Non-Associative Local Lie Groups

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Lie's Theorems

structure constants C^i_{jk}

$$\Downarrow$$

Lie algebra $\mathfrak{g}$

$$\Downarrow$$

local Lie group L

$$\Downarrow$$

global Lie group G

Theorem. Every local Lie group L is contained in a global Lie group.

$\Longrightarrow$ The result is only true for sufficiently small local Lie groups!

Some History

Local Lie Groups & Lie Algebras:

Lie, Killing, Cartan

Smoothness and Analyticity of Group Actions:

Hilbert's Fifth Problem

Global Lie Groups:

Weyl, Cartan, Chevalley

Globalizability of Topological Groups:

P.A. Smith, Mal'cev

$\Longrightarrow$ associativity

Globalizability of Transformation Groups:

Mostow, Palais

Hilbert's Fifth Problem (Global):

Gleason, Montgomery, Zippin

Hilbert's Fifth Problem (Local):

♠ Jacoby ♠

Hilbert's Fifth Problem (Semigroups):

? Brown, Houston, Hofmann, Weiss ?

Globalizability of Local Groups:

van Est, Douady, Plaut

$\Longrightarrow$ Isometries & metric convergence

Basic Definitions

Definition. *Global Lie group* G:

(i) group (ii) smooth manifold

Multiplication:

$$\mu : G \times G \longrightarrow G \qquad \mu(g,h) = g \cdot h$$

Inversion:

$$\iota : G \longrightarrow G, \qquad \iota(g) = g^{-1}$$

$\Longrightarrow$ smooth, globally defined.

Definition. *Local Lie group L*:

Multiplication:

$$\mu\colon \mathcal{U} \longrightarrow L, \qquad \mu(x,y) = x \cdot y$$
$$(\{e\} \times L) \cup (L \times \{e\}) \subset \mathcal{U} \subset L \times L$$

Inversion:

$$\iota\colon \mathcal{V} \longrightarrow L, \qquad \iota(g) = g^{-1}$$
$$e \in \mathcal{V} \subset L \qquad \mathcal{V} \times \iota(\mathcal{V}),\ \iota(\mathcal{V}) \times \mathcal{V} \subset \mathcal{U}$$

(i) *Identity:* $\quad e \cdot x = x = x \in e, \quad x \in L$

(ii) *Inverse:* $\quad x^{-1} \cdot x = e = x \cdot x^{-1}, \quad x \in \mathcal{V}$

(iii) *Associativity:* $\quad x \cdot (y \cdot z) = (x \cdot y) \cdot z$

$$(x,y),\ (y,z),\ (x \cdot y, z),\ (x, y \cdot z) \in \mathcal{U}.$$

Key Example of a Local Lie Group

$$\{e\} \subset N \subset G$$

$\Longrightarrow$ Open neighborhood of the identity in a global Lie group.

Globalizability

Definition. A local Lie group L is called *globalizable* if there exists a local group homeomorphism $\Phi : L \to N$ mapping L onto a neighborhood of the identity of a global Lie group G.

$$\boxed{\Phi(x \cdot y) = \Phi(x) \cdot \Phi(y) \qquad \Phi(x^{-1}) = \Phi(x)^{-1}}$$

Infinite Elements

Example. $L = \mathbb{R}$. Identity: $e = 0$

$$\mathcal{U} = \{ (x,y) \mid |x\,y| \neq 1 \} \subset L \times L$$

$$\mathcal{V} = \left\{ x \;\middle|\; x \neq \tfrac{1}{2},\ x \neq 1 \right\} \subset L$$

$$\boxed{\mu(x,y) = \frac{2xy - x - y}{xy - 1} \qquad \iota(x) = \frac{x}{2x-1}}$$

$\Longrightarrow$ $\widetilde{L} = \left\{ \ |x| < \frac{1}{2} \ \right\}$ is globalizable via

$$\Phi(x) = \frac{x}{x-1} \ : \ \widetilde{L} \longrightarrow \left\{ \ -1 < x < \tfrac{1}{3} \ \right\} \subset \mathbb{R}$$

$$\Phi(\mu(x,y)) = \Phi(x) + \Phi(y) \qquad \Phi(\iota(x)) = -\,\Phi(x)$$

$$\boxed{\mu(x,y) = \frac{2xy - x - y}{xy - 1} \qquad \iota(x) = \frac{x}{2x-1}}$$

But:

$\mu(x,1) = \mu(1,x) = 1$ for all $x \neq 1$

$\Longrightarrow$ *infinite group element*

Also: $\iota(1) = 1$, but $\mu(1, \iota(1))$ not defined.

$\mu(x,y) = 1$ if and only if $x = 1$ or $y = 1$

$\Longrightarrow$ *inaccessible*

Note: $L \subset \mathbb{RP}^1$, which is also a local Lie group with an infinite group element, containing a global Lie group as a dense open subset.

Regularity

Definition. A local Lie group L is called *regular* if, for each $x \in L$, the left and right multiplication maps

$$\lambda_x(y) = \mu(x, y), \quad \rho_x(y) = \mu(y, x).$$

are diffeomorphisms on their respective domains of definition.

Inversional Local Groups

Given $U \subset L$, let $U^{(n)}$ denote the set of all well-defined n-fold products of elements $x_1, \ldots, x_n \in U$.

Definition. U *generates* L if $L = \bigcup_{n=1}^{\infty} U^{(n)}$.

Definition. A local Lie group L is called *globally inversional* if the inversion map ι is defined everywhere, so that $\mathcal{V} = L$.

Definition. A local Lie group L is called *inversional* if $\mathcal{V}$ generates L, i.e., every $x \in L$ can be written as a product of invertible elements.

Theorem. Every inversional local Lie group is regular.

Definition. L is a *connected local Lie group* if

(i) L is a connected manifold,

(ii) the domains of definition of the multi- plication and inversion maps are connected,

(iii) if $U \subset L$ is any neighborhood of the identity, then U generates L.

$\Longrightarrow$ *Plaut*

Proposition. Any connected local Lie group is inversional, and hence regular.

$\Longrightarrow$ From now on all local Lie groups are be assumed to be connected.

Higher Associativity

Definition. A local Lie group is

$$\boxed{\textit{associative to order } n}$$

if, for every $3 \leq m \leq n$, and every $(x_1, \ldots, x_m) \in L^{\times m}$, all well-defined m-fold products are equal.

Example.

$$x_1 \cdot (x_2 \cdot (x_3 \cdot x_4)) = x_1 \cdot ((x_2 \cdot x_3) \cdot x_4)$$
$$= (x_1 \cdot x_2) \cdot (x_3 \cdot x_4) = (x_1 \cdot (x_2 \cdot x_3)) \cdot x_4$$
$$= ((x_1 \cdot x_2) \cdot x_3) \cdot x_4$$

$$\Longrightarrow \quad \text{Catalan number } C_n = \frac{1}{n}\binom{2n-2}{n-1}$$

A local group is called *globally associative* if it is associative to every order $n \geq 3$.

Globalizability

Theorem. A connected local Lie group L is globalizable if and only if it is globally associative.

$\Longrightarrow$ Mal'cev

$\star\ \star\ \star$ There exist local Lie groups that are associative to order n but not order $n+1$!

The Simplest non-Globalizable Example

$$G = \mathbb{R}^2 \simeq \mathbb{C} \qquad\qquad M = G \setminus \{-1\}$$

$$L = \widetilde{M} \simeq \mathbb{R}^2$$

— simply connected covering space

$\pi : L \longrightarrow M$ — covering map.

$$\pi(\widehat{z}) = z \qquad\qquad \widehat{z} = (z, n)$$

$$L \;\simeq\; \{\, (r, \theta) \mid\; r > 0 \,\}$$

$$z = \pi(r, \theta) = r e^{i\theta} - 1 \qquad\qquad (2n-1)\pi < \theta \leq (2n+1)\pi$$

$$L_0 = \{ (r,\theta) \mid \tfrac{1}{2}\sec\theta < r < \tfrac{3}{2}\sec\theta, -\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi \}$$

lies above $M_0 = \{ -\tfrac{1}{2} < \operatorname{Re} z < \tfrac{1}{2} \}$

$$L_1 = \{ (r,\theta) \mid -\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi \}$$

lies above $M_1 = \{\operatorname{Re} z > -1\}$

$$\alpha(z,w) = \arg(w+1) - \arg(z+1)$$

$$-\pi < \alpha(z,w) \le \pi$$

$\Longrightarrow$ angle from z to w wrt -1

$$H_z = \{ \widehat{w} \in L_0 \mid -\tfrac{1}{2}\pi < \alpha(z, z+w) < \tfrac{1}{2}\pi \}$$

Domain of definition of multiplication:

$$\mathcal{U} = \{ (\widehat{z}, \widehat{w}) \in L \times L \mid \widehat{z} \in H_w \quad \text{or} \quad \widehat{w} \in H_z \}.$$

Domain of definition of inversion: $\mathcal{V} = L_0$

Theorem. Under the above constructions, the product $\mu\colon \mathcal{U} \to L$ and inversion $\iota\colon \mathcal{V} \to L$ endow L with the structure of a regular, connected, associative, local Lie group which is not globally associative.

General Examples

G — connected, simply connected global Lie group

$e \notin S \subset G$ — closed subset

$M = G \setminus S$ — globalizable local Lie group

$L = \widetilde{M}$ — nontrivial covering group

$\Longrightarrow$ non-globalizable local Lie group

$\Longrightarrow$ A (generalized) *covering map* is a local diffeomorphism $\Phi : L \to \widetilde{M}$

Frames

M — smooth m-dimensional manifold.

Definition. A *frame* is an ordered set of vector fields $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ that form a basis for the tangent space $TM|_x$ at each $x \in M$.

Structure equations:

$$[\,\mathbf{v}_i, \mathbf{v}_j\,] = \sum_{k=1}^{m} C_{ij}^k\,\mathbf{v}_k, \quad i, j = 1, \ldots, m.$$

The frame has *rank* 0 if the structure coefficients C_{ij}^k are all constant, and are hence the structure constants of a Lie algebra $\mathfrak{g}$.

Theorem. If L is a regular, locally associative, local Lie group, then it admits a right-invariant frame of rank 0. Conversely, if M is a manifold that admits a rank 0 frame, then M can be endowed with the structure of a regular, locally associative local Lie group having the given frame as right-invariant Lie algebra elements.

Coframes

Definition. A *coframe* on M is an ordered set of one-forms $\boldsymbol{\theta} = \{\theta^1, \ldots, \theta^m\}$ which form a basis for the cotangent space $T^*M|_x$ at each $x \in M$:

$$\theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^m \neq 0$$

Structure equations:

$$d\theta^k = - \sum_{1 \leq i < j \leq m} C^k_{ij}\, \theta^i \wedge \theta^j,$$

$\Longrightarrow$ Maurer–Cartan forms

Main Theorem

Theorem. Let L be a connected local Lie group. Then there exists a local covering group $\bar{L} \to L$ which is also a local covering group $\bar{L} \to M$ of an open subset $e \in M \subset G$ of a global Lie group G.

$\Longrightarrow$ The proof is based on the Cartan equivalence method, using the Frobenius Existence Theorem for first order systems of partial differential equations and Cartan's technique of the graph.

Another Example

$$L = \{\, (r,\varphi) \mid \quad r > 0 \ \,\}$$

Frame vector fields:

$$\mathbf{v}_1 = \cos\varphi\,\frac{\partial}{\partial r} - \frac{\sin\varphi}{r}\,\frac{\partial}{\partial\varphi} \qquad \mathbf{v}_2 = \sin\varphi\,\frac{\partial}{\partial r} + \frac{\cos\varphi}{r}\,\frac{\partial}{\partial\varphi}$$

$\Longrightarrow$ in rectangular coordinates: $\mathbf{v}_1 \longmapsto \dfrac{\partial}{\partial x}, \ \mathbf{v}_2 \longmapsto \dfrac{\partial}{\partial y}$

The vector fields commute: $[\,\mathbf{v}_1, \mathbf{v}_2\,] = 0$

but their flows do not commute!

$$\exp(\sqrt{2}\,\mathbf{v}_1)\exp(\sqrt{2}\,\mathbf{v}_2)\left(1, \tfrac{5}{4}\pi\right) = \exp(\sqrt{2}\,\mathbf{v}_1)\left(1, \tfrac{3}{4}\pi\right) = \left(1, \tfrac{1}{4}\pi\right)$$

$$\exp(\sqrt{2}\,\mathbf{v}_2)\exp(\sqrt{2}\,\mathbf{v}_1)\left(1, \tfrac{5}{4}\pi\right) = \exp(\sqrt{2}\,\mathbf{v}_2)\left(1, \tfrac{7}{4}\pi\right) = \left(1, \tfrac{9}{4}\pi\right)$$

Indeed,

$$\boxed{\exp(s\mathbf{v}_1)\exp(t\mathbf{v}_2)x_0 = \exp(t\mathbf{v}_2)\exp(s\mathbf{v}_1)x_0,}$$

only for (s,t) in the *connected component* of $V = \{\,(s,t) \mid \text{both sides are defined}\,\} \subset \mathbb{R}^2$.