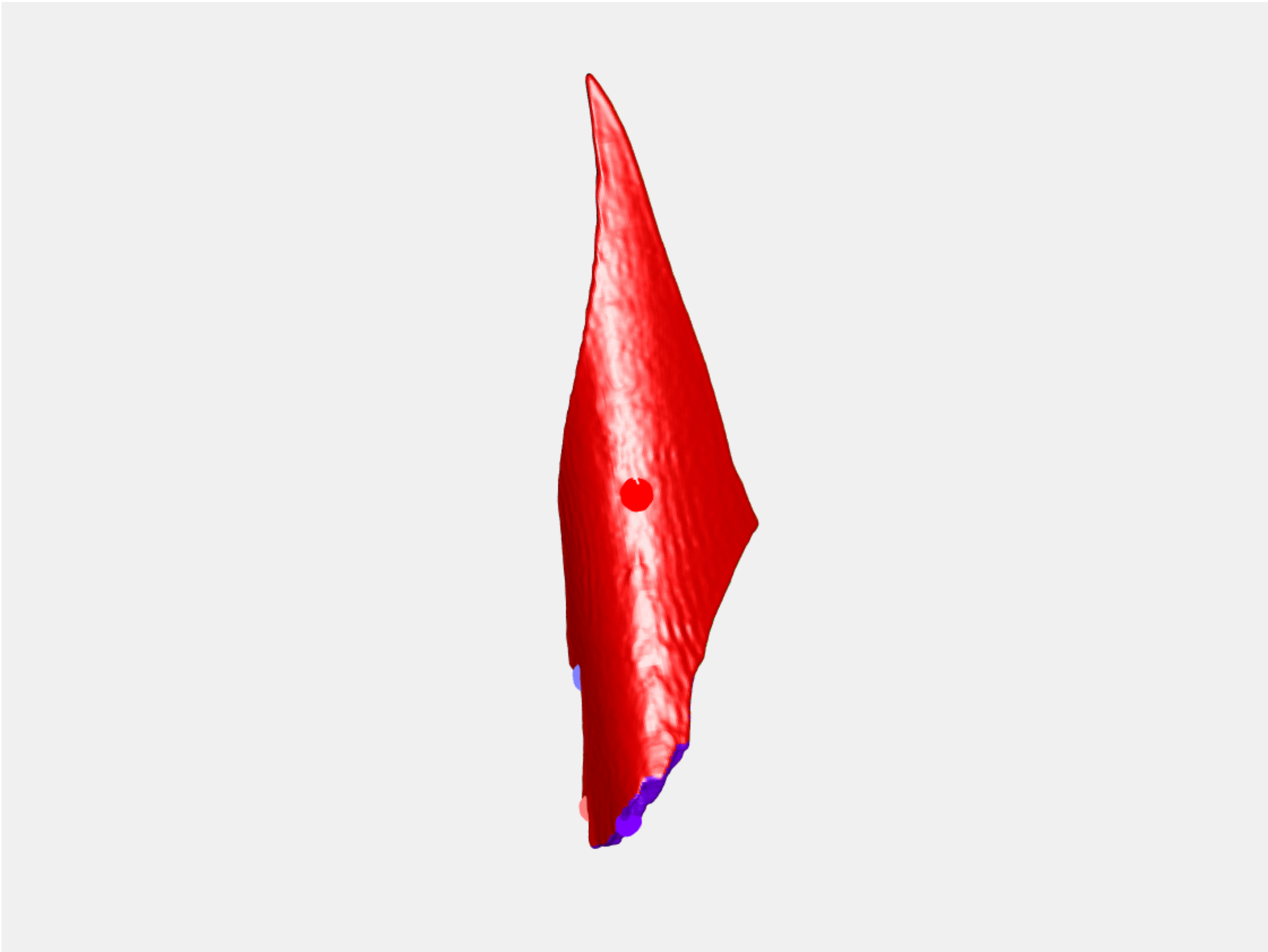
Segmentation



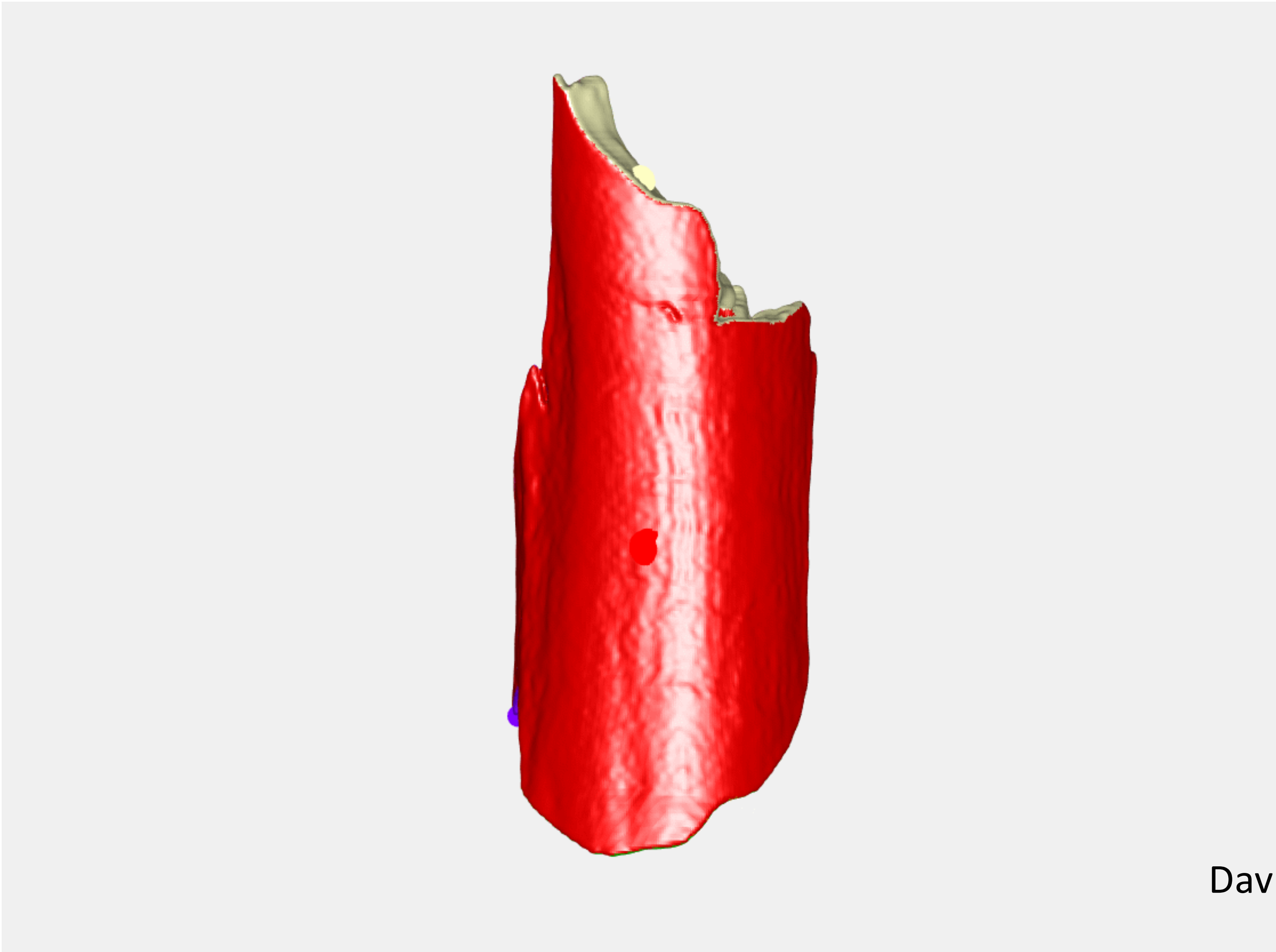


(a) Bone fragment (b) Face segmentation (c) Edge tracing

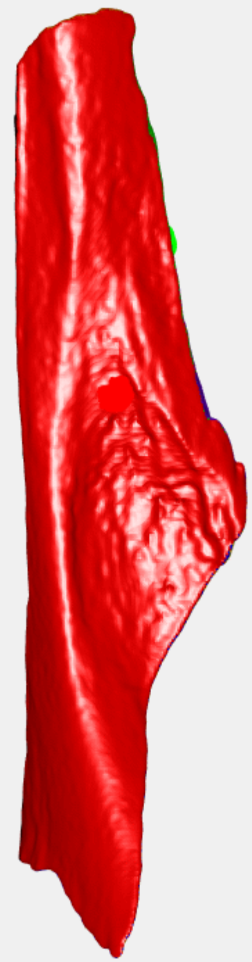
Bone Fragment Segmentation using Semi-supervised Graph-based Poisson Learning



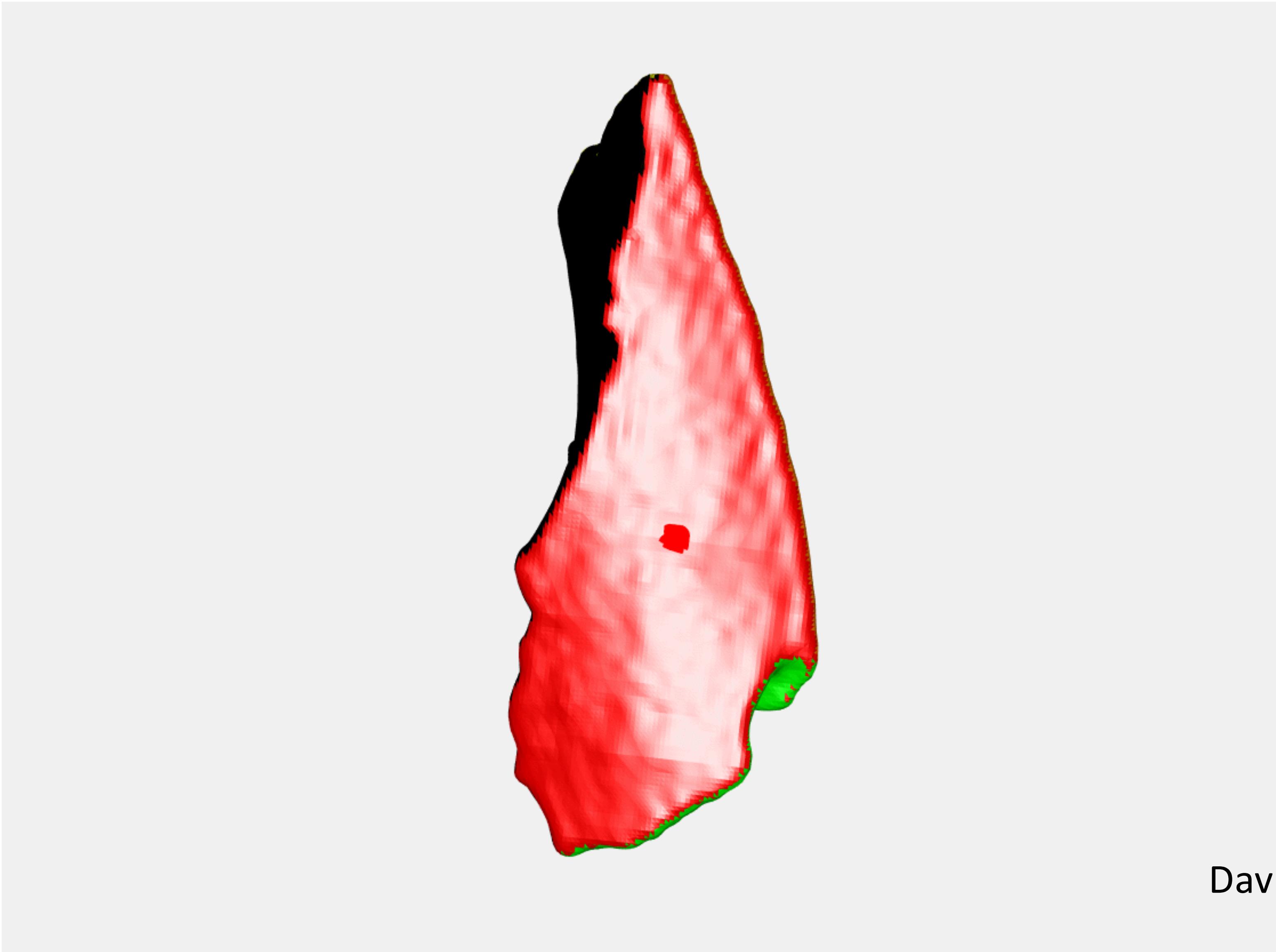
David Floeder



David Floeder



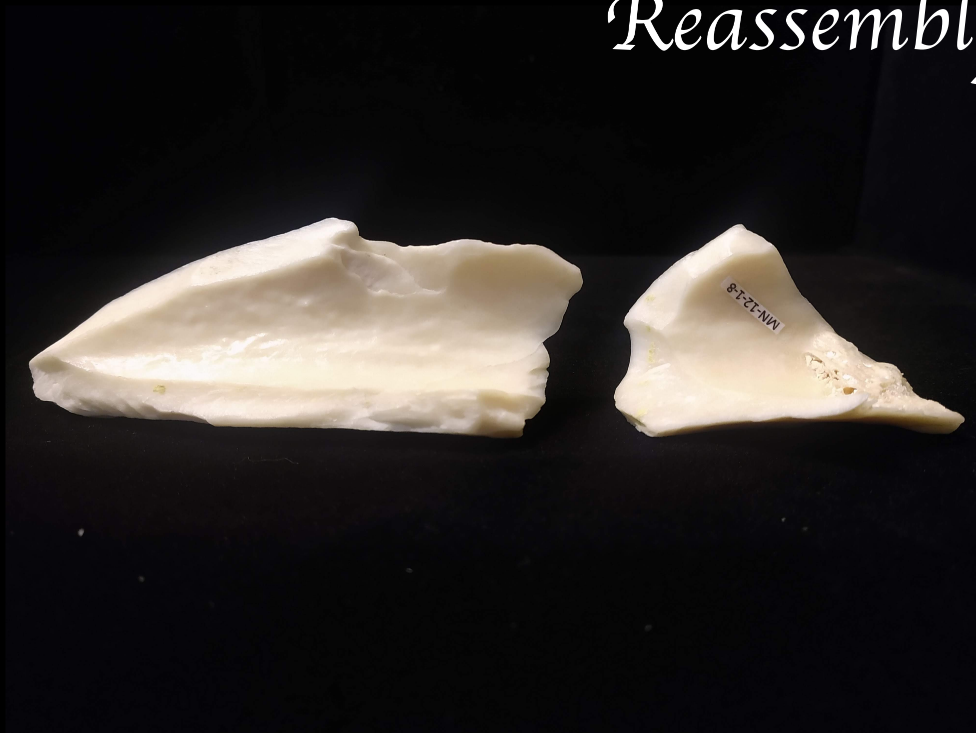
David Floeder

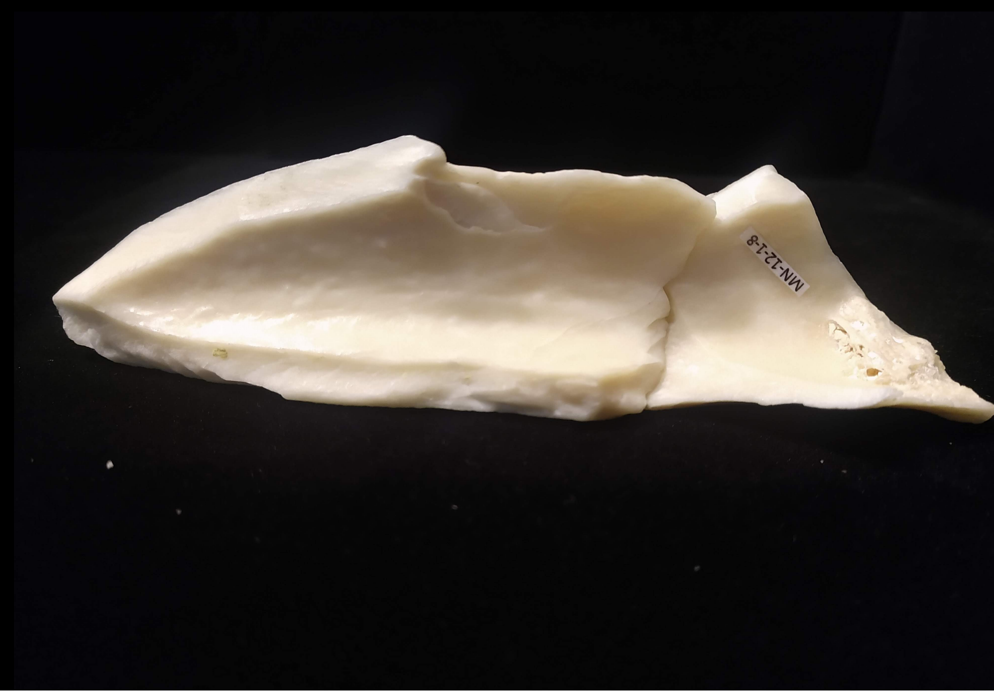


David Floeder

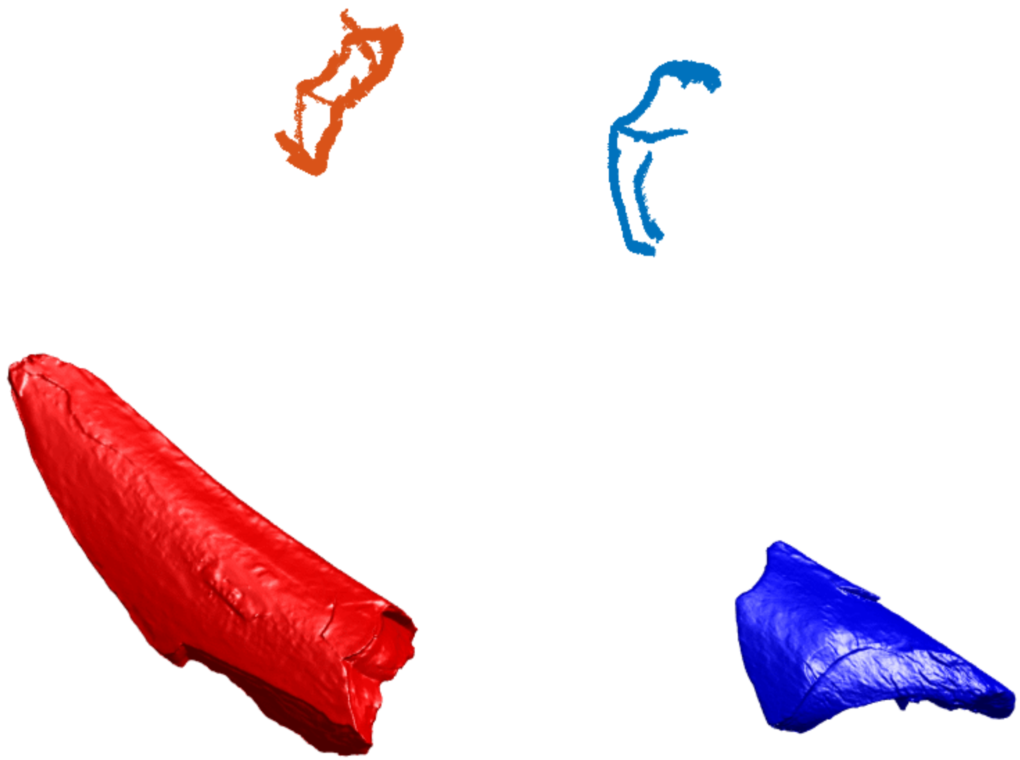
Reassembly (Refit)

Reassembly (Refit)





Gradient descent on SE(3) using an objective function based on segmented break edges and surface normals



Riley O'Neill

Thanks for your attention!