Convergence of Normal Forms for Submanifolds under Lie Pseudo–Group Actions

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Normal Forms

A *normal form*, also known as a *canonical form*, is defined as a simple representative element chosen from an equivalence class of objects.

- Simplifies the treatment of such objects
- Solves the equivalence problem: two objects are equivalent if and only if they have the same normal form.

Examples:

- Algebra: Jordan canonical form
- Dynamical systems of ODEs.

Normal Forms of Submanifolds

Normal forms of (analytic) submanifolds under a group action can be identified with normalized power series. The non-constant Taylor coefficients provide a complete set of independent differential invariants.

Applications include:

- Differential geometry
- Differential equations
- Calculus of variations
- Control theory
- Classical invariant theory
- Image processing

Normal Forms & Moving Frames

The equivariant moving frame normalizations based on a cross-section to the pseudo-group orbits in jet space can be reinterpreted as placing the submanifold in normal form, meaning that one uses successive group transformations to move it to a distinguished location and then normalize certain coefficients in the associated Taylor expansion. Once these are fixed, the remaining unnormalized coefficients are the differential invariants of
the original submanifold.

Normal Forms for Lie Group Actions

Assume the Lie group G acts freely, regularly, analytically, and transitively on the independent variables. Let $r = \dim G$ and $p = \dim S$. Then the moving frame fixes the center of the normal form series and normalizes $r - p$ of its coefficients to constants by applying a sequence of group transformations. The remaining unnormalized coefficients, when expressed in terms of the original submanifold, form a complete system of functionally independent differential invariants.

Since the normal form results from applying a finite number of group transformations, it is automatically convergent.

Normal Forms of Plane Curves

Euclidean normal form: $G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$

$$u_0(x) = \frac{1}{2}\,\kappa\,x^2 + \frac{1}{6}\,\kappa_s\,x^3 + \frac{1}{24}\,(\,\kappa_{ss} + 3\,\kappa^3\,)\,x^4 + \;\cdots$$

$\Longrightarrow$ κ curvature; ds arc length

Equi-affine normal form: $G = \mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$

$$u_0(x) = \frac{1}{2}\,x^2 + \frac{1}{4!}\,\kappa\,x^4 + \frac{1}{5!}\,\kappa_s\,x^5 + \frac{1}{6!}\,(\,\kappa_{ss} + 5\,\kappa^2\,)\,x^6 + \;\cdots$$

$\Longrightarrow$ κ equi-affine curvature; ds equi-affine arc length

$\Longrightarrow$ The formulas for the higher order coefficients are algorithmically found using the equivariant moving frame recurrence formulas.

Normal Forms for Lie Pseudo-group Actions

Again one normalizes the Taylor coefficients of the normal form to constants by recursively applying a sequence of group transformations. The remaining unnormalized coefficients, when expressed in terms of the original submanifold, form a complete system of functionally independent differential invariants. However, since an infinite number of pseudo-group transformations are required, convergence of the normal form is no longer assured.

The Chern–Moser Normal Form

★★ Basic problem: Equivalence of real hypersurfaces in $\mathbb{C}^n$ for the Lie pseudo-group of biholomorphic transformations

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- Poincaré (1907) was the first to observe that not all hypersurfaces are equivalent.
- Élie Cartan (1932) solves the $n = 2$ dimensional case
- Chern and Moser (1974) solve the n-dimensional case, and prove the existence of a convergent normal form power series for suitably nondegenerate real hypersurfaces.
- Kolář (2012) produced examples of singular hypersurfaces whose normal form power series are divergent.

Real Hypersurfaces in Complex Two-dimensional Space

Coordinates on $\mathbb{C}^2$: $z = x + \mathrm{i}\, y, \quad w = u + \mathrm{i}\, v$

$S \subset \mathbb{C}^2$ —

real three-dimensional hypersurface, the graph of

$$v = F(z, \bar{z}, u)$$

Assume S is everywhere *Levi nondegenerate*: $\dfrac{\partial^2 F}{\partial z\, \partial \bar{z}} \neq 0$

Preliminary normal form:

$$v = z\,\bar{z} + 8\,\mathrm{Re}\,(\,C\,z^4\,\bar{z}^2\,) + \ \cdots$$

C — *Cartan curvature* of the hypersurface

Umbilic point: $C = 0$

Theorem. If the hypersurface is everywhere umbilic, then it is locally biholomorphically equivalent to the Heisenberg sphere, which is equivalent to the usual sphere:

$$v = |\,z\,|^2 \qquad \text{or} \qquad |\,z\,|^2 + |\,w\,|^2 = 1$$

Moreover, it admits an 8-dimensional holomorphic symmetry group locally isomorphic to $\mathfrak{sl}(3, \mathbb{R})$.

At a non-umbilic point, the Chern–Moser normal form is

$$v = z\,\bar{z} + 2\,\mathrm{Re}\,(\,z^4\,\bar{z}^2 + J\,z^5\,\bar{z}^2 + \mathrm{i}\,K\,z^4\,\bar{z}^2 u) + L\,z^4\,\bar{z}^4\,) + \ \cdots$$

Coefficients of the higher order terms $z^j\,\bar{z}^k\,u^l$ with $j+k \geq 7$ and $\min\{\,j,k\,\} \geq 2$, when expressed in terms of the original hypersurface, form a complete system of functionally independent differential invariants (modulo changes in sign).

As a consequence of the moving frame algorithm, the entire algebra of differential invariants is generated by:

J — complex-valued differential invariant of order 7

K — real-valued differential invariant of order 7

L — real-valued differential invariant of order 8

In fact, generically, for a "K-nondegenerate hypersurface", the algebra is generated by just one Chern–Moser invariant K

In the Chern–Moser paper, convergence was established using a somewhat mysterious construction of “chains” which are certain distinguished curves contained in the submanifold. The chains satisfy analytic systems of ordinary differential equations, and are hence analytic curves, which can be used to prove analyticity of the pseudo-group transformation taking the submanifold to its normal form, which therefore is analytic.

Sabzevari, Valiquette, PJO (2023) used equivariant moving frames, as developed by PJO–Juha Pohjanpelto, to

(a) determine a generating set of differential invariants; and

(b) construct new normal forms at singularly umbilic points.

However, convergence of the normal forms continued to rely on the original Chern–Moser chain method.

The Main Theorem

Theorem. Under suitable hypotheses (which are satisfied in a very broad range of applications), the normal form for an analytic submanifold under an analytic Lie pseudo-group is a convergent power series.

Basic idea behind the proof: The normal form is a solution to an involutive system of partial differential equations, and the moving frame cross-section provides appropriate initial conditions. Hence, by the Cartan–Kähler Existence Theorem, the normal form is an analytic function.

$\star\star$ We utilize the purely PDE form of Cartan–Kähler.

No exterior differential systems!

$\Longrightarrow$ Werner Seiler, *Involution*, Springer, 2010

The Cartan–Kähler Theorem

Theorem. The solution to the non-characteristic initial value problem for an involutive analytic system of partial differential equations is unique and analytic.

Proof:

Repeatedly use the Cauchy–Kovalevskaya Theorem to solve a series of initial value problems.

$\Longrightarrow$ Analyticity is essential.
(Lewy-type counterexamples in the C^∞ category.)

The Normal Form Determining Equations

⋆ The construction of the involutive system of partial differential equations whose solution contains the normal form proceeds in four steps:

Basic Steps

(1) By definition, the Lie pseudo-group satisfies an involutive system of determining equations.

(2) Restricting the pseudo-group determining equations to a fixed submanifold produces the reduced determining equations. If the pseudo-group is reducible, these are also involutive.

(3) A simple change of variables converts the reduced determining equations into the normal form determining equations whose involutivity follows immediately.

(4) Compatibility of involutivity with the initial conditions prescribed by the equivariant moving frame construction requires the notion of a well-posed cross-section.

$\Longrightarrow$ Kolář's divergent normal forms are due to a non-well-posed cross-section.

Jet Bundles

$x = (x^1, \ldots, x^p), \quad u = (u^1, \ldots, u^q)$

— independent and dependent variables

$J = (j_1, \ldots, j_k)$ — symmetric multi-index with $1 \leq j_\nu \leq p$

$$u_J^\alpha = \frac{\partial^k u^\alpha}{\partial x^{j_1} \ \ldots \ \partial x^{j_k}}$$

— partial derivative of order $k = |\,J\,|$

$(x, u^{(n)}) = (\ldots x^i \ldots u^\alpha \ldots u_J^\alpha \ldots) \quad |\,J\,| \leq n$ — jet coordinates

$$\dim \mathrm{J}^n = p + q \binom{p+n}{n}$$

Systems of Partial Differential Equations

n^{th} order system: $\qquad\Longrightarrow\quad$ regularity

$$\mathcal{R}^{(n)} = \{\Delta(x, u^{(n)}) = 0\} \subset \mathrm{J}^n$$

Prolongation = (total) differentiation:

$$\mathcal{R}^{(n+k)} = \{\mathrm{D}_J \Delta(x, u^{(n)}) = 0, \quad 0 \le |\, J \,| \le k\} \subset \mathrm{J}^{n+k}$$

Projection: $\qquad \pi_n^{n+k} : \mathrm{J}^{n+k} \longrightarrow \mathrm{J}^n$

If $\pi_n^{n+k}(\mathcal{R}^{(n+k)}) \subsetneq \mathcal{R}^{(n)}$

it means there are integrability conditions of order k.

Formal Integrability

A system of order $n \geq 1$ is formally integrable if, for all $j, k \geq 0$,

$$\pi^{n+k+j}_{n+k}(\mathcal{R}^{(n+k+j)}) = \mathcal{R}^{(n+k)}$$

In other words, a system of differential equations is formally integrable if, at all orders of prolongation, no additional integrability conditions arise.

However, formal integrability does not suffice to establish existence of solutions! For this we need involutivity.

The Symbol Matrix

$\Delta = 0$ — a system of partial differential equations, including all equations obtained by differentiation (prolongation).

$\Delta_n = 0$ — those of order $= n$

M_n — n^{th} order symbol matrix

$=$ Jacobian matrix of the order n equations with respect to the derivatives of order $= n$

$r_n = \operatorname{rank} M_n$.

How are the ranks computed?

♡ Depends on the ordering of variables (derivatives).

Involutivity

Involutivity = Gaussian Elimination + term ordering.

More specifically, the rank of symbol matrix equals the number of pivots following Gaussian elimination.

$\star\star$ When performing elimination, the columns of the symbol matrix are arranged using an intelligent ordering of the jet variables.

Row Echelon Form:

$$
\begin{pmatrix}
0 & \circledast & * & \dots & * & * & \dots & * & * & \dots & \dots & * & * & * & \dots & * \\
0 & 0 & 0 & \dots & 0 & \circledast & \dots & * & * & \dots & \dots & * & * & * & \dots & * \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \circledast & \dots & \dots & * & * & * & \dots & * \\
 & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & & \ddots & & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \dots & 0 & \circledast & * & \dots & * \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \dots & 0 & 0 & 0 & \dots & 0 \\
 & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \dots & 0 & 0 & 0 & \dots & 0
\end{pmatrix}
$$

The entries indicated by $\circledast$ are the *pivots* and must be nonzero.

Rank = # pivots

Reduced Row Echelon Form:

$$
\begin{pmatrix}
0 & \circledast & \ast & \dots & \ast & 0 & \dots & \ast & 0 & \dots & \dots & \ast & 0 & & \ast\dots & \ast \\
0 & 0 & 0 & \dots & 0 & \circledast & \dots & \ast & 0 & \dots & \dots & \ast & 0 & \ast & \dots & \ast \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \circledast & \dots & \dots & \ast & 0 & \ast & \dots & \ast \\
 & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & & \ddots & & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \dots & 0 & \circledast & \ast & \dots & \ast \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \dots & 0 & 0 & 0 & \dots & 0 \\
 & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \dots & 0 & 0 & 0 & \dots & 0
\end{pmatrix}
$$

Key remark: under permutations of the columns (reordering the variables), the rank does not change but the columns in which the pivots appear can change.

Term Ordering

The rows of M_n are indexed by the equations.

The columns of M_n are indexed by the jet variables u_J^α representing derivatives of order $|J| = n$.

$J = (j_1, \dots, j_n)$ is a symmetric multi-index with $1 \le j_\nu \le p$

Definition. The *class* of J is the smallest index that appears:

$$\operatorname{cls} J = \min\{j_1, \dots, j_n\}$$

Assuming $x = x^1 \prec y = x^2 \prec z = x^3$. At order $n = 2$

Class 1: $(xx), (xy), (xz)$ Class 2: $(yy), (yz)$ Class 3: (zz)

The columns in M_n are ordered so that those of higher class are to the *left*:

$$u_{zz}, u_{yz}, u_{yy}, u_{xz}, u_{xy}, u_{xx}$$

Once the symbol matrix is fully row reduced, the pivot columns correspond to principal derivatives.

$$r_n = \operatorname{rank} M_n = \#\,\text{pivots} = \#\,\text{principal derivatives}$$

The non-pivot columns correspond to parametric derivatives.

By the Implicit Function Theorem, the system can be locally solved for the principal derivatives in terms of the parametric derivatives (and the independent variables).

$\Longrightarrow$ (reduced) Cartan normal form.

First order system in reduced Cartan normal form:

$$
\begin{aligned}
u_p^\alpha &= \Delta_p^\alpha(x^1,\dots,x^p,\dots,u_k^\beta,\dots), && 1 \le \alpha \le b_1^{(p)},\\
u_{p-1}^\alpha &= \Delta_{p-1}^\alpha(x^1,\dots,x^p,\dots,u_k^\beta,\dots), && 1 \le \alpha \le b_1^{(p-1)},\\
&\vdots \\
u_1^\alpha &= \Delta_1^\alpha(x^1,\dots,x^p,\dots,u_k^\beta,\dots), && 1 \le \alpha \le b_1^{(1)},\\
u^\alpha &= \Delta^\alpha(x^1,\dots,x^p,u^\delta), && 1 \le \alpha \le b_0,
\end{aligned}
$$

with indices (see below)

$$
0 \le b_0 \le b_1^{(1)} \le \cdots \le b_1^{(p-1)} \le b_1^{(p)} \le q,
$$

and where all the derivatives appearing on the right hand side of each equation are parametric of class smaller than or equal to the class of the principal derivative occurring on the left hand side. If $b_0 = 0$ there are no algebraic equations.

Formally well-posed initial conditions:

$$\begin{aligned}
u^\beta(0,\ldots,0) &= f^\beta, & b_0 < \beta \le b_1^{(1)},\\
u^\beta(x^1,0,\ldots,0) &= f^\beta(x^1), & b_1^{(1)} < \beta \le b_1^{(2)},\\
&\vdots\\
u^\beta(x^1,\ldots,x^{p-1},0) &= f^\beta(x^1,\ldots,x^{p-1}), & b_1^{(p-1)} < \beta \le b_1^{(p)},\\
u^\beta(x^1,\ldots,x^p) &= f^\beta(x^1,\ldots,x^p), & b_1^{(p)} < \beta \le q.
\end{aligned}$$

When involutive, uniquely solve the corresponding initial value problem using Cauchy–Kovalevskaya, thereby establishing Cartan–Kähler.

Indices and Cartan Characters

$p = \#$ independent variables; $q = \#$ dependent variables

Number of derivatives u^α_J of order $n = |\,J\,|$ and class k

$$t_n^{(k)} = q \binom{p+n-k-1}{n-1}$$

Indices:

$$\begin{aligned} b_n^{(k)} &= \#\text{ pivots of class } k \text{ in } M_n \\ &= \#\text{ principal derivatives of class } k \text{ and order } n \end{aligned}$$

Cartan characters:

$$\begin{aligned} c_n^{(k)} &= t_n^{(k)} - b_n^{(k)} \\ &= \#\,\text{parametric derivatives of class } k \text{ and order } n \end{aligned}$$

Rank of symbol matrix: $r_n = \operatorname{rank} M_n$

Definition. The symbol matrix M_n is involutive if

$$r_{n+1} = \operatorname{rank} M_{n+1} = \sum_{k=1}^{p} k\, b_n^{(k)}.$$

$\Longrightarrow$ Requires working in δ regular coordinates (maximize r.h.s.)!

Definition. The n^{th} order system of partial differential equations is involutive if it is (a) formally integrable, and (b) has involutive symbol matrix M_n.

Rough Idea: all equations of order $n+1$ should be attained by differentiating the $b_n^{(k)}$ class k equations in the Cartan normal form with respect to the k variables of class $\le k$.

★ Involutivity at order n implies involutivity at all higher orders.

Chern–Moser Chains

Suppose that only the first Cartan character is nonzero:

$$c_n^{(1)} \neq 0 \qquad\qquad c_n^{(2)} = \cdots = c_n^{(p)} = 0 \qquad\qquad (*)$$

Then the integration of the initial value problem for the first order Cartan normal form reduces to a system of analytic first order ordinary differential equations with initial values

$$\begin{array}{ll}
u^\beta(0,\ldots,0) = f^\beta, & b_0 < \beta \leq b_1^{(1)}, \\
u^\beta(x^1,0,\ldots,0) = f^\beta(x^1), & b_1^{(1)} < \beta \leq b_1^{(2)}, \\
\vdots & \\
u^\beta(x^1,\ldots,x^{p-1},0) = f^\beta(x^1,\ldots,x^{p-1}), & b_1^{(p-1)} < \beta \leq b_1^{(p)}, \\
u^\beta(x^1,\ldots,x^p) = f^\beta(x^1,\ldots,x^p), & b_1^{(p)} < \beta \leq q.
\end{array}$$

Chern–Moser Chains

Suppose that only the first Cartan character is nonzero:

$$c_n^{(1)} \neq 0 \qquad\qquad c_n^{(2)} = \cdots = c_n^{(p)} = 0 \qquad\qquad (*)$$

Then the integration of the initial value problem for the first order Cartan normal form reduces to a system of analytic first order ordinary differential equations.

★★ In the Chern–Moser problem, the normal form determining equations satisfy $(*)$, and the curves defined by their solutions are the chains.

This construction indicates how to formulate a concept of chains, including higher dimensional versions, for rather general pseudo-group actions.

Pseudo-groups

M — analytic manifold

Definition. A pseudo-group is a collection of local analytic diffeomorphisms $\phi\colon \operatorname{dom}\phi \subset M \to M$ such that

- *Identity:* $\mathbf{1}_M \in \mathcal{G}$
- *Inverses:* $\phi^{-1} \in \mathcal{G}$
- *Restriction:* $U \subset \operatorname{dom}\phi \implies \phi \,|\, U \in \mathcal{G}$
- *Continuation:* $\operatorname{dom}\phi = \bigcup U_\kappa$ and $\phi \,|\, U_\kappa \in \mathcal{G} \implies \phi \in \mathcal{G}$
- *Composition:* $\operatorname{im}\phi \subset \operatorname{dom}\psi \implies \psi \circ \phi \in \mathcal{G}$

Sur la théorie, si importante sans doute, mais pour nous si obscure, des ≪groupes de Lie infinis≫, nous ne savons rien que ce qui se trouve dans les mémoires de Cartan, première exploration à travers une jungle presque impénétrable; mais celle-ci menace de se refermer sur les sentiers déjà tracés, si l'on ne procède bientôt à un indispensable travail de défrichement.

— André Weil, 1947

Why an "Impenetrable Jungle"?

- Lie invented Lie groups to study symmetry and solution of differential equations.

♢ In Lie's time, there were no abstract Lie groups. All groups were realized by their action on a space.

♠ Therefore, Lie saw no essential distinction between finite-dimensional and infinite-dimensional group actions.

However, with the advent of abstract Lie groups, the two subjects have gone in radically different directions.

♡ The general theory of finite-dimensional Lie groups has been rigorously formalized and applied throughout mathematics.

♣ But there is still no generally accepted abstract object that represents an infinite-dimensional Lie pseudo-group!

⋆ essential invariants

Lie Pseudo-groups — Applications

- CR geometry and Chern–Moser Theory
- Normal forms for submanifolds
- Symmetry groups of differential equations
- Calculus of variations
- Gauge theories
- Invariant geometric flows
- Computer vision and mathematical morphology
- Geometric numerical integration
- Control theory
- Cartan equivalence problems
- *Lie groups*!

Symmetry Lie Pseudo-groups

- Linear and linearizable PDEs
- Relativity
- Noether's Second Theorem
- Gauge theory and field theories:
  Maxwell, Yang–Mills, conformal, string, . . .
- Fluid mechanics, metereology:
  Navier–Stokes, Euler, boundary layer, quasi-geostropic, . . .
- Solitons (in $2 + 1$ dimensions):
  Kadomtsev–Petviashvili, Davey–Stewartson, . . .
- Vessiot group splitting; explicit solutions

The Diffeomorphism Pseudo-group

M — $m = p + q$ dimensional manifold

Local coordinates on M: $z = (x, u) = (x^1, \ldots, x^p, u^1, \ldots, u^q)$

$\mathcal{D} = \mathcal{D}(M)$ — pseudo-group of all local analytic diffeomorphisms.

Cartan's notation:

$$\begin{cases} z = (z^1, \ldots, z^m) & \text{— source coordinates} \\ Z = (Z^1, \ldots, Z^m) & \text{— target coordinates} \end{cases}$$

Diffeomorphism:

$$Z = \phi(z)$$

Jets of Diffeomorphisms

$\mathrm{J}^n(M,M)$ — n^{th} order jet bundle for maps $\phi\colon M \to M$.

Local coordinates on $\mathrm{J}^n(M,M)$:

$$(z, Z^{(n)}) = (\ \ldots\ z^a\ \ldots\ Z^b\ \ldots\ Z^b_A\ \ldots\ ) \qquad Z^b_A = \frac{\partial^k Z^b}{\partial z^{a_1} \cdots \partial z^{a_k}}$$

Diffeomorphism subbundle: $\mathcal{D}^{(n)} \subset \mathrm{J}^n(M,M)$ consists of all jets with non-singular Jacobian matrix.

Lie Pseudo-groups

Any pseudo-group $\mathcal{G} \subset \mathcal{D}$ defines a subvariety (system of partial differential equations) $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ consisting of all its jets.

Definition. $\mathcal{G}$ is regular if, for all $n \gg 0$, $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ forms an embedded subbundle and the projection $\pi_n^{n+1} : \mathcal{G}^{(n+1)} \to \mathcal{G}^{(n)}$ is a fibration.

Definition. A regular, analytic pseudo-group $\mathcal{G}$ is called a Lie pseudo-group of order $n \geq 1$ if *every* local diffeomorphism $\phi \in \mathcal{D}$ satisfying $\mathrm{j}_n\phi \subset \mathcal{G}^{(n)}$ belongs it: $\phi \in \mathcal{G}$.

In local coordinates, $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ forms a system of differential equations

$$F^{(n)}(z, Z^{(n)}) = 0$$

called the determining system of the pseudo-group. The Lie condition requires that *every* local solution to the determining system belongs to the pseudo-group.

What about involutivity?

Lemma. In the analytic category, for sufficiently large $n \gg 0$ the determining system $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ of a regular pseudo-group is an involutive system of partial differential equations.

Proof: Cartan–Kuranishi + local solvability.

Steps in the Proof of the Normal Form Convergence Theorem

In slightly more detail, let us go through the four steps of the proof using a relatively simple illustrative example.

Start with the pseudo-group transformations:

$$X^i = X^i(x,u), \quad U^\alpha = U^\alpha(x,u)$$

written in terms of the independent and dependent variables defining our submanifolds.

★★ We use Cartan's notation, where x^j, u^β denote the source variables while the corresponding upper case letters X^i, U^α denote the target variables.

Step 1: By definition, the Lie pseudo-group satisfies an involutive system of determining equations.

To find the determining equations, successively differentiate the pseudo-group transformations

$$X^i = X^i(x,u), \qquad U^\alpha = U^\alpha(x,u).$$

with respect to the x^j and u^β.

Use implicitization to determine all the equations relating their derivatives X^i_A, U^α_B of order $|\,A\,|, |\,B\,| \le n$.

Place the resulting equations in Cartan normal form by solving for the principal derivatives in terms of the parametric derivatives. Check involutivity by computing the indices or the Cartan characters.

A (Relatively) Simple Example

$$X = f(x) \qquad Y = f'(x)\, y + g(x) \qquad U = u + \frac{f''(x)\, y + g'(x)}{f'(x)}$$

Here $f(x)$ and $g(x)$ are analytic scalar functions with $f'(x) \neq 0$, so that f defines a local diffeomorphism.

We consider the action of the pseudo-group on surfaces (graphs)

$u = f(x, y)$ — source submanifold (normal form)

$U = F(X, Y)$ — target submanifold (given)

In other words, the pseudo-group transformation maps the normal form to the given submanifold (left moving frame).

A (Relatively) Simple Example

$$X = f(x) \qquad Y = f'(x)\,y + g(x) \qquad U = u + \frac{f''(x)\,y + g'(x)}{f'(x)}$$

Determining equations of order ≤ 2 (in normal form):

$$X_y = X_u = 0, \quad Y_x = (U - u)X_x, \quad Y_y = X_x, \quad Y_u = 0, \quad U_u = 1,$$

$$X_{xx} = U_y X_x, \quad X_{xy} = X_{xu} = X_{yy} = X_{yu} = X_{uu} = 0, \quad Y_{xx} = (U_x + (U - u)U_y)X_x,$$

$$Y_{xy} = U_y X_x, \quad Y_{xu} = Y_{yy} = Y_{yu} = Y_{uu} = 0, \quad U_{xu} = U_{yy} = U_{yu} = U_{uu} = 0.$$

Parametric derivatives: $X,\ Y,\ U,\ X_x,\ U_x,\ U_y,\ U_{xx},\ U_{xy}.$

ranks = # equations: $r_1 = 6, \quad r_2 = 16$

Using the ordering $x \prec y \prec u$, the second order indices are

$$b_2^{(1)} = 7 \qquad b_2^{(2)} = 6 \qquad b_2^{(3)} = 3 \qquad b_2^{(1)} + b_2^{(2)} + b_2^{(3)} = r_2$$

Determining equations of order 3:

$$X_{xxx} = (U_{xy} + U_y^2)X_x,$$

$$X_{xxy} = X_{xxu} = X_{xyy} = X_{xyu} = X_{xuu} = X_{yyy} = X_{yyu} = X_{yuu} = X_{uuu} = 0,$$

$$Y_{xxx} = (U_{xx} + (U - u)(U_{xy} + U_y^2) + 2U_xU_y)X_x, \qquad Y_{xxy} = (U_{xy} + U_y^2)X_x,$$

$$Y_{xxu} = Y_{xyy} = Y_{xyu} = Y_{xuu} = Y_{yyy} = Y_{yyu} = Y_{yuu} = Y_{uuu} = 0,$$

$$U_{xxu} = U_{xyy} = U_{xyu} = U_{xuu} = U_{yyy} = U_{yyu} = U_{yuu} = U_{uuu} = 0.$$

Involutivity:

$$b_2^{(1)} + 2\,b_2^{(2)} + 3\,b_2^{(3)} = 7 + 2 \times 6 + 3 \times 3 = 28 = r_3 = \#\ \text{equations}$$

Step 2: Restricting the pseudo-group determining equations to a fixed submanifold produces the reduced determining equations.

If the pseudo-group is reducible, these are also involutive.

To find the reduced determining equations start with the reduced pseudo-group transformations

$$\overline{X}^i(x) = X^i(x, u(x)), \quad \overline{U}^\alpha(x) = U^\alpha(x, u(x)).$$

Use the chain rule to successively differentiate with respect to the x^j and simplify using the determining equations. Eliminate all parametric pseudo-group derivatives (implicitization) to determine all the equations relating their derivatives $\overline{X}^i_J, \overline{U}^\alpha_K$.

Reducible Lie Pseudo-groups

Definition. A Lie pseudo-group is reducible on a submanifold s if, at sufficiently large order $n \gg 0$, the fiber dimension of its reduced determining equations equals the fiber dimension of its determining equations. Such a submanifold is called reducible.

$\Longrightarrow$ The determining equations are for functions $X(x,u), U(x,u)$ of $p+q$ variables, whereas the reduced determining equations are for functions $\overline{X}(x) = X(x, f(x))$, $\overline{U}(x) = U(x, f(x))$ of only p variables.

★ All Lie pseudo-groups that act eventually freely, which are precisely the ones amenable to the equivariant moving frame construction, are reducible.

$\Longrightarrow$ A Lie pseudo-group is reducible if it is "not too large". In particular, it cannot depend on functions of more than p variables. More precisely:

Theorem. Let $\mathcal{G}$ be a reducible Lie pseudo-group on a given submanifold. Then, for sufficiently large n, the Cartan characters of its determining equations satisfy

$$c_n^{(i)} = \bar{c}_n^{(i)}, \quad i = 1, \ldots, p, \qquad c_n^{(p+\alpha)} = 0, \quad \alpha = 1, \ldots, q,$$

where $\bar{c}_n^{(i)}$ are the Cartan characters of the reduced determining equations.

Moreover, the reduced determining equations are formally integral and hence involutive at a sufficiently high order.

Freeness

For Lie group actions, freeness means trivial isotropy:

$$G_z = \{\, g \in G \mid g \cdot z = z \,\} = \{e\}.$$

For infinite-dimensional pseudo-groups, this definition cannot work, and one must restrict to the transformation jets of order n, using the n^{th} order isotropy subgroup:

$$\mathcal{G}^{(n)}_{z^{(n)}} = \left\{\, \bar{g}^{(n)} \in \mathcal{G}^{(n)}_z \;\middle|\; g^{(n)} \cdot z^{(n)} = z^{(n)} \,\right\}$$

Definition. At a jet $z^{(n)} \in \mathrm{J}^n$, the pseudo-group $\mathcal{G}$ acts

- freely if $\mathcal{G}^{(n)}_{z^{(n)}} = \{\, \mathbf{1}^{(n)}_z \,\}$
- locally freely if
  - $\mathcal{G}^{(n)}_{z^{(n)}}$ is a discrete subgroup of $\mathcal{G}^{(n)}_z$
  - the orbits have dimension $r_n = \dim \mathcal{G}^{(n)}_z$

$\Longrightarrow$ Kumpera's growth bounds on Spencer cohomology.

For our example, set

$$\overline{X} = X(x, y, u(x, y)) \qquad \overline{Y} = Y(x, y, u(x, y)) \qquad \overline{U} = U(x, y, u(x, y))$$

Differentiate and use the determining equations:

$$\begin{aligned}
\overline{X}_x &= X_x + X_u u_x = X_x, & \overline{X}_y &= X_y + X_u u_y = 0,\\
\overline{Y}_x &= Y_x + Y_u u_x = Y_x = (U - u)X_x, & \overline{Y}_y &= Y_y + Y_u u_y = X_x,\\
\overline{U}_x &= U_x + U_u u_x = U_x + u_x, & \overline{U}_y &= U_y + U_u u_y = U_y + u_y,
\end{aligned}$$

$$\begin{aligned}
\overline{X}_{xx} &= U_y X_x, & \overline{X}_{xy} &= 0, & \overline{X}_{yy} &= 0,\\
\overline{Y}_{xx} &= (U_x + (U - u)U_y)X_x, & \overline{Y}_{xy} &= U_y X_x, & \overline{Y}_{yy} &= 0,\\
\overline{U}_{xx} &= U_{xx} + u_{xx}, & \overline{U}_{xy} &= U_{xy} + u_{xy}, & \overline{U}_{yy} &= u_{yy}.
\end{aligned}$$

Eliminate the parametric variables $X, Y, U, X_x, U_x, U_y, U_{xx}, U_{xy}$:

The Reduced Determining Equations

$$\overline{X}_y = 0, \qquad \overline{Y}_x = (\overline{U} - u)\overline{X}_x, \qquad \overline{Y}_y = \overline{X}_x, \qquad \overline{X}_{xx} = (\overline{U}_y - u_y)\overline{X}_x,$$

$$\overline{X}_{xy} = \overline{X}_{yy} = 0, \qquad \overline{Y}_{xx} = (\,\overline{U}_x - u_x + (\overline{U} - u)(\overline{U}_y - u_y)\,)\overline{X}_x,$$

$$\overline{Y}_{xy} = (\overline{U}_y - u_y)\overline{X}_x, \qquad \overline{Y}_{yy} = 0, \qquad \overline{U}_{yy} = u_{yy},$$

$$\boxed{X = f(x) \qquad Y = f'(x)\,y + g(x) \qquad U = u + \frac{f''(x)\,y + g'(x)}{f'(x)}}$$

Step 3: A simple change of variables converts the reduced determining equations into the normal form determining equations whose involutivity follows.

Replace $\overline{U}^\alpha(x)$ by $\widehat{U}^\alpha(X)$ defined by

$$\overline{U}(x) = \widehat{U}(\overline{X}(x))$$

or, more explicitly,

$$U(x, u(x)) = \widehat{U}(X(x, u(x)))$$

Use the chain rule to rewrite the derivatives of $\overline{U}^\alpha$ with respect to the x^j in terms of the derivatives of $\widehat{U}^\alpha$ with respect to the X^j and the derivatives of $\overline{X}^i$ with respect to the x^j.

Substitute the resulting formulas into the reduced determining equations — the result is the normal form determining equations which take the form

$$\Delta(x, u^{(n)}, \overline{X}^{(n)}, \widehat{U}^{(n)}) = 0$$

Given a prescribed function $\widehat{U} = \widehat{U}(\overline{X})$ defining a submanifold, with known derivatives $\widehat{U}^{(n)}$, we can view these equations as a system of differential equations for the unknown functions $\overline{X}(x), u(x)$ — the latter prescribing the normal form of the given submanifold.

In our example,

$$U(x, y, u(x, y)) = \widehat{U}(\, X(x, y, u(x, y)), Y(x, y, u(x, y))\,)$$

Differentiate:

$$\overline{U}_x = \widehat{U}_X \overline{X}_x + \widehat{U}_Y \overline{Y}_x, \qquad \overline{U}_y = \widehat{U}_X \overline{X}_y + \widehat{U}_Y \overline{Y}_y,$$

$$\overline{U}_{xx} = \widehat{U}_{XX} \overline{X}_x^2 + 2\,\widehat{U}_{XY} \overline{X}_x \overline{Y}_x + \widehat{U}_{YY} \overline{Y}_x^2 + \widehat{U}_X \overline{X}_{xx} + \widehat{U}_Y \overline{Y}_{xx},$$

$$\overline{U}_{xy} = \widehat{U}_{XX} \overline{X}_x \overline{X}_y + \widehat{U}_{XY} (\overline{X}_x \overline{Y}_y + \overline{X}_y \overline{Y}_x) + \widehat{U}_{YY} \overline{Y}_x \overline{Y}_y + \widehat{U}_X \overline{X}_{xy} + \widehat{U}_Y \overline{Y}_{xy},$$

$$\overline{U}_{yy} = \widehat{U}_{XX} \overline{X}_y^2 + 2\,\widehat{U}_{XY} \overline{X}_y \overline{Y}_y + \widehat{U}_{YY} \overline{Y}_y^2 + \widehat{U}_X \overline{X}_{yy} + \widehat{U}_Y \overline{Y}_{yy}.$$

Substitute into the reduced determining equations, to produce:

The Normal Form Determining Equations

$$\overline{X}_y = 0, \qquad \overline{X}_{xx} = \widehat{U}_Y \overline{X}_x^2 - u_y \overline{X}_x, \qquad \overline{X}_{xy} = \overline{X}_{yy} = 0,$$

$$\overline{Y}_x = (\widehat{U} - u)\overline{X}_x, \qquad \overline{Y}_{xx} = (\widehat{U}_X + 2\,(\widehat{U} - u)\widehat{U}_Y)\overline{X}_x^2 - (u_x + (\widehat{U} - u)u_y)\overline{X}_x,$$

$$\overline{Y}_y = \overline{X}_x, \qquad \overline{Y}_{xy} = \widehat{U}_Y \overline{X}_x^2 - u_y \overline{X}_x, \qquad \overline{Y}_{yy} = 0, \qquad u_{yy} = \widehat{U}_{YY} \overline{X}_x^2,$$

where $\widehat{U}(\overline{X}, \overline{Y})$ is the prescribed submanifold, while the solution

$$\overline{X}(x,y), \;\; \overline{Y}(x,y), \;\; u(x,y)$$

includes the normal form $u(x,y)$.

Step 4: Compatibility of involutivity with the initial conditions prescribed by the equivariant moving frame construction requires the notion of a well-posed cross-section.

A coordinate cross-section is well-posed if the derivatives being normalized of sufficiently high order coincide with the parametric derivatives in the normal form determining equations.

⋆⋆ This can be checked algebraically without needing to explicitly construct the normal form determining equations using the existence of an algebraic Rees decomposition of the corresponding monomial ideal.

To construct a (reduced) moving frame

I. Compute the prolonged reduced pseudo-group action

$$u_K^\alpha \quad \longmapsto \quad U_K^\alpha = F_K^\alpha(x, u^{(n)}, \bar{g}^{(n)})$$

on submanifold jets by implicit differentiation.

II. Choose a cross-section to the pseudo-group orbits:

$$x^i = c^i, \qquad i = 1, \ldots, p$$

$$u_{J_\kappa}^{\alpha_\kappa} = c_\kappa, \qquad \kappa = p + 1, \ldots, r_n = \text{fiber dim } \mathcal{G}^{(n)}$$

III. Solve the normalization equations

$$X^i = H^i(x, u, \bar{g}^{(0)}) = c_i$$

$$U^{\alpha_\kappa}_{J_\kappa} = F^{\alpha_\kappa}_{J_\kappa}(x, u^{(n)}, \bar{g}^{(n)}) = c_\kappa$$

for the n^{th} order reduced pseudo-group parameters

$$\bar{g}^{(n)} = \rho^{(n)}(x, u^{(n)})$$

IV. Substitute the moving frame formulas into the un-normalized jet coordinates $u^\alpha_K = F^\alpha_K(x, u^{(n)}, \bar{g}^{(n)})$.

$$I^\alpha_K(x, u^{(n)}) = F^\alpha_K(x, u^{(n)}, \rho^{(n)}(x, u^{(n)}))$$

The resulting functions form a complete system of n^{th} order differential invariants

Our Favorite Example

$$\boxed{X = f(x) \qquad Y = f'(x)\,y + g(x) \qquad U = u + \frac{f''(x)\,y + g'(x)}{f'(x)}}$$

Compute prolonged action from surfaces $u = u(x,y)$ to surfaces $U = \widehat{U}(X,Y)$ in terms of the reduced pseudo-group parameters using the reduced determining equations to simplify.

Apply the implicit differentiation operators:

$$\mathrm{D}_x = \overline{X}_x\,\mathrm{D}_X + \overline{Y}_x\,\mathrm{D}_Y = \overline{X}_x\,[\,\mathrm{D}_X + (u - \overline{U})\,\mathrm{D}_Y\,] \qquad \mathrm{D}_X = \frac{\mathrm{D}_x + (u - \overline{U})\,\mathrm{D}_y}{\overline{X}_x}$$

$$\mathrm{D}_y = \overline{X}_y\,\mathrm{D}_X + \overline{Y}_y\,\mathrm{D}_Y = \overline{X}_x\,\mathrm{D}_Y \qquad \mathrm{D}_Y = \frac{1}{\overline{X}_x}\,\mathrm{D}_y$$

to $\widehat{U}$.

Prolonged action to order three — free at order ≥ 2

$$\widehat{U}_X = \frac{\overline{U}_x + (u - \overline{U})\overline{U}_y}{\overline{X}_x} \qquad \widehat{U}_Y = \frac{\overline{U}_y}{\overline{X}_x}$$

$$\widehat{U}_{XX} = \frac{\begin{array}{l}\overline{U}_{xx} - (u_y - \overline{U}_y)\overline{U}_x + (u_x - \overline{U}_x)\overline{U}_y \\ \qquad + (u - \overline{U})(2\,\overline{U}_{xy} + 2(u - \overline{U})u_{yy} + (u_y - \overline{U}_y)\overline{U}_x)\end{array}}{\overline{X}_x^2}$$

$$\widehat{U}_{XY} = \frac{\overline{U}_{xy} + (u_y - \overline{U}_y)\overline{U}_y + (u - \overline{U})u_{yy}}{\overline{X}_x^2} \qquad \widehat{U}_{YY} = \frac{u_{yy}}{\overline{X}_x^2}$$

$$\widehat{U}_{XXX} = *** \qquad \widehat{U}_{XXY} = ***$$

$$\widehat{U}_{XYY} = \frac{u_{xyy} + 2(u_y - \overline{U}_y)u_{yy} + (u - \overline{U})u_{yyy}}{\overline{X}_x^3} \qquad \widehat{U}_{YYY} = \frac{u_{yyy}}{\overline{X}_x^3}$$

Cross-section (assuming $u_{yy} > 0$):

$$x = 0, \quad y = 0, \quad u_{yy} = 1, \quad u_{x^k} = c_k, \quad u_{x^k y} = d_k, \quad k \geq 0$$

Normal form:

$$u = c(x) + y\, d(x) + \frac{y^2}{2} w(x, y), \qquad \text{where} \qquad w(0,0) = 1$$

★ If the cross-section defines analytic functions $c(x), d(x)$
i.e., it is well-posed, then the normal form is analytic.

For simplicity, set all $c_k = d_k = 0$, i.e., $c(x) = d(x) \equiv 0$.

Normalization equations:

$$0 = \widehat{U}_X = \frac{\overline{U}_x + (u - \overline{U})\overline{U}_y}{\overline{X}_x} \qquad 0 = \widehat{U}_Y = \frac{\overline{U}_y}{\overline{X}_x}$$

$$0 = \widehat{U}_{XX} = \frac{\begin{array}{r}\overline{U}_{xx} - (u_y - \overline{U}_y)\overline{U}_x + (u_x - \overline{U}_x)\overline{U}_y \\ + (u - \overline{U})(2\overline{U}_{xy} + 2(u - \overline{U})u_{yy} + (u_y - \overline{U}_y)\overline{U}_x)\end{array}}{\overline{X}_x^2}$$

$$0 = \widehat{U}_{XY} = \frac{\overline{U}_{xy} + (u_y - \overline{U}_y)\overline{U}_y + (u - \overline{U})u_{yy}}{\overline{X}_x^2} \qquad 1 = \widehat{U}_{YY} = \frac{u_{yy}}{\overline{X}_x^2}$$

$$0 = \widehat{U}_{XXX} = *** \qquad 0 = \widehat{U}_{XXY} = ***$$

$$\widehat{U}_{XYY} = \frac{u_{xyy} + 2(u_y - \overline{U}_y)u_{yy} + (u - \overline{U})u_{yyy}}{\overline{X}_x^3} \qquad \widehat{U}_{YYY} = \frac{u_{yyy}}{\overline{X}_x^3}$$

Reduced moving frame:

$$\overline{X} = 0, \qquad \overline{Y} = 0, \qquad \overline{U} = 0, \qquad \overline{X}_x = \sqrt{u_{yy}},$$

$$\overline{U}_x = 0, \quad \overline{U}_y = 0, \quad \overline{U}_{xx} = 0, \quad \overline{U}_{xy} = -\,u u_{yy}.$$

Substitute in preceding formulae to obtain the differential invariants.

At order 3:

$$\widehat{U}_{XYY} \;\longmapsto\; I = \frac{u u_{yyy} + (1 + 2\,u_y) u_{yy}}{u_{yy}^{3/2}}, \qquad \widehat{U}_{YYY} \;\longmapsto\; J = \frac{u_{yyy}}{u_{yy}^{3/2}}.$$

Also the invariant differentiations

$$\mathrm{D}_X \;\longmapsto\; \mathcal{D}_1 = \frac{\mathrm{D}_x + u\,\mathrm{D}_y}{\sqrt{u_{yy}}} \qquad \mathrm{D}_Y \;\longmapsto\; \mathcal{D}_2 = \frac{\mathrm{D}_y}{\sqrt{u_{yy}}}$$

★★ The higher order differential invariants can be obtained by invariantly differentiating I and J.

Reduced moving frame:

$$\overline{X} = 0, \qquad \overline{Y} = 0, \qquad \overline{U} = 0, \qquad \overline{X}_x = \sqrt{u_{yy}},$$

$$\overline{U}_x = 0, \quad \overline{U}_y = 0, \quad \overline{U}_{xx} = 0, \quad \overline{U}_{xy} = -\,u u_{yy}.$$

Substitute in preceding formulae to obtain the differential invariants.

At order 3:

$$\widehat{U}_{XYY} \longmapsto I = \frac{u u_{yyy} + (1 + 2\,u_y) u_{yy}}{u_{yy}^{3/2}}, \qquad \widehat{U}_{YYY} \longmapsto J = \frac{u_{yyy}}{u_{yy}^{3/2}}.$$

Also the invariant differentiations

$$\mathrm{D}_X \longmapsto \mathcal{D}_1 = \frac{\mathrm{D}_x + u\,\mathrm{D}_y}{\sqrt{u_{yy}}} \qquad \mathrm{D}_Y \longmapsto \mathcal{D}_2 = \frac{\mathrm{D}_y}{\sqrt{u_{yy}}}$$

★ ★ Actually, by further inspection of the moving frame recurrence formulae, you only need to invariantly differentiate J to generate all differential invariants.

In order to match the moving frame computations with the normal form determining equations, we need to switch the role of the source and target submanifolds, so that $\widehat{U}(X,Y)$ is the given submanifold while $u(x,y)$ is its normal form.

Thus, the differential invariants (and invariant differentiations) should be rewritten in terms of $X, Y, \widehat{U}$:

$$I = \frac{\widehat{U}_{XYY} + 2\,\widehat{U}_Y \widehat{U}_{YY} + \widehat{U}\widehat{U}_{YYY}}{\widehat{U}_{YY}^{3/2}}, \qquad J = \frac{\widehat{U}_{YYY}}{\widehat{U}_{YY}^{3/2}}.$$

One could, of course, do this in advance, but it seems to make everything more confusing!

Step 4: Compatibility of involutivity with the initial conditions prescribed by the equivariant moving frame construction requires the notion of a well-posed cross-section.

⋆ ⋆ Technical complication in the final step — the moving frame normalization equations do not respect the term ordering used to demonstrate the involutivity of the normal form equations.

In our example, this complication shows up in the normal form determining equation

$$u_{yy} = \widehat{U}_{YY} \overline{X}_x^2$$

In the involutivity framework, it must be solved for the principal derivative u_{yy} in terms of the parametric derivative $\overline{X}_x$, whereas in the moving "framework", it is solved for the pseudo-group parameter $\overline{X}_x$.

Fortunately, the two methods are compatible once the order is sufficiently large, meaning beyond the orders of involutivity and freeness. Technical details can be found in the paper.

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