Reconstruction of Convex Bodies from their Projections

Peter J. Olver
School of Mathematics
University of Minnesota
Minneapolis, MN 55455
olver@umn.edu
http://www.math.umn.edu/∼olver

Abstract. The method of envelopes is used to deduce formulas that enable one to reconstruct two- and three-dimensional convex bodies from the outlines formed by their orthogonal projections onto, respectively, lines and planes.

Orthogonally projecting a solid three-dimensional body onto a plane results in a two-dimensional region. In this paper we derive formulas for reconstructing a convex body from its projections. Before analyzing the three-dimensional case, we first discuss the simpler case of a planar body that is orthogonally projected onto lines in the plane. Our planar formulas are well known, [**6**, **7**], while some of the three-dimensional counterparts appear to be new. For this purpose, we make use of the method of envelopes, and, in particular, prove that a convex body coincides with the envelope of its outline cylinders, as defined below. For additional results, including analysis of the more challenging nonconvex case, we refer the reader to [**3**, **5**, **6**, **8**, **11**, **12**].

In image processing, orthogonal (also known as parallel or orthographic) projection corresponds to photographing the body with a camera that has infinite focal length. The case of finite cameras will be the subject of a tentatively planned sequel.

1. Two-dimensional Bodies.

We begin by recalling the definition of the envelope of a one-parameter family

$$\mathbf{p}(t,\theta) = (x(t,\theta), y(t,\theta)) \tag{1}$$

of plane curves. For fixed θ, we regard t as the curve parameter. Vice versa, we can fix t and regard θ as the curve parameter; the latter curves forming the *dual family*. We assume that all curves, in both families, are *nonsingular*, meaning they are simple (without self-intersections), and possess a tangent line at each point. The latter condition is guaranteed

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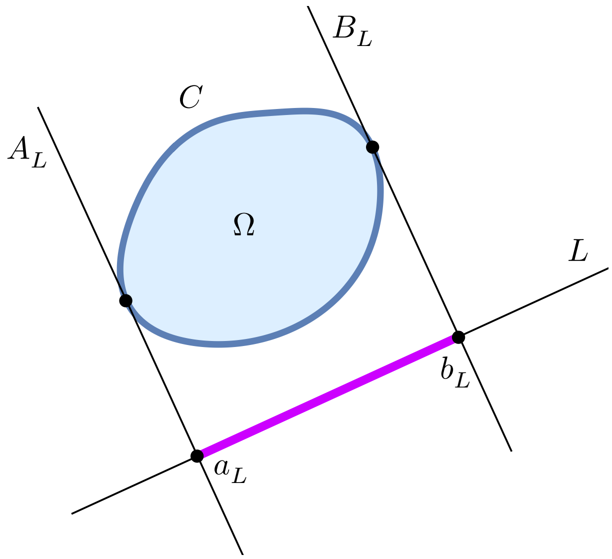

Figure 1. Projection of a Planar Body.

by the non-vanishing of their respective tangent vectors: $\partial\mathbf{p}/\partial t, \partial\mathbf{p}/\partial\theta \neq 0$. Given a parameter value $q = (t, \theta)$, the *dual tangent lines* of the family are denoted, respectively, by L_q and $\widetilde{L}_q$. Following [**4**, **15**], we state the standard definition of the envelope of the curve family.

Definition 1. The *envelope* of a nonsingular curve family (1) is defined as the set of points $p(q) = p(t, \theta) \in \mathbb{R}^2$ where the dual tangent lines coincide: $L_q = \widetilde{L}_q$.

In other words, the envelope condition requires that the dual tangent vectors be parallel, so

$$\frac{\partial\mathbf{p}}{\partial t} \wedge \frac{\partial\mathbf{p}}{\partial\theta} = \frac{\partial x}{\partial t}\,\frac{\partial y}{\partial\theta} - \frac{\partial x}{\partial\theta}\,\frac{\partial y}{\partial t} = 0, \tag{2}$$

where $\wedge$ refers to the two-dimensional cross product: $(v_1, v_2) \wedge (w_1, w_2) = v_1 w_2 - v_2 w_1$.

Next, let us define the class of planar bodies to be treated.

Definition 2. A simple closed C^2 plane curve $C \subset \mathbb{R}^2$ is called *strictly convex* if its signed Euclidean curvature, based on the counterclockwise orientation, is everywhere strictly positive: $\kappa > 0$. A *strictly convex planar body* is a compact region $\Omega \subset \mathbb{R}^2$ whose boundary $C = \partial\Omega$ is a strictly convex curve.

Given such a body, we choose coordinates so that the origin $\mathbf{0}$ lies in the interior of Ω, for example at its center of mass. Then $C = \partial\Omega$ can be parametrized in polar coordinates (r, θ) by a single-valued function $r = h(\theta)$.

Orthogonally projecting the body Ω onto a line $L \subset \mathbb{R}^2$ results in a line segment, which depends only on the slope of the line. Thus, we can assume that the line passes through the origin, and identify $L \in \mathbb{RP}^1$ as an element of the one-dimensional real projective space. The task is to reconstruct Ω or, equivalently, its boundary C, from all of its projected line segments; the geometry underlying the reconstruction is illustrated in Figure 1.

Theorem 3. *Given a line $L \subset \mathbb{R}^2$, let $[a_L, b_L] \subset L$ denote the line segment obtained by orthogonally projecting the body Ω onto L. Let A_L, B_L be the lines orthogonal to L passing through the endpoints a_L, b_L of the projection. Then the boundary curve $C = \partial\Omega$ can be realized as the envelope of the collection of lines $\{ A_L, B_L \mid L \in \mathbb{RP}^1 \}$.*

Proof: The lines A_L, B_L are tangent at their points of intersection with the boundary curve C, which is hence contained in their envelope. To prove that C coincides with the envelope, we introduce coordinates such that C can be locally identified as the graph of a function $y = f(x)$. The tangent line through $(x, f(x))$ is thus parametrized by

$$\mathbf{p}(t,x) = \bigl(t, f(x) + f'(x)(t-x) \bigr).$$

Thus,

$$\frac{\partial \mathbf{p}}{\partial t} = \bigl(1, f'(x) \bigr), \qquad \frac{\partial \mathbf{p}}{\partial x} = \bigl(0, f''(x)(t-x) \bigr).$$

The strict convexity assumption implies $f''(x) \neq 0$, and hence the family of tangent lines is nondegenerate. Equation (2) implies that the envelope is given by $t = x$ and hence $y = f(x)$, which coincides with that portion of the boundary of the body. *Q.E.D.*

To derive an explicit formula for the reconstruction of the body, for $-\pi < \theta \leq \pi$, let L_θ be the (oriented) line passing through the origin and the point $(\cos\theta, \sin\theta)$ on the unit circle. We use the signed radial coordinate r to parametrize the line $L_\theta = \{ (r\cos\theta, r\sin\theta) \mid r \in \mathbb{R} \}$, noting that $L_{\theta+\pi}$ is the same line, but with the opposite orientation. Let $\sigma(\theta) < 0 < \rho(\theta)$ be the radial coordinates of the two endpoints $a_\theta = a_{L_\theta}$, $b_\theta = b_{L_\theta}$ of the projection of Ω onto L_θ. These functions are 2π periodic. Moreover, $\rho(\theta) = -\sigma(\theta+\pi)$, and hence we only need one of them, say $\rho(\theta)$, to reconstruct C.

The line $B_\theta = B_{L_\theta}$ perpendicular to L_θ passing through the endpoint b_θ of the projection is thus parametrized by

$$x = \rho(\theta)\cos\theta + t\sin\theta, \qquad y = \rho(\theta)\sin\theta - t\cos\theta, \qquad t \in \mathbb{R}. \tag{3}$$

Thus, according to Theorem 3, the curve C is the envelope of this one-parameter family of lines, as illustrated in Figure 2.

In our case, for the family (3),

$$\begin{aligned} \frac{\partial x}{\partial t} &= \sin\theta, & \frac{\partial x}{\partial \theta} &= \rho'(\theta)\cos\theta - \rho(\theta)\sin\theta + t\cos\theta, \\ \frac{\partial y}{\partial t} &= -\cos\theta, & \frac{\partial y}{\partial \theta} &= \rho'(\theta)\sin\theta + \rho(\theta)\cos\theta + t\sin\theta, \end{aligned}$$

and hence, after simplification, (2) reduces to

$$\rho'(\theta) + t = 0.$$

Substituting back into (3), we are led to a formula that can be found in a paper by Giblin and Weiss, [**7**].

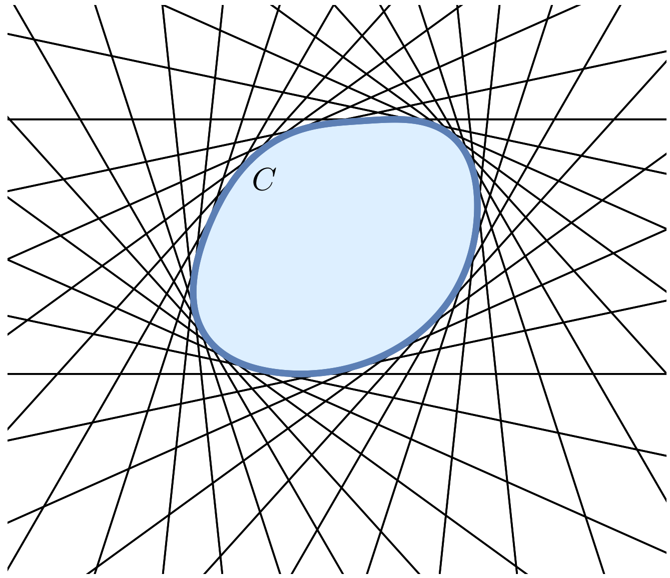

Figure 2. The Projection Envelope.

Theorem 4. *Given the projections of* Ω *with endpoint* $b_\theta \in L_\theta$ *parametrized by its radial coordinate* $\rho(\theta) > 0$*, the boundary* $C = \partial\Omega$ *is parametrized by*

$$x = \rho(\theta)\cos\theta - \rho'(\theta)\sin\theta, \qquad y = \rho(\theta)\sin\theta + \rho'(\theta)\cos\theta, \qquad -\pi < \theta \leq \pi. \tag{4}$$

Note that the result is local, in that if one only knows the endpoints over a range of angles θ, one can still use formula (4) to reconstruct the corresponding part of the boundary. Thus, if the boundary is not C^1, the smooth, strictly convex parts can be reconstructed over the sections where $\rho(\theta)$ is continuously differentiable. Corners of $\rho(\theta)$, meaning points where it is continuous but not differentiable, correspond to corners of the body, while discontinuities of $\rho(\theta)$ correspond to straight line segments in C.

If the body is connected, but not convex, then the reconstruction algorithm will produce its convex hull. Indeed, concave indentations cannot be detected by mere projection onto lines. Projections of disconnected bodies are interesting, since one part will occlude another when the direction of projection has certain orientations. But, given the locality of the result, one can reconstruct those parts of the boundary of a nonconvex two-dimensional body having the property the tangent line does not intersect the body at any other point.

2. Three-dimensional Bodies.

Let us now discuss the three-dimensional case. Let us first characterize the envelope, [**4**, **15**], of a two-parameter families of surfaces

$$\mathbf{p}(q) = \mathbf{p}(s,t,\theta,\varphi) = \big(\, x(s,t,\theta,\varphi), y(s,t,\theta,\varphi), z(s,t,\theta,\varphi)\,\big), \tag{5}$$

where we abbreviate $q = (s,t,\theta,\varphi) \in \mathbb{R}^4$. As before, for fixed θ,φ, we regard t,s as surface parameters; vice versa, we can fix t,s and regard θ,φ as parametrizing a dual surface.

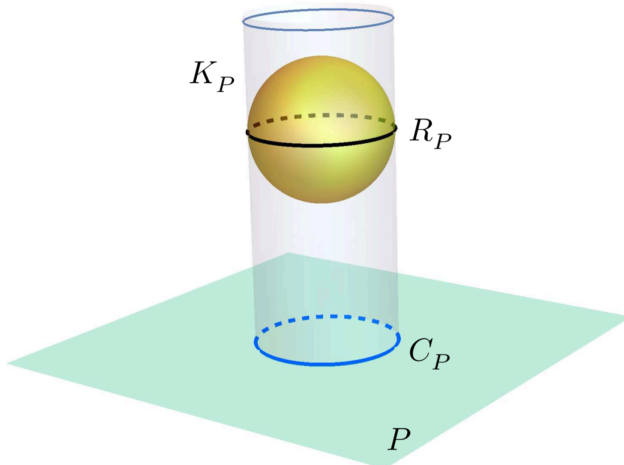

Figure 3. The Outline, Outline Cylinder, and Rim of a Convex Body.

We assume that the family is *nonsingular*, meaning that each surface in both the original and dual families is simple and possesses a tangent plane at each point. Given q, the *dual tangent planes* at the point $\mathbf{p}(q)$ are denoted by T_q and $\widetilde{T}_q$, and are spanned by the respective pairs of tangent vectors $\partial \mathbf{p}/\partial s, \partial \mathbf{p}/\partial t$ and $\partial \mathbf{p}/\partial \theta, \partial \mathbf{p}/\partial \varphi$. The tangent plane condition is implied by the non-vanishing of the respective surface normals:

$$\frac{\partial \mathbf{p}}{\partial s} \wedge \frac{\partial \mathbf{p}}{\partial t} \neq 0, \qquad\qquad \frac{\partial \mathbf{p}}{\partial \theta} \wedge \frac{\partial \mathbf{p}}{\partial \varphi} \neq 0, \tag{6}$$

where $\wedge$ now denotes the usual cross product in $\mathbb{R}^3$.

Definition 5. A point $\mathbf{p}(q) \in \mathbb{R}^3$ lies in the *envelope* of the nonsingular surface family (5) whenever the dual tangent planes coincide: $T_q = \widetilde{T}_q$.

Thus, a point lies in the envelope of the family if and only if the dual surface normals (6) are parallel:

$$\left(\frac{\partial \mathbf{p}}{\partial s} \wedge \frac{\partial \mathbf{p}}{\partial t} \right) \wedge \left(\frac{\partial \mathbf{p}}{\partial \theta} \wedge \frac{\partial \mathbf{p}}{\partial \varphi} \right) = 0. \tag{7}$$

Let $\Omega \subset \mathbb{R}^3$ be a convex three-dimensional *body*, meaning that it is compact, with non-empty interior and smooth (C^2) boundary $S = \partial\Omega$. We will call Ω *strictly convex* if the principal curvatures (relative to the outwards normal) of S are everywhere strictly positive.

The orthogonal projection of Ω onto a plane P will be a convex two-dimensional body $D_P \subset P$ with boundary curve $C_P = \partial D_P$, known as the *outline* of Ω on the plane P, [**11**, **12**], also called its *apparent contour*, *profile*, or *silhouette*, [**2**, **3**, **6**]. Again, parallel planes produce identical projections (or, more accurately, parallel projections) and so we need only consider the planes passing through the origin, i.e., the Grassmannian of two-dimensional subspaces of $\mathbb{R}^3$, [**13**], which, by identifying each plane with its normal line, is isomorphic to the two-dimensional projective space $\mathbb{RP}^2$. The task is to reconstruct Ω or, equivalently, S from these projections.

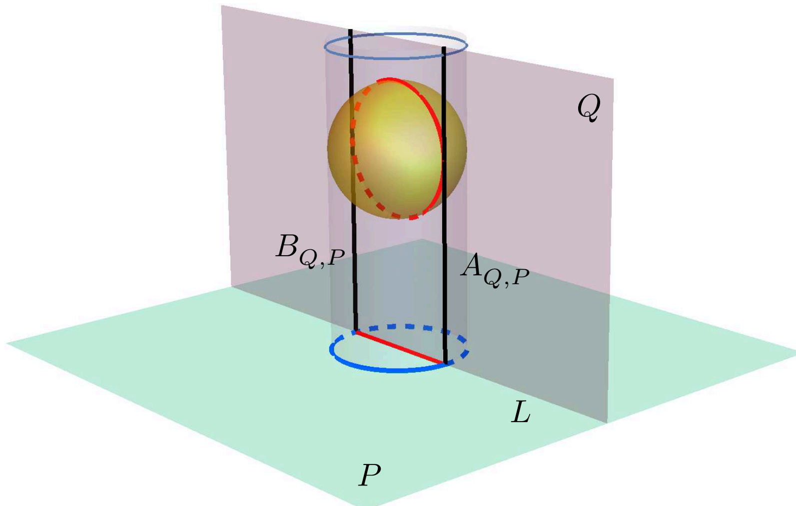

Figure 4. *Outline Cylinder Envelope Proof Construction.*

The outline curve C_P is the projection of a curve $R_P \subset S$ on the boundary of the body, known as the *rim*, [**12**], or *contour generator*, [**3**], associated with P. Let $K_P \subset \mathbb{R}^3$ denote the corresponding *outline cylinder*†, consisting of all points in $\mathbb{R}^3$ that orthogonally project to the outline curve C_P. The case when Ω is a sphere is illustrated in Figure 3. By convexity, $R_P = K_P \cap S$. Moreover, K_P is tangent to S at each point on the rim:

$$TK_P|_{\mathbf{p}} = TS|_{\mathbf{p}} \qquad \text{for all} \qquad \mathbf{p} \in R_P.$$

Theorem 6. *If $\Omega \subset \mathbb{R}^3$ is a strictly convex body, then its boundary $S = \partial\Omega$ is the envelope of the family of cylindrical surfaces K_P corresponding to its orthogonal projections onto planes $P \in \mathbb{RP}^2$.*

Proof: Let $Q \subset \mathbb{R}^3$ be a plane whose intersection with Ω is a planar body $\Omega_Q = \Omega \cap Q$, i.e., Q does not lie outside of Ω and is not tangent to $S = \partial\Omega$. (See Figure 4 for an illustration of the construction.) Keep in mind that the intersection boundary $S_Q = S \cap Q = \partial\Omega_Q$, which is the red curve in the figure, is not usually a rim curve. If $P \subset \mathbb{R}^3$ is any plane orthogonal to Q, meaning they intersect at right angles, then the intersection $K_P \cap Q$ of the corresponding outline cylinder is a pair of parallel lines $A_{Q,P}, B_{Q,P}$ that are tangent to Ω_Q and orthogonal to the line $L = P \cap Q$. Moreover, it is not hard to see that the intersection of the projection D_P of Ω onto P with Q, namely, $D_L = D_P \cap Q \subset L = P \cap Q$, is a line segment that coincides with the orthogonal projection of Ω_Q onto the line L; in the figure, D_L is the red line segment. Thus, as P ranges over all planes orthogonal to Q, the intersection of the outline cylinders $K_P \cap Q$ can be identified with the families of lines $A_{Q,P}, B_{Q,P} \subset Q$ determining the orthogonal

† Not necessarily a circular cylinder.

projections of Ω_Q onto the corresponding lines $L = P \cap Q$. As a consequence of the planar envelope Theorem 3, the boundary $\partial\Omega_Q$ is the envelope of this family of lines.

For the same reason as in the proof of the planar Theorem 3, every point in S belongs to the envelope of the family of outline cylinders K_P. Conversely, suppose $\mathbf{p}(q) \in \mathbb{R}^3$ is a point in their envelope, which, in terms of the parameters[†] $q = (s, t, \theta, \varphi)$ of the outline cylinder family, requires that the dual tangent planes coincide: $T_q = \widetilde{T}_q$. Let us choose a transverse plane $Q \neq T_q = \widetilde{T}_q$ containing $\mathbf{p}(q)$ that has the preceding property, namely $\Omega_Q = \Omega \cap Q$ is a planar body. The envelope condition implies that the lines obtained by intersecting the dual tangent planes with Q coincide: $L_q = T_q \cap Q = \widetilde{L}_q = \widetilde{T}_q \cap Q$. But $L_q, \widetilde{L}_q$ are clearly the dual tangent lines at the point $\mathbf{p}(q) \in Q$ associated with the family of lines $A_{Q,P}, B_{Q,P}$ constructed above. Since $L_q = \widetilde{L}_q$, we conclude that $\mathbf{p}(q)$ lies in their envelope, and thus, by the planar result, $\mathbf{p}(q) \in \partial\Omega_Q \subset \partial\Omega = S$. *Q.E.D.*

Using this result, let us determine an explicit reconstruction formula for a convex three-dimensional body from its orthogonal projections. We assume that the origin $\mathbf{0} \in \mathbb{R}^3$ lies in the interior of Ω, for example at its center of mass. We then adopt spherical coordinates:

$$x = r\cos\theta\,\sin\varphi, \qquad y = r\sin\theta\,\sin\varphi, \qquad z = r\cos\varphi, \tag{8}$$

using the mathematician's convention, [**14**], so $-\pi < \theta \leq \pi$ is the *azimuthal angle* (longitude), while $0 \leq \varphi \leq \pi$ is the *zenith angle* (latitude). (Keep in mind the inherent singularities at $\varphi = 0, \pi$, i.e., at the north and south poles, where the value of θ is irrelevant.) Let $\mathbb{S}^2 \subset \mathbb{R}^3$ denote the two-dimensional unit sphere, i.e., where $r = 1$. Let $P_{\theta,\varphi}$ denote the plane with unit normal

$$\mathbf{n}_{\theta,\varphi} = (\cos\theta\,\sin\varphi,\ \sin\theta\,\sin\varphi,\ \cos\varphi) \in \mathbb{S}^2, \tag{9}$$

which serves to orient $P_{\theta,\varphi}$. Observe that $P_{\theta+\pi,\pi-\varphi}$ represents the same plane, but endowed with the opposite orientation, so $\mathbf{n}_{\theta+\pi,\pi-\varphi} = -\mathbf{n}_{\theta,\varphi}$.

We now select an orthonormal basis of the plane $P_{\theta,\varphi}$, and, for this purpose use that provided by the unit vectors

$$\mathbf{u}_{\theta,\varphi} = (\cos\theta\,\cos\varphi,\ \sin\theta\,\cos\varphi,\ -\sin\varphi), \qquad \mathbf{v}_{\theta,\varphi} = (-\sin\theta, \cos\theta, 0). \tag{10}$$

Note that $\mathbf{n}_{\theta,\varphi}, \mathbf{u}_{\theta,\varphi}, \mathbf{v}_{\theta,\varphi}$ form a right handed orthonormal frame. We employ polar coordinates (r,ψ) on $P_{\theta,\varphi}$ with respect to the prescribed orthonormal basis, whereby points $\mathbf{p} \in P_{\theta,\varphi}$ have the form

$$\mathbf{p} = (r\cos\psi)\,\mathbf{u}_{\theta,\varphi} + (r\sin\psi)\,\mathbf{v}_{\theta,\varphi} \in P \qquad \text{for} \qquad r \geq 0, \quad -\pi < \psi \leq \pi.$$

Note that r coincides with the spherical radial coordinate of the point $\mathbf{p}$.

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† See equation (12) below for the explicit formulas for the parametrization of the outline cylinders in terms of spherical coordinates, which also implies nonsingularity of the family.

The outline curve $C_{\theta,\varphi} \subset P_{\theta,\varphi}$ will thereby be determined by a function

$$r = \rho(\theta, \varphi, \psi) > 0$$

which is strictly positive owing to our assumption that the origin lies inside the body, and so its outline curves do not pass through the origin of the plane. Thus, in three-dimensional space, $C_{\theta,\varphi}$ is parametrized by

$$\mathbf{p} = \rho(\theta, \varphi, \psi) \cos\psi \, \mathbf{u}_{\theta,\varphi} + \rho(\theta, \varphi, \psi) \sin\psi \, \mathbf{v}_{\theta,\varphi}, \qquad -\pi < \psi \leq \pi. \tag{11}$$

The corresponding outline cylinder $K_{\theta,\varphi}$ is the surface parametrized by

$$K_{\theta,\varphi} = \left\{ \, t \, \mathbf{n}_{\theta,\varphi} + \rho(\theta, \varphi, \psi) \cos\psi \, \mathbf{u}_{\theta,\varphi} + \rho(\theta, \varphi, \psi) \sin\psi \, \mathbf{v}_{\theta,\varphi} \; \middle| \; -\pi < \psi \leq \pi, \; t \in \mathbb{R} \, \right\}. \tag{12}$$

By construction, the outline cylinder intersects the body surface S tangentially at the rim curve associated with the plane $P_{\theta,\varphi}$:

$$R_{\theta,\varphi} = S \cap K_{\theta,\varphi} = \left\{ \, p \in S \; \middle| \; \mathbf{n}_{\theta,\varphi} \in TS|_p \, \right\}.$$

According to Theorem 6, we can reconstruct the surface S as the envelope of the two parameter family of outline cylinders

$$\mathbf{p}(\theta, \varphi; t, \psi) = t \, \mathbf{n}_{\theta,\varphi} + \rho(\theta, \varphi, \psi) \cos\psi \, \mathbf{u}_{\theta,\varphi} + \rho(\theta, \varphi, \psi) \sin\psi \, \mathbf{v}_{\theta,\varphi}. \tag{13}$$

To apply (7), we must compute the derivatives of (13) with respect to the family parameters θ, φ, and the individual surface parameters t, ψ. We begin by noting the following basic formulae:

$$\begin{aligned}
&\frac{\partial \mathbf{n}_{\theta,\varphi}}{\partial \theta} = \sin\varphi \, \mathbf{v}_{\theta,\varphi}, \qquad &&\frac{\partial \mathbf{u}_{\theta,\varphi}}{\partial \theta} = \cos\varphi \, \mathbf{v}_{\theta,\varphi}, \qquad &&\frac{\partial \mathbf{v}_{\theta,\varphi}}{\partial \theta} = -\sin\varphi \, \mathbf{n}_{\theta,\varphi} - \cos\varphi \, \mathbf{u}_{\theta,\varphi}, \\
&\frac{\partial \mathbf{n}_{\theta,\varphi}}{\partial \varphi} = \mathbf{u}_{\theta,\varphi}, &&\frac{\partial \mathbf{u}_{\theta,\varphi}}{\partial \varphi} = -\mathbf{n}_{\theta,\varphi}, &&\frac{\partial \mathbf{v}_{\theta,\varphi}}{\partial \varphi} = 0.
\end{aligned} \tag{14}$$

Thus, differentiating (13),

$$\begin{aligned}
\frac{\partial \mathbf{p}}{\partial \theta} &= -\rho \sin\varphi \sin\psi \, \mathbf{n}_{\theta,\varphi} + \left(\frac{\partial \rho}{\partial \theta} \cos\psi - \rho \cos\varphi \, \sin\psi \right) \mathbf{u}_{\theta,\varphi} \\
&\qquad\qquad + \left(t \sin\varphi + \frac{\partial \rho}{\partial \theta} \sin\psi + \rho \cos\varphi \, \cos\psi \right) \mathbf{v}_{\theta,\varphi}, \\
\frac{\partial \mathbf{p}}{\partial \varphi} &= -\rho \cos\psi \, \mathbf{n}_{\theta,\varphi} + \left(t + \frac{\partial \rho}{\partial \varphi} \cos\psi \right) \mathbf{u}_{\theta,\varphi} + \frac{\partial \rho}{\partial \varphi} \sin\psi \, \mathbf{v}_{\theta,\varphi}, \\
\frac{\partial \mathbf{p}}{\partial \psi} &= \left(\frac{\partial \rho}{\partial \psi} \cos\psi - \rho \sin\psi \right) \mathbf{u}_{\theta,\varphi} + \left(\frac{\partial \rho}{\partial \psi} \sin\psi + \rho \cos\psi \right) \mathbf{v}_{\theta,\varphi}, \\
\frac{\partial \mathbf{p}}{\partial t} &= \mathbf{n}_{\theta,\varphi}.
\end{aligned} \tag{15}$$

These satisfy the envelope condition (7) if and only if the corresponding coefficient matrix has rank 2. In view of the fourth equation, this will be the case if and only if the 2×3

matrix obtained from the coefficients of $\mathbf{u}_{\theta,\varphi}, \mathbf{v}_{\theta,\varphi}$ in the first three right hand sides has rank 1. This clearly requires

$$\begin{aligned}
\frac{\partial\rho}{\partial\theta}\cos\psi - \rho\cos\varphi\,\sin\psi &= \lambda\left(t\sin\varphi + \frac{\partial\rho}{\partial\theta}\sin\psi + \rho\cos\varphi\,\cos\psi\right), \\
t + \frac{\partial\rho}{\partial\varphi}\cos\psi &= \lambda\,\frac{\partial\rho}{\partial\varphi}\sin\psi, \\
\frac{\partial\rho}{\partial\psi}\cos\psi - \rho\sin\psi &= \lambda\left(\frac{\partial\rho}{\partial\psi}\sin\psi + \rho\cos\psi\right),
\end{aligned} \tag{16}$$

for some scalar $\lambda(t,\theta,\varphi,\psi)$. Assuming its right hand side is nonzero, the last equation prescribes

$$\lambda = \frac{\rho_\psi\cos\psi - \rho\sin\psi}{\rho_\psi\sin\psi + \rho\cos\psi}, \tag{17}$$

where, for brevity, we now employ subscripts to denote partial derivatives, so $\rho_\psi = \partial\rho/\partial\psi$ and so on. Substituting this expression into the second equation gives an explicit formula for t in terms of θ, φ, ψ:

$$t = \frac{\rho_\psi\cos\psi - \rho\sin\psi}{\rho_\psi\sin\psi + \rho\cos\psi}\,\rho_\varphi\sin\psi - \rho_\varphi\cos\psi = -\,\frac{\rho\,\rho_\varphi}{\rho_\psi\sin\psi + \rho\cos\psi}. \tag{18}$$

The resulting first equation in (16) produces an implicit trigonometric equation for the planar angle ψ in terms of θ, φ; clearing the denominator yields

$$(\rho_\theta - \rho_\psi\cos\varphi)(\rho_\psi\sin\psi + \rho\cos\psi) + \rho_\varphi\sin\varphi\,(\rho_\psi\cos\psi - \rho\sin\psi) = 0. \tag{19}$$

Solving for ψ and then substituting all the formulas into (13) produces a parametrization of the body surface S in terms of the spherical angles θ, φ that is prescribed by its outline curves, as determined by the function $\rho(\theta,\varphi,\psi)$.

The implicit nature of the final equation (19) makes it potentially challenging to work with in practice. On the other hand, given a point $\mathbf{q}(\theta,\varphi,\psi)$ lying on the outline curve $C_{\theta,\varphi} \subset P_{\theta,\varphi}$ at spherical angles θ, φ, we can use (18) to directly find the corresponding point on the rim $R_{\theta,\varphi} \subset S$ lying on the normal line through $\mathbf{q}$, whereby

$$\mathbf{p}(\theta,\varphi,\psi) = \mathbf{q}(\theta,\varphi,\psi) + t\,\mathbf{n}_{\theta,\varphi} = \mathbf{q}(\theta,\varphi,\psi) - \frac{\rho\,\rho_\varphi}{\rho_\psi\sin\psi + \rho\cos\psi}\,\mathbf{n}_{\theta,\varphi}, \tag{20}$$

is the rim point that projects onto the outline point $\mathbf{q}(\theta,\varphi,\psi)$. Thus, substituting (18) back into (13) serves to explicitly parametrize the rim curve $R_{\theta,\varphi}$ knowing its corresponding outline curve $C_{\theta,\varphi}$:

$$\begin{aligned}
\mathbf{p}_{\theta,\varphi}(\psi) &= \mathbf{p}(\theta,\varphi,\psi) \qquad\qquad -\pi < \psi \le \pi. \\
&= -\,\frac{\rho\,\rho_\varphi}{\rho_\psi\sin\psi + \rho\cos\psi}\,\mathbf{n}_{\theta,\varphi} + \rho(\theta,\varphi,\psi)\cos\psi\,\mathbf{u}_{\theta,\varphi} + \rho(\theta,\varphi,\psi)\sin\psi\,\mathbf{v}_{\theta,\varphi},
\end{aligned} \tag{21}$$

It is noteworthy that this only involves the zenith variations of ρ, although we note that its azimuthal variations are dependent on the zenith variations as a consequence of equation (19). As θ, φ vary over the unit sphere, we obtain a doubly redundant three-parameter representation of the body surface S, since each point $\mathbf{p} \in S$ belongs to a one-parameter family of rims obtained by rotation around the surface normal at $\mathbf{p}$. On the other hand, in data-driven applications, one could thus practically use this formula to numerically reconstruct the surface S as a collection of points, e.g., a point cloud or triangulated mesh, [**1**, **9**, **10**].

Now, it is usually impractical to produce the entire set of outline curves, which, after all, are redundant as noted above. However, one could envision a drone shooting a movie of the body whilst following a prescribed space curve† $\mathbf{x}(s)$ around it; in particular, we require $\mathbf{x}(s) \neq \mathbf{0}$ since we are assuming $\mathbf{0} \in \Omega$. The resulting video will contain a one-parameter family of outline curves. If the path followed by the drone is sufficiently robust, one could expect to reconstruct the body from this data alone. We will assume that the drone camera is always pointed at the origin; the more complicated case of general camera orientations is deferred to a subsequent project. Thus, we can identify the images captured by its camera with the outlines corresponding to a one-parameter family of normals, or, equivalently, a smooth spherical curve $\mathbf{n}(s) = \mathbf{x}(s)/\| \mathbf{x}(s) \| \in \mathbb{S}^2$ for $a < s < b$. More explicitly, in view of equation (9),

$$\mathbf{n}(s) = \mathbf{n}_{\theta(s),\varphi(s)} = (\cos\theta(s)\, \sin\varphi(s),\, \sin\theta(s)\, \sin\varphi(s),\, \cos\varphi(s)), \tag{22}$$

so that $\theta(s), \varphi(s)$ denote the spherical angles of the camera axis when the drone is at position $\mathbf{x}(s)$.

We expect the body surface S — or at least the part of it that was filmed by the drone — to be the envelope of the corresponding one-parameter family of outline cylinders. In this case, the outline curve $C_s \subset P_{\theta(s),\varphi(s)}$ is determined by a function $r = \rho(s,\psi) > 0$, and thus parametrized by

$$\mathbf{q}(s,\psi) = \rho(s,\psi)\cos\psi\, \mathbf{u}(s) + \rho(s,\psi)\sin\psi\, \mathbf{v}(s), \qquad -\pi < \psi \leq \pi, \tag{23}$$

where $\mathbf{u}(s) = \mathbf{u}_{\theta(s),\varphi(s)}$, $\mathbf{v}(s) = \mathbf{v}_{\theta(s),\varphi(s)}$ are determined by (10). The corresponding outline cylinder K_s is parametrized by

$$\mathbf{p}(t,s,\psi) = t\,\mathbf{n}(s) + \rho(s,\psi)\cos\psi\, \mathbf{u}(s) + \rho(s,\psi)\sin\psi\, \mathbf{v}(s) \qquad -\pi < \psi \leq \pi, \qquad t \in \mathbb{R}. \tag{24}$$

The envelope of this one-parameter family of cylinders is obtained by requiring that the partial derivatives of $\mathbf{p}$ with respect to t, s, ψ be linearly dependent. Since

$$\frac{\partial \mathbf{p}}{\partial s} = \frac{\partial \mathbf{p}}{\partial \theta}\, \frac{\partial \theta}{\partial s} + \frac{\partial \mathbf{p}}{\partial \varphi}\, \frac{\partial \varphi}{\partial s}\,,$$

the system defining the current envelope is obtained from (16) by taking the sum of the first equation times θ_s plus the second equation times φ_s (again using subscript notation

† The parameter $s \in \mathbb{R}$ is not necessarily arc length.

for partial derivatives) along with the third equation as it stands. Thus λ has the same expression (17) as above, while the other equation

$$t\,\varphi_s + \rho_s \cos\psi - \rho\,\theta_s \cos\varphi\ \sin\psi = \lambda\,(\,t\,\theta_s \sin\varphi + \rho_s \sin\psi + \rho\,\theta_s \cos\varphi\ \cos\psi\,)\,. \tag{25}$$

can be solved for t in terms of s, ψ, yielding

$$t = \frac{\rho\,(\rho_s - \rho_\psi \theta_s \cos\varphi)}{(\rho_\psi \cos\psi - \rho \sin\psi)\,\theta_s \sin\varphi - (\rho_\psi \sin\psi + \rho\cos\psi)\,\varphi_s}\,. \tag{26}$$

Substituting this formula back into (24) serves to reconstruct the body surface S parametrized by s, ψ.

In particular, if the drone follows an equatorial great semicircle, so $\theta(s) = s$, $\varphi(s) = \frac{1}{2}\pi$, where $0 \le s \le \pi$ (since the back and the front of the body have the same outline), then the reconstructed body is given by (26) with

$$t = \frac{\rho\,\rho_s}{\rho_\psi \cos\psi - \rho\sin\psi}\,. \tag{27}$$

Alternatively, sending the drone on a polar great semicircle, e.g., setting $\theta(s) = 0$, $\varphi(s) = s$, where $0 \le s \le \pi$, produces

$$t = -\,\frac{\rho\,\rho_s}{\rho_\psi \sin\psi + \rho\cos\psi}\,, \tag{28}$$

which recovers (18). These can be compared with formulas in [**7**].

As stated this analysis only works in full for convex bodies, although the formulas are local and so can apply in certain directions to some of the visible parts of non-convex bodies. See [**3**, **6**, **8**, **11**, **12**] for further details.

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